

Social Planning for Hedonistic Altruists

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Abstract. We study the welfare consequences of social planning for citizens who display hedonistic altruism towards the next generation. A social planner must effectively be more patient than its citizens, even if *every* citizen has the same discount factor, but the planner also exhibits time inconsistency and has systematic future bias. We presume that successive generations of planners cannot bind one another to a course of action. Our main result states that every equilibrium policy with positive consumption that is nondecreasing in initial output must be either *fully* Pareto optimal, or “consumption inefficient” in the sense of Malinvaud (1953) and Phelps (1962, 1965). Moreover, we provide sufficient conditions for either outcome to occur.

1. Introduction

Should a government be more patient than the citizens it serves? And if it cannot bind successor governments to its plans, what is lost?

The first question is as old as the theory of optimal saving. Ramsey (1928) famously described the social discounting of future utilities as “ethically indefensible.” So did Stern (2007) in the context of climate policy, a particularly energetic forum for such debates.¹ Nordhaus (2007) countered this viewpoint by asserting that social discounting should be informed by observed market returns and savings; that is, by what people reveal about their own patience. This exchange is often read as a contest between ethics and revealed preference. A government that overrides the impatience of its citizens is paternalistic, while one that respects their preferences must adopt their discount rate.

We argue that the second half of this reading is incorrect. Whether respect for individual preferences calls for more patience than individuals display depends on *why* individuals care about the future. It rests on a distinction, often obscured in formal models, between *hedonistic* and *obligatory altruism*. Under the former, a parent derives genuine pleasure from the well-being of her child, felt in the same way that the consumption of a good yields direct satisfaction. Under obligatory altruism, by contrast, transfers from one generation to the next arise from a sense of social duty.

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¹On the basis of this sharply lower rate of time preference, the *Stern Review on the Economics of Climate Change* (Stern 2007) concluded that immediate environmental action was warranted.

Parents save for their children because that is what parents are *supposed* to do. They do not so much enjoy their offspring's well-being as fulfill an obligation.

At a behavioral level, the two forms of altruism can be modeled in the same way. Both can be represented by a utility function that includes the child's well-being. In a normative setting, however, matters are different. Suppose that every parent feels exactly the same degree of obligation and no hedonistic altruism at all. Then each parent *exactly* embodies the society's welfare instincts. The child's well-being is already accounted for via the parent's obligatory compliance, and a welfarist planner would err in tinkering with parental actions. But under hedonistic altruism, the child's well-being enters the planner's calculus *twice*: once for its own sake, and once via parental utility. This is not an accounting error. If the parent genuinely experiences pleasure from the child's welfare, that pleasure is a real component of her well-being. A planner who respects individual utilities must count it, *and* count the child's utility separately.

It follows that under hedonistic altruism, a (weighted) utilitarian planner must be more patient than the individuals themselves. This is true even when each parent uses the same discount factor δ and the planner uses discount factor β , whether or not β exceeds δ , as long as $\beta > 0$. The planner values generation t 's material utility directly, but also indirectly via every earlier generation's altruistic regard for generation t . Stern's patience can therefore be had on Nordhaus's premises, provided that altruism is hedonistic.²

This double-counting argument appears in Bernheim (1989) and in Caplin and Leahy (2004), while Ray (2013) connects it to the distinction between the two forms of altruism. Here, we study a related implication: if the utilities of deceased ancestors are no longer counted in social welfare, the resulting welfare criterion is dynamically inconsistent. The planner exhibits *future* bias: she initially prefers some future generations to consume more than the current generation wishes them to, but as that future generation becomes the present, she wishes to revise that generation's consumption downward. Bernheim (1989) introduces quite demanding conditions under which dynamic consistency can be restored, and discusses how commitment can approximate welfare optimality. But the question of how a planner, sophisticated in the sense of Strotz (1955), might behave without commitment while being fully aware of her own inconsistency, is left open. That is our theme.

Such a planner can be viewed as a sequence of entities, none of which can bind the next. She plays a game with her future selves. We describe her policy by a stationary Markov perfect equilibrium of that game. We restrict attention throughout to *regular* equilibria, in which every generation

²Suppose that $\beta = \delta$, so that the planner employs the very discount factor that her citizens do. She then places weight $(n+1)\delta^n$ on generation n 's felicity (a citizen places δ^n). If a generation spans 25 years, the planner's weight on felicity a century ahead is five times the citizen weight. That is equivalent to lowering the annual rate of time preference by $(\ln 5)/100$, or about 1.6 percentage points, comparable to the entire gap between Stern and Nordhaus.

enjoys positive consumption, and consumption is nondecreasing in initial output. Our examples possess such equilibria, but whether one exists in general is a question that we leave open.

What is lost when such a planner cannot commit? The answer, from Kydland and Prescott (1977) on, is that discretion is costly: the time-consistent outcome is worse than what commitment could achieve. In the literature on present-biased individuals that follows Phelps and Pollak (1968) and Laibson (1997), the outcome is typically worse for *every* self, which is why commitment devices are valued. We ask whether the same is true here.

Say that a consumption plan is *Pareto optimal* if no other feasible plan makes every generation at least as well off, in terms of its own hedonistic utility, and some generation strictly better off. This is an unusual definition in that the generations are not the strategic actors in the model. The *planners* are. Efficiency is therefore judged by those on whose behalf the game is played, and not by those who play it. In contrast, in the literature on consistent plans (see Section 7), players and beneficiaries coincide. There, Goldman (1979), Pearce (1983) and Ray (1987) show that equilibria may be inefficient from the standpoint of the active participants in those equilibria.

Our main result, Theorem 1, presents a dichotomy. The consumption plan generated by any regular equilibrium is *either* Pareto optimal, *or* it exhibits a dramatic form of consumption inefficiency: there is another plan with consumption higher at every date. Plans that are inefficient in this sense are unambiguously bad: they are not only utility-dominated but *consumption*-dominated (Malinvaud 1953, Phelps 1962, Phelps 1965, Cass and Yaari 1971, Cass 1972, Mitra 1979a).

This dichotomy varies across environments. When the technology is “strongly productive,” with the limit marginal product of capital in excess of 1, every plan generated by a regular equilibrium is Pareto optimal (Theorem 2). When the production function is “neoclassical” — i.e., strictly concave, with limit marginal product below 1 — sufficient conditions can be given for either pattern to emerge (Theorem 3). Specifically, for every degree of social concern β exhibited by the planner, equilibrium plans must be consumption-inefficient if individuals are patient enough, with discount factor δ close to 1. Conversely, when $\beta + \delta < 1$, every such plan must be Pareto optimal.

In summary, in certain production environments (e.g., strongly productive technologies), equilibrium plans redistribute across generations, favoring the future at the expense of the present, without any failure of Pareto optimality. But in environments where Pareto optimality fails, as it might in the neoclassical case, it fails with a vengeance. Even basic notions of consumption efficiency are violated, so that not just generational utility but consumption can be made to improve at every date.

Our exercise is deliberately abstract. It isolates the philosophical problem of the welfare implications of hedonistic altruism, and it asks the most basic question one can ask of the resulting

equilibrium: is it efficient? The same problem invites applied questions as well, on expenditure taxation, paternalism, and public debt; we mention these briefly in Section 8, but have not worked on them. Rather, we present this simple model as a fable in the sense of Rubinstein (2012).

2. Model

Generational time t is discrete, spanning all nonnegative integers. A growth model is parameterized by its initial condition y_0 and a technology f , where for all $t \geq 0$ we have:

$$(1) \quad y_t = c_t + k_{t+1} \text{ for all } t \geq 0 \text{ and } y_t = f(k_t) \text{ for all } t \geq 0,$$

where c is consumption, k is capital and y is output (including undepreciated capital). Equation (1) yields a class of *feasible programs* $\{c_t, y_t, k_t\}$ from y_0 . A sequence $\mathbf{c} \in \mathbb{R}_+^\infty$ is a *consumption plan* (from y_0) if it is the consumption sequence generated by some feasible program from y_0 .

Each generation values its own consumption through a felicity function u . We assume that u and f are increasing and twice differentiable, and we further assume that $u''(c) < 0$ and $u'(0) = \infty$, and that $f''(k) \leq 0$ for all $(k, c) \gg 0$, with $f(0) = 0$ and $f'(0) > 1$.

Every generation additionally values the felicity of future generations using the discount factor $\delta \in (0, 1)$. The total (hedonistic) utility of a consumption plan $\mathbf{c}$ (with tail $\mathbf{c}^t$) for generation t is:

$$(2) \quad U(\mathbf{c}^t) = \sum_{s=t}^{\infty} \delta^{s-t} u(c_s) = u(c_t) + \delta U(\mathbf{c}^{t+1}).$$

A utilitarian social planner seeks to maximize the discounted sum of current and future hedonistic utilities using the discount factor $\beta \in (0, 1)$. Deceased ancestors are not valued. This means that the welfare induced by a consumption plan $\mathbf{c}$ at time $t \geq 0$ is:

$$(3) \quad W(\mathbf{c}^t) = \sum_{s=t}^{\infty} \beta^{s-t} U(\mathbf{c}^s) = \sum_{s=t}^{\infty} \sum_{\tau=0}^{s-t} \beta^\tau \delta^{s-\tau-t} u(c_s) = \sum_{r=t}^{\infty} \sum_{q=t}^{\infty} \beta^{r-t} \delta^{q-t} u(c_{r+q-t}).$$

The third equality uses the double change of variables $r = t + \tau$ and $q = s - \tau$. We impose summability conditions so that utilities and welfare are bounded above over all consumption plans — though not necessarily below, to accommodate felicities such as $\log$ — and upper semicontinuous in the product topology on plans. Specifically, let $\lambda := \lim_{k \rightarrow \infty} f'(k)$ be the limiting marginal product of capital, and $e(c) := u'(c)c/u(c)$ the elasticity of the felicity function. We assume:

[Sum] Either $\lambda < 1$, or $\lambda > 1$ and $\limsup_{c \rightarrow \infty} \max\{\beta, \delta, \lambda^{-1}\} \lambda^{e(c)} < 1$.

Proposition 1 in the Appendix shows how [Sum] implies the needed boundedness of W (and U). It should be noted that [Sum] is slightly stronger than what we need for that Proposition. Specifically,

it also eliminates the knife-edge case in which $\lambda = 1$. Also note that if the production function f crosses the 45° line, as it will when capital depreciates and production technologies satisfy the Inada conditions, the summability condition is automatically satisfied, as λ is then smaller than 1.

Dynamic inconsistency is captured by the fact that if a consumption plan maximizes W_t at date t , its continuation does not generally maximize W_{t+1} . While it is certainly true from (2) and (3) that

$$(4) \quad W(\mathbf{c}^t) = U(\mathbf{c}^t) + \beta W(\mathbf{c}^{t+1}),$$

a rough parallel to the single-dynasty Ramsey parent described in (2), this is not a standard Bellman equation because U is defined on sequences of consumptions. Indeed, from (4) applied to generation $t + 1$, we see that $U(\mathbf{c}^{t+1}) = W(\mathbf{c}^{t+1}) - \beta W(\mathbf{c}^{t+2})$, and using this expression in equation (2), we have $U(\mathbf{c}^t) = u(c_t) + \delta W(\mathbf{c}^{t+1}) - \beta \delta W(\mathbf{c}^{t+2})$. Substituting this in (4), we must conclude that

$$(5) \quad W(\mathbf{c}^t) = u(c_t) + (\beta + \delta)W(\mathbf{c}^{t+1}) - \beta \delta W(\mathbf{c}^{t+2}),$$

and this, not (4), is the analogue to the standard generation-based recursive formulation in (2).³ The planner behaves as if she values her immediate descendant, just as a Ramsey parent would do, but also appears to *dislike* her grand-descendant, which is a novel feature that distinguishes this model from the nonpaternalistic utility structure

$$W(\mathbf{c}^t) = \Phi(c_t, W(\mathbf{c}^{t+1}), W(\mathbf{c}^{t+2}), \dots)$$

studied in Ray (1987), where it was assumed that Φ was nondecreasing in each of its arguments. In any event, it is this negatively signed term that succinctly captures the planner's time inconsistency.

A sophisticated planner will anticipate this inconsistency: she knows that all consumption plans may well be revised in the future. In this sense, the planner must play a dynamic game with herself. We capture this interaction by viewing the planner's policy as a stationary Markov perfect equilibrium, or *equilibrium* for short. Such a policy maps starting incomes y to choices of capital $k = g(y) \in [0, y]$ that will be used for production of next period's income.

3. Equilibrium

For any policy g and $y_0 = y$, denote by $\mathbf{c}(g, y)$ the induced consumption plan. Formally, $\mathbf{c}(g, y) := \{(f \circ g)^t(y) - g((f \circ g)^t(y))\}_{t=0}^\infty$. Define the associated sequence of capital stocks $\mathbf{k}(g, y) := \{g((f \circ g)^{t-1}(y))\}_{t=1}^\infty$, so that $k_{t+1}(g, y)$ is the stock chosen at date t , in keeping with (1). For any

³When $\mathbf{c}^{t+1} = 0$, (5) is to be understood as the welfare of the induced stream $(c_t, 0, 0, \dots)$, if the corresponding recursive expression in (5) involves subtraction of infinities. The same convention applies to (6) and (7) below.

such policy g and initial y , generational utility and planner value functions

$$U^g(y) := U(c(g, y)) \text{ and } V^g(y) := W(c(g, y))$$

are well defined. A policy g is an equilibrium if for each $y > 0$, $V^g(y) > -\infty$, and $g(y)$ solves

$$(6) \quad \max_{k \in [0, y]} u(y - k) + (\beta + \delta)V^g(f(k)) - \beta\delta V^g(f(g(f(k)))).$$

Equivalently, any equilibrium is a pair (g, V) such that, for every $y > 0$, $V(y) > -\infty$, and

$$(7) \quad g(y) \in \arg \max_{k \in [0, y]} u(y - k) + (\beta + \delta)V(f(k)) - \beta\delta V(f(g(f(k)))),$$

where $V(y) = W(c(g, y))$ is the maximum value. By Blackwell's unimprovability theorem, an equilibrium reduces to the optimal policy from the standard Bellman equation in the limiting cases $\beta = 0$ or $\delta = 0$.

Note that we ask that an equilibrium should have finite values. In case felicity is unbounded below, there is always a trivial "equilibrium" with zero consumption, which we ignore.

An equilibrium g is *regular* if every generation enjoys positive consumption that is nondecreasing in initial output, or more precisely, if at every $y > 0$, $c(g, y) \gg 0$ and $y - g(y)$ is nondecreasing in y . Regularity ensures that the equilibrium satisfies the following desirable properties.

Observation 1. *If g is a regular equilibrium, then:*

- (i) g and $y - g(y)$ are both nondecreasing, and in particular, g is continuous and 1-Lipschitz.
- (ii) $c(g, y)$ and $k(g, y)$ are monotone in time (increasing or decreasing) for every y .
- (iii) V^g is differentiable and $V^{g'}(y) = u'(y - g(y))$.

The proof is in the Appendix. We note that the questions

- Is there *some* regular equilibrium?
- Is regularity of equilibrium needed for the theorems that follow?

remain unanswered in this paper. (We do not even know if at this level of generality, there is *some* equilibrium, regular or not.) That said, we will show by example that the regularity stipulation appears to be sensible.

4. Two Examples

In this section we introduce two examples, both with linear equilibria.

4.1. Constant-Elasticity Production. Suppose that $u(c) = \ln c$, and that $f(k) = Ak^a$ for $0 < a \leq 1$ and $A > 1$. This example satisfies [Sum]. If $a < 1$, then $\lambda = 0 < 1$. If $a = 1$, then $\lambda = A > 1$, and since $e(c) = 1/\ln c$ for all $c > 1$, we have $A^{e(c)} \rightarrow 1$ as $c \rightarrow \infty$, so $\limsup_{c \rightarrow \infty} \max\{\beta, \delta, A^{-1}\}A^{e(c)} = \max\{\beta, \delta, A^{-1}\} < 1$. We guess at a solution in which

$$V(y) = B \ln y + C \text{ and } g(y) = \gamma y,$$

with $(B, \gamma) \gg 0$ and $C \in \mathbb{R}$. Recall equation (7), so that in equilibrium,

$$\begin{aligned} V(y) &= \max_k [u(y - k) + (\beta + \delta)V(Ak^a) - \beta\delta V(A(\gamma Ak^a)^a)] \\ &= \max_k [\ln(y - k) + (\beta + \delta)B \ln(Ak^a) - \beta\delta B \ln(A(\gamma Ak^a)^a)] + (\beta + \delta - \beta\delta)C \\ &= \max_k [\ln(y - k) + Ba(\beta + \delta - a\beta\delta) \ln k] + \xi \\ (8) \quad &= \max_k [\ln(y - k) + D \ln k] + \xi \end{aligned}$$

for every $y > 0$, where $\xi := B[(\beta + \delta) - \beta\delta(1 + a)] \ln A - Ba\beta\delta \ln \gamma + (\beta + \delta - \beta\delta)C$ collects the terms that do not involve k , and

$$(9) \quad D := Ba(\beta + \delta - a\beta\delta).$$

This maximization problem of the sum of logarithms has a standard solution:

$$(10) \quad \frac{k}{y} = \frac{D}{1 + D} \text{ (= } \gamma \text{ in equilibrium).}$$

Using $k = \gamma y$ in (8) along with (9), and rearranging to organize the constant terms, we must conclude that

$$\begin{aligned} B \ln y + C &= \ln([1 - \gamma]y) + D \ln(\gamma y) + \xi \\ (11) \quad &= (1 + D) \ln(y) + \ln(1 - \gamma) + D \ln \gamma + \xi. \end{aligned}$$

Matching the coefficient on $\ln y$ in equilibrium, we see that $B = 1 + D$, and invoking (9), we have:

$$(12) \quad B = \frac{1}{1 - a(\beta + \delta - a\beta\delta)}.$$

We note in passing that B is well defined and strictly positive, because

$$(13) \quad 0 < a\delta < a(\beta + \delta - a\beta\delta) < 1 \text{ for } a \in (0, 1] \text{ and } \beta, \delta \in (0, 1),$$

an inequality with more information that we use below.⁴ Of particular interest is the solution for γ , which combines (10) with the fact that $B = 1 + D$, and then uses (12):

$$(14) \quad \gamma = \frac{D}{1+D} = \frac{B-1}{B} = a(\beta + \delta - a\beta\delta),$$

which lies strictly between 0 and 1, by (13). So the equilibrium we have identified is regular. Notice, again using (13), that the equilibrium savings rate is larger than the rate at which decentralized savings would have occurred, which is $a\delta$.

As an ancillary observation, C can be obtained by matching constant terms in (11), yielding:

$$C = \frac{\ln(1-\gamma) + B[(\beta + \delta) - \beta\delta(1+a)] \ln(A\gamma^a)}{(1-\beta)(1-\delta)}.$$

4.2. Linear Production. Assume u is CRRA with $u(c) = (c^{1-\sigma} - 1)/(1-\sigma)$ for $\sigma > 0$ (with $u(c) = \ln(c)$ for $\sigma = 1$), and that f is linear, with $f(k) = Ak$ for $A > 1$. The summability conditions change depending on σ as discussed below. As in the previous example, we guess that

$$V(y) = B \frac{y^{1-\sigma} - 1}{1-\sigma} + C \text{ and } g(y) = \gamma y,$$

with $(B, \gamma) \gg 0$ and $C \in \mathbb{R}$. Now recall equation (7), so that in equilibrium,

$$(15) \quad \begin{aligned} V(y) &= \max_k [u(y-k) + (\beta + \delta)V(Ak) - \beta\delta V(A\gamma Ak)] \\ &= \max_k \left[\frac{(y-k)^{1-\sigma}}{1-\sigma} + (\beta + \delta) \frac{B(Ak)^{1-\sigma}}{1-\sigma} - \beta\delta \frac{B(A^2\gamma k)^{1-\sigma}}{1-\sigma} \right] + \xi \end{aligned}$$

for every $y > 0$, where, as in the previous example, ξ collects the terms that do not involve k ; here $\xi = (\beta + \delta - \beta\delta)[C - B/(1-\sigma)] - 1/(1-\sigma)$. This maximization yields the necessary and sufficient first-order condition:

$$-(y-k)^{-\sigma} + (\beta + \delta)BA^{1-\sigma}k^{-\sigma} - \beta\delta B(A^2\gamma)^{1-\sigma}k^{-\sigma} = 0.$$

Inspecting this condition assures us that the choice of k is indeed linearly related to y , and so, imposing the equilibrium condition $k/y = \gamma$, we obtain

$$(16) \quad \left(\frac{\gamma}{1-\gamma} \right)^\sigma = (\beta + \delta)BA^{1-\sigma} - \beta\delta B(A^2\gamma)^{1-\sigma}.$$

We still need to solve out for B . Matching the appropriate coefficients in (15), we have

$$B = (1-\gamma)^{1-\sigma} + (\beta + \delta)B(A\gamma)^{1-\sigma} - \beta\delta B(A^2\gamma^2)^{1-\sigma},$$

⁴Write $a(\beta + \delta - a\beta\delta) = a\delta + a\beta(1 - a\delta)$. Since $0 < a\delta < 1$ and $0 < \beta < 1$, the second term lies strictly between 0 and $a(1 - a\delta)$, so $a\delta < a(\beta + \delta - a\beta\delta) < a\delta + a(1 - a\delta) = a + a\delta(1 - a) \leq a + (1 - a) = 1$.

so that

$$(17) \quad B = \frac{(1 - \gamma)^{1-\sigma}}{[1 - \beta(A\gamma)^{1-\sigma}][1 - \delta(A\gamma)^{1-\sigma}]}.$$

Equations (16) and (17) determine the equilibrium objects B and γ , while C is determined as an ancillary outcome by matching coefficients in (15). Using (17) in (16), we see that:

$$\frac{\gamma^\sigma}{1 - \gamma} = \frac{(\beta + \delta)A^{1-\sigma} - \beta\delta(A^2\gamma)^{1-\sigma}}{[1 - \beta(A\gamma)^{1-\sigma}][1 - \delta(A\gamma)^{1-\sigma}]},$$

which simplifies to

$$(18) \quad \gamma^\sigma = (\beta + \delta)A^{1-\sigma} - \beta\delta(A^2\gamma)^{1-\sigma}.$$

When $\beta = 0$, this solution describes the standard policy under dynamic programming with CRRA felicity. It also matches the solution of the first example when felicity is logarithmic ($\sigma = 1$) and the production function is linear ($a = 1$ in that example). To study the solution(s) of (18) for $\sigma \neq 1$, we distinguish between the cases $\sigma < 1$ and $\sigma > 1$.

Case 1 ($\sigma < 1$). The summability condition requires $\limsup_{c \rightarrow \infty} \max\{\beta, \delta, A^{-1}\}A^{e(c)} < 1$. When $\sigma < 1$, $\lim_{c \rightarrow \infty} e(c) = 1 - \sigma$, so [Sum] holds iff $A < \max\{\beta, \delta\}^{\frac{1}{\sigma-1}}$. When $\sigma < 1$, the LHS of (18) is increasing continuously from 0 to ∞ , and the RHS is decreasing continuously from $(\beta + \delta)A^{1-\sigma}$ to $-\infty$. Therefore, (18) admits a unique solution. Note that $\gamma \in (0, 1)$ since the difference between left and right sides of (18) is strictly negative at $\gamma = 0$ but strictly positive at $\gamma = 1$, by [Sum]. Moreover, $V^g > -\infty$ by $\sigma < 1$ and [Sum], so we have a regular equilibrium.

Case 2 ($\sigma > 1$). $\sigma > 1$ implies $\lim_{c \rightarrow \infty} e(c) = 0$, so $\limsup_{c \rightarrow \infty} \max\{\beta, \delta, A^{-1}\}A^{e(c)} = \limsup_{c \rightarrow \infty} \max\{\beta, \delta, A^{-1}\} < 1$, so [Sum] holds. With $\sigma > 1$, the LHS of (18) is a convex function increasing from 0 to ∞ , and the RHS is a concave function increasing from $-\infty$ to $(\beta + \delta)A^{1-\sigma}$, and so (18) can be shown to have two positive solutions. To check whether a solution is an equilibrium we must verify that the corresponding policy $g(y) = \gamma y$ induces $V^g > -\infty$. The consumption stream induced by g is $\mathbf{c}(g, y) = \{A^n \gamma^n (1 - \gamma)y\}_{n=0}^\infty$, and therefore

$$V^g(y) = \sum_{t=0}^{\infty} \sum_{s=0}^{\infty} \beta^t \delta^s \frac{(A^{t+s} \gamma^{t+s} (1 - \gamma)y)^{1-\sigma} - 1}{1 - \sigma}.$$

Thus, $V^g(y) > -\infty$ if and only if $\sum_{t=0}^{\infty} [\beta(A\gamma)^{1-\sigma}]^t \cdot \sum_{s=0}^{\infty} [\delta(A\gamma)^{1-\sigma}]^s < \infty$. In turn, this happens if and only if $\max\{\beta, \delta\}(A\gamma)^{1-\sigma} < 1$, which (by $\sigma > 1$) we can rewrite as

$$(19) \quad \gamma > \hat{\gamma} = \frac{\max\{\beta, \delta\}^{\frac{1}{\sigma-1}}}{A}.$$

We can now study the behavior of equation (18) under condition (19). At $\hat{\gamma}$, the LHS of (18) equals

$$\hat{\gamma}^\sigma = \hat{\gamma} \max\{\beta, \delta\} A^{1-\sigma},$$

and the RHS equals

$$(\beta + \delta) A^{1-\sigma} - \beta \delta (A^2 \hat{\gamma})^{1-\sigma} = (\beta + \delta) A^{1-\sigma} - \min\{\beta, \delta\} A^{1-\sigma} = \max\{\beta, \delta\} A^{1-\sigma} > \hat{\gamma}^\sigma.$$

At $\gamma = 1$, the LHS of (18) equals 1 and the RHS equals $(\beta + \delta) A^{1-\sigma} - \beta \delta A^{2(1-\sigma)} < 1$. Moreover, the difference between the LHS (convex) and the RHS (concave) is a convex function of γ that is negative at $\hat{\gamma}$ and positive at 1, so it has exactly one zero in $(\hat{\gamma}, 1)$. This unique solution corresponds to a (regular) equilibrium since it satisfies condition (19).

5. Efficiency

A consumption plan $\mathbf{c}$ (or a feasible program $\{c_t, y_t, k_t\}$) from $y > 0$ is *Pareto dominated* if there is a consumption plan $\tilde{\mathbf{c}}$ from y with

$$U(\tilde{\mathbf{c}}^t) \geq U(\mathbf{c}^t) \text{ for all } t, \text{ with strict inequality for at least one } t.$$

Otherwise, we say that the consumption plan or feasible program is *Pareto optimal*.

Notice that while the game is actually played by the social planner, our notion of Pareto optimality pertains to the (passive) agents in the society. This is not the notion studied in the literature on consistent plans, which asks whether an equilibrium is efficient for the players themselves; see Goldman (1979), Pearce (1983) and Ray (1987). We return to that comparison in Section 7.

An infinite-horizon model such as this could potentially exhibit another form of inefficiency. Say that a consumption plan $\mathbf{c}$ from $y > 0$ is *consumption inefficient* if there is a consumption plan $\tilde{\mathbf{c}}$ from y with $\tilde{c}_t \geq c_t$ for all t , with strict inequality for some t .⁵ Consumption inefficiency is a stark failure because it involves waste at the primitive consumption level, not in some misallocation of that consumption that generates utility losses for each generation. This strong efficiency failure plays a central role in an older literature going back to Malinvaud (1953), Phelps (1962, 1965), Cass and Yaari (1971), Cass (1972), and Mitra (1979a), with the Phelps–Koopmans theorem (Phelps 1965) and the subsequent Cass criterion for inefficiency (Cass 1972) serving as landmark references. We discuss the relevant efficiency literature further in Section 7.

5.1. A Dichotomy. In principle, there is a gap between the absence of consumption inefficiency, and the attainment of full Pareto optimality, as the latter is contingent on allocations of consumption

⁵Because f is strictly increasing and continuous, this is equivalent to the existence of a consumption plan $\tilde{\mathbf{c}}$ from y such that $\tilde{c}_t > c_t$ for all t .

rather than pure waste. And yet, the central result of this paper is that there is no such gap, at least under the assumption of a regular equilibrium.

Theorem 1. *Any consumption plan induced by a regular equilibrium is either consumption inefficient, or it is Pareto optimal.*

In the literature that follows Strotz (1955), time-inconsistent planners are engaged in a strategic interaction. The outcome of that interaction is often Pareto suboptimal, as in Phelps and Pollak (1968), Goldman (1979), Pearce (1983), Bernheim and Ray (1987) and Laibson (1997). That failure happens even when equilibria are consumption efficient in the sense studied by Malinvaud, Phelps, Cass, Mitra and others; see, for instance, Theorem 4.1 in Bernheim and Ray (1987).

The contrast with Theorem 1 is strong. Granted, Pareto suboptimality is indeed a possibility, but *only* via a severe failure of consumption efficiency. If that consumption inefficiency can be avoided, Pareto optimality is fully restored. The absence of a “gap” between the zones that admit consumption inefficiency and those that yield Pareto optimality is precisely the striking feature of the theorem. That feature emanates from the future bias exhibited by the planner. It causes the agent-level Euler equation to be distorted, but in a direction that can be systematically signed (see Lemma 1 in the proof below). That distortion destroys the optimality of the plan for every agent, given their initial conditions, but the initial conditions are twisted in favor of future agents in a way that compensates for this optimality failure. So there is no failure of *Pareto*-optimality, unless the future bias leads to “overaccumulation” and a consumption efficiency failure.

The proof that follows is long and involved, so it might be useful to informally connect these broad remarks to the technical arguments. Despite the Euler distortion just mentioned, if a path is interior, one can find weights on generational utilities such that the path satisfies the Euler equations of the resulting “artificial sum” of weighted utilities. That suggests a line of attack for Pareto optimality: show that the equilibrium maximizes this artificial objective. However, there are three difficulties with this approach.

First, the implied weights need not be nonnegative — at the level of abstraction just described, it is trivial to write down examples to the contrary. Second, the resulting infinite welfare sum need not be well defined, even under [Sum]. (Again, counterexamples can be found to this effect.) Finally, even when all weights are nonnegative and the weighted sum well defined, satisfying its Euler equations does not by itself establish maximization. The proof takes care of the first problem by showing that for regular equilibria, the implied weights are indeed nonnegative. This is related to the “systematic signing” of the distortion accomplished in Lemma 1, and alluded to above. It then

exploits limiting properties of finite sums, along with a classical characterization in Cass and Yaari (1971) for consumption efficient paths, to entirely circumvent the second and third problems.

In the linear-production example, Observation 3 shows how, with CRRA felicity and $\sigma \leq 1$, the supporting weights can be made explicit and seen to be nonnegative, and the formal objective becomes a well-defined maximization problem. But this approach, while suggestive, cannot be turned into a general line of argument, for the reasons given above.

Proof of Theorem 1. Let g be a regular equilibrium. Fix $y > 0$, and let $\mathbf{c}^* := \mathbf{c}(g, y)$ and $\mathbf{k}^* := \mathbf{k}(g, y)$. The following claim will imply that the implied supporting weights are nonnegative.

Lemma 1. *For all $t \geq 0$,*

$$(20) \quad u'(c_t^*) \geq \delta u'(c_{t+1}^*) f'(k_{t+1}^*).$$

Proof. Let $\Psi(y) := V^g(y) - \delta V^g(f(g(y)))$. With each series below absolutely convergent (positive parts summable by [Sum] and all values finite) we can rearrange the order of summation to obtain:

$$\begin{aligned} \Psi(y) &= V^g(y) - \delta V^g(f(g(y))) = \sum_{t=0}^{\infty} \sum_{s=0}^{\infty} \beta^t \delta^s u(c_{t+s}(g, y)) - \delta \sum_{t=1}^{\infty} \sum_{s=1}^{\infty} \beta^{t-1} \delta^{s-1} u(c_{t+s-1}(g, y)) \\ &= \sum_{t=0}^{\infty} \sum_{s=0}^{\infty} \beta^t \delta^s u(c_{t+s}(g, y)) - \sum_{t=0}^{\infty} \sum_{s=1}^{\infty} \beta^t \delta^s u(c_{t+s}(g, y)) \\ &= \sum_{t=0}^{\infty} \beta^t u(c_t(g, y)), \end{aligned}$$

which is nondecreasing in y (after induction on t) by regularity of g . Note that for any $k \in (0, y)$,

$$(21) \quad u(y - k) + (\beta + \delta)V^g(f(k)) - \beta\delta V^g(f(g(f(k)))) = u(y - k) + \delta V^g(f(k)) + \beta\Psi(f(k)).$$

By regularity, $y - g(y) > 0$. Consider any $k' > g(y)$ such that $y - k' > 0$. Because $g(y)$ solves (6), we can use (21) to conclude that

$$u(y - g(y)) + \delta V^g(f(g(y))) + \beta\Psi(f(g(y))) \geq u(y - k') + \delta V^g(f(k')) + \beta\Psi(f(k')).$$

Rearranging terms, we see that

$$(22) \quad [u(y - g(y)) + \delta V^g(f(g(y)))] - [u(y - k') + \delta V^g(f(k'))] \geq \beta[\Psi(f(k')) - \Psi(f(g(y)))] \geq 0,$$

where the second inequality uses $k' > g(y)$ and Ψ nondecreasing. It follows from (22) that

$$\frac{u(y - g(y)) - u(y - k')}{k' - g(y)} \geq \delta \frac{V^g(f(k')) - V^g(f(g(y)))}{k' - g(y)}$$

But then, using Observation 1(iii) and passing to the limit as $k' \downarrow g(y)$, we obtain:

$$(23) \quad \begin{aligned} u'(y - g(y)) &= \lim_{k' \downarrow g(y)} \frac{u(y - g(y)) - u(y - k')}{k' - g(y)} \\ &\geq \delta \lim_{k' \downarrow g(y)} \frac{V^g(f(k')) - V^g(f(g(y)))}{k' - g(y)} = \delta \cdot u'(f(g(y)) - g(f(g(y)))) \cdot f'(g(y)), \end{aligned}$$

where the last equality uses Observation 1(iii). To complete the proof, substitute $y = y_t^*$, $g(y) = k_{t+1}^*$, $c_t^* = y_t^* - k_{t+1}^*$ and $c_{t+1}^* = f(k_{t+1}^*) - g(f(k_{t+1}^*))$ in (23) to obtain (20). $\square$

Lemma 2 (Cass and Yaari 1971). *The consumption efficiency of any feasible program $\{c_t^*, y_t^*, k_t^*\}$ (with positive values of k_t^* at all dates) is equivalent to the condition*

$$(24) \quad \liminf_{T \rightarrow \infty} \sum_{t=0}^T p_t^*(c_t - c_t^*) \leq 0 \quad \text{for every feasible } \mathbf{c},$$

where $p_0^* = 1$ and $p_t^* = [\prod_{s=1}^t f'(k_s^*)]^{-1}$ for all $t \geq 1$. In particular, if $\mathbf{c}^*$ is consumption efficient and $\sum_t p_t^* c_t^* < \infty$, then

$$(25) \quad \sum_t p_t^* c_t \leq \sum_t p_t^* c_t^* \quad \text{for every feasible } \mathbf{c}.$$

Proof. Follows directly from Cass and Yaari (1971), Section 5, equation (11), p. 337, and the Corollary on p. 338. All the assumptions of their theorem are satisfied by our model. $\square$

Let $\mathbf{c}^*$ be an efficient consumption plan generated by a regular equilibrium from $y > 0$, so all consumptions and associated capital stocks are strictly positive. We claim that $\mathbf{c}^*$ is Pareto optimal. If not, there is a distinct program $\mathbf{c} \neq \mathbf{c}^*$ that Pareto dominates $\mathbf{c}^*$. Write, for all t ,

$$(26) \quad d_t = u(c_t) - u(c_t^*) \quad \text{and} \quad D_t = U(\mathbf{c}^t) - U(\mathbf{c}^{*t}) = \sum_{s=t}^{\infty} \delta^{s-t} d_s \geq 0.$$

Observe that $D_t \geq 0$ for all t and is finite: a dominating plan has utilities bounded below by the finite utilities generated by $\mathbf{c}^*$ (because an equilibrium has finite V -values) and is bounded above, by [Sum]. Because $\mathbf{c}$ Pareto dominates $\mathbf{c}^*$, $c_S > c_S^*$ at some date S . By the strict concavity of u ,

$$(27) \quad d_t = u(c_t) - u(c_t^*) \leq u'(c_t^*)[c_t - c_t^*], \quad \text{with} \quad d_S = u(c_S) - u(c_S^*) \leq u'(c_S^*)[c_S - c_S^*] - \eta$$

for some $\eta > 0$. Recall the definition of $\{p_t^*\}$ in the statement of Lemma 2. Multiplying both sides of the inequalities in (27) by the corresponding values of $w_t := p_t^*/u'(c_t^*) > 0$ and transposing terms, we have $p_t^*[c_t - c_t^*] \geq w_t d_t$ for all t and $p_S^*[c_S - c_S^*] \geq w_S d_S + \epsilon$, where $\epsilon := w_S \eta > 0$.

Adding these inequalities over dates $0, \dots, T$ for any $T \geq S$, we have

$$\begin{aligned}
 \sum_{t=0}^T p_t^*[c_t - c_t^*] &\geq \sum_{t=0}^T w_t d_t + \varepsilon = \sum_{t=0}^T w_t [D_t - \delta D_{t+1}] + \varepsilon \\
 &= w_0 D_0 + \sum_{t=1}^T [w_t - \delta w_{t-1}] D_t - \delta w_T D_{T+1} + \varepsilon \\
 (28) \quad &= \sum_{t=0}^T \pi_t D_t - \delta w_T D_{T+1} + \varepsilon,
 \end{aligned}$$

where the second equality follows from the fact that $d_t = D_t - \delta D_{t+1}$ (inspect (26)), and the last equality follows from defining $\pi_0 := w_0$ and $\pi_t := w_t - \delta w_{t-1}$ for $t \geq 1$.

Note that $\pi_0 > 0$, and $\pi_t \geq 0$ for $t \geq 1$ by virtue of Lemma 1.⁶ Moreover, $D_t \geq 0$ for all t , because $\mathbf{c}$ Pareto dominates $\mathbf{c}^*$. It follows that the first partial sum in (28) is nonnegative for all T . Because $\varepsilon > 0$, *our proof is therefore complete if we can show that*

$$(29) \quad w_T D_{T+1} \rightarrow 0 \quad \text{as} \quad T \rightarrow \infty.$$

For if (29) holds, we can pass to the limit in (28) to argue that

$$\liminf_{T \rightarrow \infty} \sum_{t=0}^T p_t^*[c_t - c_t^*] > 0,$$

which contradicts (via Lemma 2, equation (24)) our presumption that $\mathbf{c}^*$ is efficient.

To prove (29), observe that $\{k_t^*\}$ is monotone in t , so $\ell := \lim_t f'(k_t^*)$ exists, possibly in the extended reals. We now consider two cases demarcated by two ranges in which ℓ might lie. The two arguments that result take very different forms.

Case 1: $\ell > 1$. Recall the definition of λ as $\lim_{k \rightarrow \infty} f'(k)$. If $\lambda > 1$, we are in the strongly productive case studied by Benveniste (1976) and Mitra (1979c), and in particular Theorem 1 in the latter contribution implies that because $\mathbf{c}^*$ is efficient, $\sum_{s=0}^{\infty} p_s^* c_s^* < \infty$.

Otherwise (recalling that $\lambda = 1$ is excluded by [Sum]), we have $\lambda < 1$. Define $\bar{y}$ by the intersection of f with the diagonal; i.e., by $f(\bar{y}) = \bar{y}$ and $f(k) < k$ for all $k > \bar{y}$. Then it is easy to see that $c \leq \max\{y_0, \bar{y}\}$ for *any* feasible level of consumption. Moreover, for all t sufficiently large,

$$p_t^* = \left[\prod_{s=1}^t f'(k_s^*) \right]^{-1} \leq m^t$$

⁶For $t \geq 0$, $\pi_{t+1} = w_{t+1} - \delta w_t = \frac{p_{t+1}^*}{u'(c_{t+1}^*)} - \delta \frac{p_t^*}{u'(c_t^*)} = \frac{p_{t+1}^* u'(c_t^*) - \delta p_t^* u'(c_{t+1}^*)}{u'(c_t^*) u'(c_{t+1}^*)} = \frac{u'(c_t^*) - \delta f'(k_{t+1}^*) u'(c_{t+1}^*)}{\prod_{s=1}^{t+1} f'(k_s^*) u'(c_s^*) u'(c_{t+1}^*)} \geq 0$.

for some $m \in (0, 1)$, because $\ell = \lim_{s \rightarrow \infty} f'(k_s^*) > 1$. Combining these observations, it follows again that $\sum_{s=0}^{\infty} p_s^* c_s^* < \infty$.

In short, in this case ($\ell > 1$), we have $\sum_{s=0}^{\infty} p_s^* c_s^* < \infty$ whether $\lambda > 1$ or $\lambda < 1$. Equation (25) of Lemma 2 then implies that $\sum_{s=0}^{\infty} p_s^* c_s < \infty$ as well.

We complete the argument for this case by noting that the definition of $\{p_t^*\}$ and iteration of (20) gives, for any $s \geq T$,

$$(30) \quad \delta^{s-T} u'(c_s^*) / u'(c_T^*) \leq p_s^* / p_T^*.$$

Combine this with the fact that $d_s \leq u'(c_s^*)[c_s - c_s^*] \leq u'(c_s^*)c_s$ (by the concavity of u), to conclude that whenever $\sum_{s=0}^{\infty} p_s^* c_s < \infty$,

$$(31) \quad \begin{aligned} 0 \leq \delta w_T D_{T+1} &= w_T \left[\sum_{s=T+1}^{\infty} \delta^{s-T} d_s \right] = \frac{p_T^*}{u'(c_T^*)} \left[\sum_{s=T+1}^{\infty} \delta^{s-T} d_s \right] \\ &\leq \frac{p_T^*}{u'(c_T^*)} \left[\sum_{s=T+1}^{\infty} \delta^{s-T} u'(c_s^*) c_s \right] \leq \sum_{s=T+1}^{\infty} p_s^* c_s, \end{aligned}$$

where the first equality uses (26), the second equality uses the definition of w_T , and the final inequality invokes (30). Because $\sum_{s=0}^{\infty} p_s^* c_s < \infty$, this inequality implies (29), and the proof for Case 1 is complete.

Case 2: $\ell \leq 1$. Then (with $\lambda = 1$ ruled out by [Sum]), it must be that $\lambda < 1$. Define $\tilde{k}$ as the smallest maximizer of $f(k) - k$. Note that $\tilde{k} > 0$ and $f'(\tilde{k}) = 1$, with $f'(k) \leq 1$ for all $k > \tilde{k}$. By Observation 1, $\lim_{t \rightarrow \infty} k_t^*$ is well defined; we claim it equals $\tilde{k}$. Suppose not; then (because $\ell \leq 1$) there is N large enough and $\nu > 0$ such that for all $t \geq N$, $k_t^* \geq \tilde{k} + \nu$. Consume an extra ν at any such date and leave all later consumptions unchanged. By a standard inductive argument, this new plan is indeed feasible, which contradicts the efficiency of c^* .

Define $\tilde{y} := f(\tilde{k})$ and $\tilde{c} := \tilde{y} - \tilde{k}$. Since $\lim_{t \rightarrow \infty} y_t^* = \tilde{y} = f(\tilde{k})$, we have $g(\tilde{y}) = \tilde{k}$ (by the continuity of g in Observation 1(i)). That is, if y_t^* reaches $\tilde{y}$, it stays there, and the desired result in (34) below follows right away. Otherwise, $e_t = y_t^* - \tilde{y}$ has unchanging sign, and $|e_t|$ monotonically decreases to zero. In what follows, our objective is to show that the convergence of k_t^* to $\tilde{k}$ is geometric. Because $f'(\tilde{k}) = 1$,

$$(32) \quad k_{t+1}^* - \tilde{k} = e_{t+1} + o(|e_{t+1}|) \quad \text{and therefore} \quad c_t^* - \tilde{c} = e_t - e_{t+1} + o(|e_{t+1}|).$$

Equilibrium value evolves as $V(y_t^*) = u(c_t^*) + (\beta + \delta)V(y_{t+1}^*) - \beta\delta V(y_{t+2}^*)$. Subtracting the steady state version of this equation yields:

$$V(y_t^*) - V(\tilde{y}) = [u(c_t^*) - u(\tilde{c})] + (\beta + \delta)[V(y_{t+1}^*) - V(\tilde{y})] - \beta\delta[V(y_{t+2}^*) - V(\tilde{y})].$$

We expand the difference at $(\tilde{y}, \tilde{c})$ using (32) and the fact that $V'(\tilde{y}) = u'(\tilde{c})$, by Observation 1(iii). By monotone convergence, $|e_{t+1}|, |e_{t+2}| \leq |e_t|$, so all remainders are $o(|e_t|)$. The cancellation of common terms, the use of (32) for $c_t^* - \tilde{c}$ and some transposition yields

$$(33) \quad \beta\delta e_{t+2} = (\beta + \delta - 1)e_{t+1} + o(|e_t|).$$

If $\beta + \delta - 1 \geq 0$, set $q = (\beta + \delta - 1)/(\beta\delta) < 1$, where the strict inequality comes from $(1 - \beta)(1 - \delta) > 0$. Equation (33) and monotonicity imply that for all t large, $|e_{t+2}| \leq \rho|e_t|$ for some $\rho \in (q, 1)$. If $\beta + \delta - 1 < 0$, multiplying (33) by the common sign of e_t, e_{t+1}, e_{t+2} yields $\beta\delta|e_{t+2}| = (\beta + \delta - 1)|e_{t+1}| + o(|e_t|) \leq o(|e_t|)$; but then, we obtain the same contraction $|e_{t+2}| \leq \rho|e_t|$ for some $\rho < 1$ and all large t . Thus convergence of y_t^* to $\tilde{y}$ is geometric, and this is therefore also true of k_t^* to $\tilde{k}$ (since $k_{t+1}^* = g(y_t^*)$ and g is Lipschitz).

Twice differentiability of f at $\tilde{k}$ implies $|f'(k_t^*) - 1| = |f''(\tilde{k}) \cdot (k_t^* - \tilde{k}) + o(|k_t^* - \tilde{k}|)| = O(|k_t^* - \tilde{k}|)$, and so by the geometric convergence just established, $\sum_t |f'(k_t^*) - 1| < \infty$. It follows from a well-known result on infinite products that

$$(34) \quad p_t^* = \left[\prod_{s=1}^t f'(k_s^*) \right]^{-1} \rightarrow \left[\prod_{s=1}^{\infty} f'(k_s^*) \right]^{-1} := p_{\infty} \in (0, \infty),$$

and so $w_t \rightarrow p_{\infty}/u'(\tilde{c}) := w_{\infty} \in (0, \infty)$.

Additionally, because $U(\mathbf{c}^{*t}) \rightarrow u(\tilde{c})/(1 - \delta)$, it must be that

$$(35) \quad 0 \leq D_t \leq \frac{u(\max\{y_0, \bar{y}\})}{1 - \delta} - U(\mathbf{c}^{*t}) \rightarrow \frac{u(\max\{y_0, \bar{y}\}) - u(\tilde{c})}{1 - \delta},$$

where $\bar{y}$ was already defined by the intersection of f with the diagonal in Case 1 (subcase $\lambda < 1$). So D_t is bounded. Combining (34) and (35), it follows that $w_T D_{T+1}$ is bounded in T .

With this in mind, return to the decomposition in (28), rewriting it here for convenience:

$$\sum_{t=0}^T p_t^* [c_t - c_t^*] \geq \sum_{t=0}^T \pi_t D_t - \delta w_T D_{T+1} + \varepsilon.$$

Recall that the summation term on the right hand side is nonnegative. If that term were unbounded, the boundedness of $w_T D_{T+1}$ established above would imply that $\sum_{t=0}^T p_t^* [c_t - c_t^*] \rightarrow \infty$, contradicting the efficiency of $\mathbf{c}^*$ via (24) of Lemma 2. Therefore $\sum_{t=0}^{\infty} \pi_t D_t < \infty$. But since

$$\pi_t = w_t - \delta w_{t-1} \longrightarrow (1 - \delta)w_{\infty} > 0,$$

this is only possible if $D_{T+1} \rightarrow 0$, and hence it must be that $w_T D_{T+1} \rightarrow 0$. But then we must conclude that $\liminf_{T \rightarrow \infty} \sum_{t=0}^T p_t^* [c_t - c_t^*] > 0$, which again, under the presumption that $\mathbf{c}^*$ is efficient, contradicts (24) of Lemma 2. $\square$

5.2. The Two Possibilities of Theorem 1. Theorem 1 outlines two sharply contrasting possibilities: that a regular equilibrium is either Pareto optimal or consumption inefficient. Of course, the contrast has little value unless each can be shown to be a real possibility. In this section, we provide a pair of theorems to this effect. The limiting behavior of the marginal product of capital provides a useful separation of possibilities.

5.2.1. Pareto Optimality When the Technology is Strongly Productive. In the case of a “strongly productive” technology with $\lambda = \lim_{k \rightarrow \infty} f'(k) > 1$, a regular equilibrium *ensures* full Pareto optimality.

Theorem 2. *Suppose $\lambda = \lim_{k \rightarrow \infty} f'(k) > 1$. Then every consumption plan generated by a regular equilibrium is Pareto optimal (and a fortiori, consumption efficient).*

So, as far as regular equilibrium is concerned, there is no dichotomy in the strongly productive case. Matters are quite different, though, when the technology is “neoclassical”, with a strictly concave production function satisfying suitable end-point conditions. This is what we turn to next.

5.2.2. Both Possibilities When the Technology is Neoclassical. Say that the technology is *neoclassical* if, in addition to the assumptions already made on f , we assume that $f''(k) < 0$ for all $k > 0$, and $\lambda = \lim_{k \rightarrow \infty} f'(k) < 1$. These imply the existence of a unique “golden rule” stock $\tilde{k}$ that maximizes $f(k) - k$ and is characterized by $f'(\tilde{k}) = 1$, and a unique $K > \tilde{k}$ such that $f(K) = K$, with $f(k) < k$ for all $k > K$. The Ramsey–Cass–Koopmans model, in which the production function satisfies standard Inada end-point conditions, uses a neoclassical technology.

Fix β , the discount factor of the planner. The theorem below asserts that inefficiency always arises if the generations are sufficiently patient; that is, if δ is close enough to 1.

Theorem 3. *Suppose that f is neoclassical. Then:*

(i) *For any $\beta > 0$, there is $\bar{\delta} \in (0, 1)$ such that if $\delta \in (\bar{\delta}, 1)$, every consumption plan generated by a regular equilibrium is consumption inefficient.*

(ii) *On the other hand, if $\beta + \delta < 1$, then every consumption plan generated by a regular equilibrium is Pareto optimal.*

In contrast to the strongly productive case, a neoclassical technology admits both possibilities: consumption inefficiency as well as Pareto optimality, and sufficient conditions can be provided

for each, thereby showing conclusively that the dichotomy presented more generally in Theorem 1 is not an empty possibility.⁷

5.3. Proofs of Theorems 2 and 3.

We begin with the strongly productive case.

Proof of Theorem 2. Fix a regular equilibrium g with consumption policy $c(z) = z - g(z)$ and a starting income $y_0 > 0$, and write $(c_t^*, y_t^*, k_{t+1}^*)$ for its induced program from y_0 . (So $c_t^* = c_t(g, y_0)$.) Recall that $p_t^* = [\prod_{s=1}^t f'(k_s^*)]^{-1}$ for all t . Under strongly productive f , Benveniste (1976, Proposition 1 and Theorem 1, pp. 341–343) informs us that

$$(36) \quad q_t := p_t^* k_{t+1}^* \longrightarrow L \in [0, \infty), \text{ and } c^* \text{ is consumption efficient if and only if } L = 0.$$

We claim that $L = 0$, so that (36) will give us the desired result.

Suppose on the contrary that $L > 0$. Then $q_{t+1}/q_t \rightarrow 1$ as $t \rightarrow \infty$, so that

$$(37) \quad \frac{q_{t+1}}{q_t} = \frac{k_{t+2}^* p_{t+1}^*}{k_{t+1}^* p_t^*} = \frac{k_{t+2}^*}{k_{t+1}^* f'(k_{t+1}^*)} = \frac{f(k_{t+1}^*)}{k_{t+1}^* f'(k_{t+1}^*)} \left(1 - \frac{c_{t+1}^*}{y_{t+1}^*}\right) \rightarrow 1$$

as well, as $t \rightarrow \infty$. At the same time, because $p_t^* \leq \lambda^{-t}$ and $\lambda > 1$, and $L > 0$, it follows from (36) that $k_{t+1}^* \rightarrow \infty$, so $f(k_{t+1}^*)/(k_{t+1}^* f'(k_{t+1}^*))$ converges to 1 as $t \rightarrow \infty$. Using this information in (37), we must conclude that

$$\frac{c_t^*}{y_t^*} \rightarrow 0, \quad \text{and} \quad \frac{y_{t+1}^*}{y_t^*} = \frac{f(k_{t+1}^*)}{y_t^*} = \frac{f(k_{t+1}^*)}{k_{t+1}^*} \left(1 - \frac{c_t^*}{y_t^*}\right) \rightarrow \lambda > 1.$$

In particular, $\{y_t^*\}$ is eventually increasing and unbounded. For $y_t^* \leq z \leq y_{t+1}^*$, monotonicity of the consumption policy $c(y)$ gives

$$0 < \frac{c(z)}{z} \leq \frac{c(y_{t+1}^*)}{y_t^*} = \frac{c_{t+1}^*}{y_{t+1}^*} \frac{y_{t+1}^*}{y_t^*} \longrightarrow 0,$$

so that our starting presumption $L > 0$ implies the tail property

$$(38) \quad \lim_{y \rightarrow \infty} \frac{c(y)}{y} = 0.$$

We now show that (38) leads to a contradiction. Intuitively, consumption becomes too low relative to income for it to be maintained as an equilibrium outcome. Specifically, we use the envelope property in Observation 1 to obtain a lower bound on the equilibrium value function, and the feasibility of consumption to obtain an upper bound on the same function, and then argue that the two bounds thus derived cannot be compatible with each other at all sufficiently large incomes.

⁷We remark that part (ii) of Theorem 3 does not invoke the strict concavity of f .

Formally, given [Sum], choose ζ, ρ, r with $\zeta > \lambda$ and $0 < \rho < r < 1$ such that⁸

$$\max\{\delta, \beta\}\zeta^r < 1 \quad \text{and} \quad u(x)^+ \in O(1 + x^\rho) \quad \text{as } x \rightarrow \infty.$$

As two immediate consequences, (a) if we define $\theta_t := \sum_{j=0}^t \beta^j \delta^{t-j}$ for all $t \geq 0$, then

$$\Theta := \sum_{t=0}^{\infty} \theta_t \zeta^{tr} = \frac{1}{(1 - \beta\zeta^r)(1 - \delta\zeta^r)} < \infty,$$

and (b) for every $b > 0$,

$$(39) \quad 0 < \int_b^{\infty} [u(x) - u(b)]x^{-r-1} dx < \infty.$$

Because (38) implies $f(g(y))/y \rightarrow \lambda > 1$, we can choose Y_0 sufficiently large so that $f(g(y)) > y$ and $f(y) \leq \zeta y$ for $y \geq Y_0$, and so that

$$(40) \quad \varepsilon := \max_{y \geq Y_0} \frac{c(y)}{y} < 1 \text{ satisfies } \varepsilon^{r-1} > \Theta.$$

The maximum in (40) is attained at some $Y \geq Y_0$, because $c(y)/y$ is continuous, positive, and converges to zero. Therefore:

$$(41) \quad c(Y) = \varepsilon Y \quad \text{while} \quad c(y) \leq \varepsilon y \quad \text{for all } y \geq Y.$$

Set $b := c(Y)$. Let $V(y) = V^g(y) = \sum_{t=0}^{\infty} \theta_t u(c_t(g, y))$ be the equilibrium value function. Normalize felicity and value by

$$h(x) := u(x) - u(b), \quad \widehat{V}(y) := V(y) - \frac{u(b)}{(1 - \beta)(1 - \delta)},$$

Note that $\sum_{t=0}^{\infty} \theta_t = [(1 - \beta)(1 - \delta)]^{-1}$, so $\widehat{V}$ is the equilibrium value computed net of $u(b)$ at every date. From any initial income $y \geq Y$, future incomes increase and all future consumptions are at least b . So $\widehat{V}(y) \geq 0$. For $y \geq Y$, define two candidate lower and upper bounds for $\widehat{V}(y)$:

$$\underline{V}(y) := \frac{h(\varepsilon y)}{\varepsilon}, \quad \overline{V}(y) := \sum_{s=0}^{\infty} \theta_s h(\zeta^s y).$$

The envelope formula from Observation 1(iii) and concave felicity applied to (41) imply

$$\widehat{V}(y) = \widehat{V}(Y) + \int_Y^y u'(c(z)) dz \geq \int_Y^y u'(\varepsilon z) dz = \underline{V}(y).$$

On the other hand, pure accumulation gives $c_s(g, y) \leq \zeta^s y$ for every $s \geq 0$. Hence

$$\widehat{V}(y) = \sum_{s=0}^{\infty} \theta_s h(c_s(g, y)) \leq \overline{V}(y).$$

⁸The power bound follows as in Proposition 1, by integrating the eventual upper bound on the elasticity where $u > 0$.

Together, these inequalities require

$$\underline{V}(y) \leq \widehat{V}(y) \leq \overline{V}(y) \quad \text{for every } y \geq Y.$$

We now show that these two bounds are incompatible. Write

$$J := \int_b^\infty h(x)x^{-r-1} dx \in (0, \infty),$$

where finiteness follows from (39). Because $b = \varepsilon Y$, a change of variables yields

$$\int_Y^\infty \underline{V}(y)y^{-r-1} dy = \varepsilon^{r-1} J.$$

All terms defining $\overline{V}$ are nonnegative on $[Y, \infty)$, so by interchanging sums and integrals (Tonelli's theorem) and performing another change of variables,

$$\int_Y^\infty \overline{V}(y)y^{-r-1} dy = \sum_{s=0}^\infty \theta_s \zeta^{sr} \int_{\zeta^s Y}^\infty h(x)x^{-r-1} dx \leq \Theta J.$$

Because $J > 0$ and $\varepsilon^{r-1} > \Theta$, we obtain

$$\int_Y^\infty \underline{V}(y)y^{-r-1} dy = \varepsilon^{r-1} J > \Theta J \geq \int_Y^\infty \overline{V}(y)y^{-r-1} dy,$$

contradicting $\underline{V}(y) \leq \overline{V}(y)$ for every $y \geq Y$. Therefore $L = 0$, so $p_t^* k_{t+1}^* \rightarrow 0$ and Benveniste's characterization implies consumption efficiency. Theorem 1 then gives Pareto optimality. $\square$

Proof of Theorem 3.

Part (i). The proof of this part is somewhat intricate, so we provide the main argument here and relegate the finer technical details to the Appendix (at points indicated below). Fix a regular equilibrium g , an initial $y_0 > 0$, and its induced plan $\{y_t^*, c_t^*, k_{t+1}^*\}$, with $c_t^* = c_t(g, y_0)$. By Observation 1(ii), $\{k_t^*\}$ is monotone. Since $\lambda < 1$, it is also bounded. Therefore, $\lim_{t \rightarrow \infty} k_t^* = \bar{k} < \infty$, and $g(f(\bar{k})) = \bar{k}$ by the continuity of g (Observation 1(i)). Let $\ell = f'(\bar{k})$.

We claim that for δ large enough, any regular equilibrium has $\ell < 1$. With that in hand, the rest is a simple matter of invoking the celebrated Phelps–Koopmans theorem (Phelps 1962, 1965):

Let $\tilde{k}$ be the maximizer of $f(k) - k$. If a feasible program satisfies

$$\liminf_{t \rightarrow \infty} k_t > \tilde{k},$$

then its consumption plan is consumption inefficient.

Because $\ell < 1$, we have $\bar{k} > \tilde{k}$. It follows that $\{c_t^*\}$ must be consumption inefficient. With that in mind, we establish the claim above.

If the claim is false, then there is $\{\delta(n)\} \rightarrow 1$, along with a sequence of regular equilibria $\{g^n\}$ and initial conditions $y_0(n)$, which generate paths with steady states $\{\bar{k}(n), \bar{y}(n), \bar{c}(n)\}$, all of which satisfy $\ell(n) = f'(\bar{k}(n)) \geq 1$. For n large enough, we have $\delta(n) > 1/f'(0)$, so $\bar{k}(n) > 0$ and $u'(\bar{c}(n)) < \infty$.⁹ Write $\bar{y}(n) = f(\bar{k}(n))$ and $\bar{c}(n) = \bar{y}(n) - g^n(\bar{y}(n))$. Passing to the time limit in (20) and using the continuous differentiability of u and f , we have $1 \leq \ell(n) \leq \delta(n)^{-1}$ for all n , so $\ell(n) \rightarrow 1$ and $(\bar{k}(n), \bar{y}(n), \bar{c}(n)) \rightarrow (\tilde{k}, \tilde{y}, \tilde{c})$, where $\tilde{k}$ is the golden rule stock characterized by $f'(\tilde{k}) = 1$, with $\tilde{y} := f(\tilde{k})$ and $\tilde{c} := \tilde{y} - \tilde{k}$.

Extend each g^n to zero by $g^n(0) = 0$. Since these policies are 1-Lipschitz (Observation 1(i)) and satisfy $0 \leq g^n(y) \leq y$, we can use the Arzelà–Ascoli theorem to find a subsequence converging uniformly on $[0, K]$ to a continuous, nondecreasing function g with $g(\tilde{y}) = \tilde{k}$, where recall that K solves $f(K) = K$. This is a limit object and has no direct equilibrium meaning, but it inherits key properties from the equilibrium sequence $\{g^n\}$. Specifically, Lemma 1 tells us that $u'(c_0(g^n, y)) \geq \delta(n)f'(g^n(y))u'(c_1(g^n, y))$ for all $y > 0$. The same is true of g for y sufficiently close to $\tilde{y}$ (see Appendix for details), so we have a nonnegative *Euler gap function*:

$$(42) \quad E(y) := u'(c_0(g, y)) - f'(g(y))u'(c_1(g, y)) \geq 0,$$

At $y = \tilde{y}$, though, $f'(g(\tilde{y}))$ equals 1, and $c_0(g, \tilde{y}) = c_1(g, \tilde{y})$, so

$$(43) \quad E(\tilde{y}) = u'(c_0(g, \tilde{y})) - f'(g(\tilde{y}))u'(c_1(g, \tilde{y})) = 0.$$

Let's pretend that g is differentiable (this issue will be fully circumvented in the Appendix), and differentiate $E(y)$ in (42) with respect to y , evaluating the outcome at $y = \tilde{y}$. We do so by remembering that $c_0(g, y) = y - g(y)$ and $c_1(g, y) = f(g(y)) - g(f(g(y)))$, so that

$$(44) \quad \begin{aligned} E'(\tilde{y}) &= u''(\tilde{c})[1 - g'(\tilde{y})] - f'(\tilde{k})u''(\tilde{c})[f'(\tilde{k})g'(\tilde{y})][1 - g'(\tilde{y})] - u'(\tilde{c})f''(\tilde{k})g'(\tilde{y}) \\ &= u''(\tilde{c})[1 - g'(\tilde{y})]^2 - u'(\tilde{c})f''(\tilde{k})g'(\tilde{y}), \end{aligned}$$

where we use the fact that $f'(\tilde{k}) = 1$.

A second feature that g inherits at the limit is a local optimality property. To get there, subtract, for each n , steady state value from equilibrium value to obtain “value differences” defined by

$$\Delta V^n(y) \equiv V^{g^n}(y) - V^{g^n}(\bar{y}(n)).$$

These value differences converge near $\tilde{y}$ (as $n \rightarrow \infty$) to a function ΔV satisfying

$$\Delta V'(y) = u'(y - g(y)).$$

⁹Fix n and suppress its index on the induced path. If $\bar{k}(n) = 0$, then $\{k_t^*\}$ is nonincreasing, and then so is $\{c_t^*\}$ by regularity, so Lemma 1 gives $u'(c_t^*) \geq \delta(n)f'(k_{t+1}^*)u'(c_{t+1}^*) \geq \delta(n)f'(k_{t+1}^*)u'(c_t^*)$, which implies $\delta(n) \leq 1/f'(k_{t+1}^*)$ for all t , a contradiction.

Subtracting constants does not change optimal choices. Passing to the limit using the uniform convergence of policies and value differences near $\tilde{y}$, the limit choice $\tilde{k} = g(\tilde{y})$ locally maximizes

$$u(\tilde{y} - k) + (1 + \beta)\Delta V(f(k)) - \beta\Delta V(f(g(f(k)))).$$

Continuing to suppose that g is differentiable, the first-order condition at $\tilde{k}$, together with $f'(\tilde{k}) = 1$ and $\Delta V'(\tilde{y}) = u'(\tilde{c})$, gives

$$u'(\tilde{c}) = u'(\tilde{c})[1 + \beta - \beta g'(\tilde{y})].$$

Hence $g'(\tilde{y}) = 1$. Substitution into (44) yields

$$E'(\tilde{y}) = -u'(\tilde{c})f''(\tilde{k}) > 0,$$

but given (43), that contradicts $E(y) \geq 0$ slightly to the left of $\tilde{y}$.

A complete proof must handle the possible failure of differentiability of g , which necessitates a subtler argument. See details in the Appendix.

In summary, note that we have established the claim that $\ell < 1$ for all δ close enough to 1. As already mentioned, the proof can now be completed by applying the Phelps-Koopmans theorem.

Part (ii). Suppose $\beta + \delta < 1$. Fix a regular equilibrium g , an initial income $y_0 > 0$, and the induced path $\{y_t^*, c_t^*, k_{t+1}^*\}$ from y_0 , with $c_t^* = c_t(g, y_0)$. We first establish an “upper Euler inequality”

$$(45) \quad u'(c_t^*) \leq (\beta + \delta)f'(k_{t+1}^*)u'(c_{t+1}^*) \quad \text{for all } t \geq 0.$$

For any $y > 0$, set $k = g(y)$. Regularity implies strictly positive consumption at the current date and the next, so $0 < k < y$. Define

$$A_y(x) = u(y - x) + (\beta + \delta)V^g(f(x)), \quad B(x) = \beta\delta V^g(f(g(f(x)))).$$

By Observation 1, B is nondecreasing and A_y is differentiable near k . The optimality of k implies, for all sufficiently small $h > 0$,

$$A_y(k) - A_y(k - h) \geq B(k) - B(k - h) \geq 0.$$

Dividing by h and letting $h \downarrow 0$, the envelope identity in Observation 1(iii) gives

$$u'(c_0(g, y)) \leq (\beta + \delta)f'(g(y))u'(c_1(g, y)).$$

Applying this inequality at $y = y_t^*$ establishes (45) (no differentiability of g is required).

As in Part (i), $\{k_t^*\}$ is monotone and bounded, so $k_t^* \rightarrow \bar{k} \geq 0$. If $\bar{k} > 0$, continuity of g gives $g(f(\bar{k})) = \bar{k}$, and regularity gives $\bar{c} = f(\bar{k}) - \bar{k} > 0$. Passing to the limit in (45) and cancelling

$u'(\bar{c})$ yields

$$f'(\bar{k}) \geq \frac{1}{\beta + \delta} > 1.$$

If instead $\bar{k} = 0$, the maintained assumption $f'(0) > 1$ implies that marginal products are eventually bounded strictly above one. Thus, in either case, there exist $R > 1$ and N such that $f'(k_t^*) \geq R$ for all $t \geq N$.

Recall that $p_0^* = 1$ and $p_t^* = [\prod_{s=1}^t f'(k_s^*)]^{-1}$ for $t \geq 1$. These prices decay geometrically, and boundedness of capital therefore implies $p_T^* k_{T+1}^* \rightarrow 0$. Let $\{y_t, c_t, k_{t+1}\}$ be any other plan from y_0 . Feasibility at date 0 and concavity of f at subsequent dates give

$$c_0 - c_0^* = -(k_1 - k_1^*), \quad c_t - c_t^* \leq f'(k_t^*)(k_t - k_t^*) - (k_{t+1} - k_{t+1}^*) \quad (t \geq 1).$$

Multiplying by p_t^* and summing, we obtain

$$\sum_{t=0}^T p_t^*(c_t - c_t^*) \leq p_T^*(k_{T+1}^* - k_{T+1}) \leq p_T^* k_{T+1}^* \rightarrow 0.$$

By Lemma 2, the path $\{y_t^*, c_t^*, k_{t+1}^*\}$ is therefore consumption efficient, and Theorem 1 implies that it is Pareto optimal. $\square$

6. The Examples Revisited

6.1. Constant-Elasticity Production. In our first example, the felicity function is logarithmic and the production function has constant elasticity. In this example we can give exact conditions that characterize Pareto optimality and consumption inefficiency.

Observation 2. *Suppose that felicity is given by $u(c) = \ln(c)$ and that the production function is of the form $f(k) = Ak^a$, for $A > 1$ and $a \in (0, 1)$. The following holds:*

- (1) *If $\beta + \delta - a\beta\delta > 1$, every consumption plan generated under the linear equilibrium is consumption inefficient.*
- (2) *If $\beta + \delta - a\beta\delta < 1$, every consumption plan generated under the linear equilibrium is Pareto optimal.*

Proof. Let $\tilde{k}$ be the “golden-rule” value of k that maximizes $Ak^a - k$. It is easy to see that

$$(46) \quad \tilde{k} = (Aa)^{1/(1-a)}.$$

Now consider any consumption plan generated by the policy $g(y) = \gamma y$, where $\gamma \in (0, 1)$. This path generates a monotone sequence of capital stocks $\{k_t\}_{t=1}^\infty$ that converges to the limit k_γ which

solves the equation

$$\gamma A k_\gamma^a = k_\gamma,$$

and solving this equation, we see that

$$(47) \quad k_\gamma = (A\gamma)^{1/(1-a)}.$$

Comparing (46) and (47), we must conclude that $k_\gamma > \tilde{k}$ if $\gamma > a$, or — using (14) — if $\beta + \delta - a\beta\delta > 1$. But then, the Phelps–Koopmans theorem implies that the plan must be consumption inefficient. Conversely, if $\beta + \delta - a\beta\delta < 1$ then $k_\gamma < \tilde{k}$. Theorem 2 of Cass (1972) then implies that the plan cannot be consumption inefficient, so by our Theorem 1 it must be Pareto optimal. $\square$

Remark 1. To strengthen this result substantially, one would have to show that no other Markov equilibrium exists, except the trivial one with $g(y) = y$, which we do not take seriously. (In any case, that equilibrium is strongly inefficient as well.)

Observation 2 is related to the inefficiency example in Ray (1987). However, in that case, the counterexample is presented in a setting in which planner value functions are nondecreasing in all future planner values, which does not hold here as already discussed.

6.2. Linear Production. In the second example, production is linear and felicity has constant elasticity. Since $f(k) = Ak$ with $A > 1$ is strongly productive, Theorem 2 applies directly. Nevertheless, because that proof is complex, we illustrate part of it by a direct argument here. This returns to the supporting-weight interpretation described before the proof of Theorem 1. In the example at hand with a restriction on the elasticity of the felicity function, the implied weights on felicity form a geometric sequence, and the required sums converge at the equilibrium and have summable positive tails over feasible plans. The supporting-weight construction can therefore be completed as an actual welfare-maximization argument.

Observation 3. *Suppose that felicity is given by $u(c) = (c^{1-\sigma} - 1)/(1 - \sigma)$, for $\sigma > 0$ (with $u(c) = \ln(c)$ for $\sigma = 1$), and that the production function is of the form $f(k) = Ak$, for $A > 1$. Then every consumption plan generated under the linear equilibrium is Pareto optimal.*

Proof. The linear equilibrium $g(y) = \gamma y$, with $\gamma \in (0, 1)$, is regular. Its production function satisfies $f'(k) = A > 1$. The conclusion therefore follows from Theorem 2. Nevertheless, we provide a direct illustration for the case $\sigma \leq 1$, which includes the logarithmic felicity function.

We show that there exists a sequence of Pareto weights $\{\pi_n\}_{n=0}^\infty$ such that $\mathbf{c}(g, y)$ and $\mathbf{k}(g, y)$ jointly solve:¹⁰

$$\max_{\mathbf{c} \in \mathbb{R}_+^\infty, \mathbf{k} \in \mathbb{R}_+^\infty} \sum_{n=0}^{\infty} \pi_n U(\mathbf{c}^n) \quad \text{subject to} \quad \begin{cases} k_{n+1} + c_n = Ak_n & \text{for all } n \geq 0 \\ y = Ak_0 & \text{given} \\ \lim_{n \rightarrow \infty} A^{-n} k_n = 0 \end{cases}$$

The Euler equation of this problem is:

$$c_{n+1} = \left(A\delta \cdot \frac{\sum_{i=0}^{n+1} \pi_i \delta^{-i}}{\sum_{i=0}^n \pi_i \delta^{-i}} \right)^{\frac{1}{\sigma}} \cdot c_n.$$

Let $q = A^{\sigma-1} \gamma^\sigma$, and define

$$\pi_0 = 1, \quad \pi_n = (q - \delta)q^{n-1} \quad (n \geq 1).$$

Equation (18) implies $q - \delta = \beta[1 - \delta(A\gamma)^{1-\sigma}] > 0$: for $\sigma < 1$ this follows from the maintained summability condition, and for $\sigma = 1$ it follows from $\delta < 1$. Therefore, $\pi_n > 0$ for all $n \geq 0$. Moreover, $A > 1$, $\gamma < 1$ and $\sigma \leq 1$ imply that $q < 1$. Next, we verify that the problem is well defined, i.e., $\sum_{n=0}^\infty \pi_n U(\mathbf{c}^n)$ does not explode to $+\infty$ along feasible paths. Since the weights $\{\pi_n\}$ decay geometrically at the same rate as $\{q^n\}$, this comes down to checking that $\limsup_{c \rightarrow \infty} qA^{e(c)} < 1$, where $e(c)$ is the elasticity of felicity with respect to consumption (see Proposition 1).

(1) If $\sigma < 1$, then $\lim_{c \rightarrow \infty} e(c) = 1 - \sigma$, so $\limsup_{c \rightarrow \infty} qA^{e(c)} = \gamma^\sigma < 1$.

(2) If $\sigma = 1$, then $\lim_{c \rightarrow \infty} e(c) = 0$, so $\limsup_{c \rightarrow \infty} qA^{e(c)} = \gamma^\sigma < 1$.

Finally, one can verify by induction that the following holds for all $n \geq 0$:

$$\sum_{i=0}^n \pi_i \delta^{-i} = \left(\frac{q}{\delta} \right)^n.$$

But then, the Euler equation simplifies to:

$$c_{n+1} = \left(A\delta \cdot \frac{q}{\delta} \right)^{\frac{1}{\sigma}} c_n = A\gamma c_n.$$

The Euler equation implies $c_n = (A\gamma)^n c_0$ for all $n \geq 0$, so $c_0 = (1 - \gamma) \sum_{n=0}^\infty (c_n / A^n)$. The resource constraint implies $c_n = Ak_n - k_{n+1}$ for all $n \geq 0$, so $\sum_{n=0}^\infty (c_n / A^{n+1})$ telescopes to

¹⁰Imposing $\lim_{n \rightarrow \infty} A^{-n} k_n = 0$ as an explicit restriction to the maximization problem is harmless: any candidate plan failing the transversality condition cannot maximize the objective.

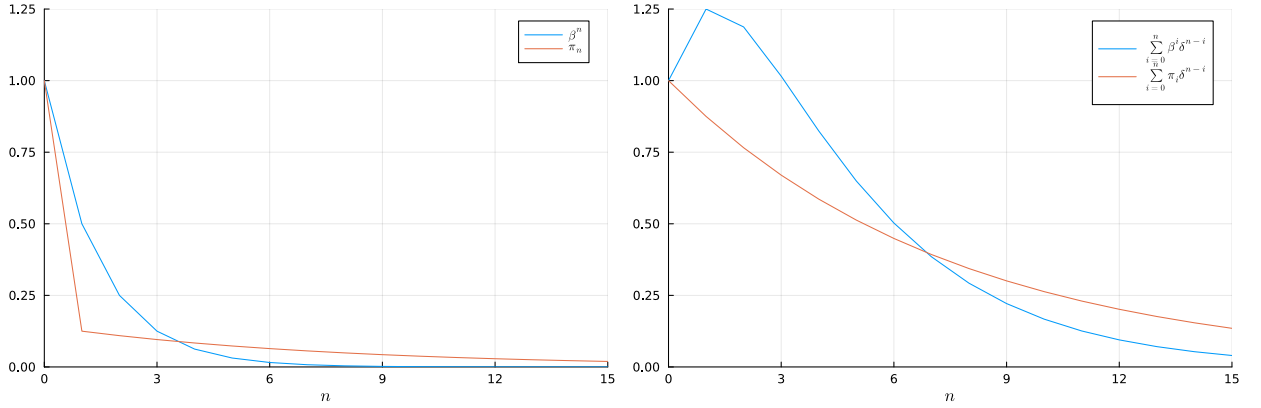


Figure 1. Plot of the true (blue) and implied (red) Pareto (left) and material (right) coefficients, for $\sigma = 1$, $\beta = 1/2$ and $\delta = 3/4$.

$k_0 - \lim_{n \rightarrow \infty} A^{-n} k_n$. But then, using $\lim_{n \rightarrow \infty} A^{-n} k_n = 0$ we get:

$$c_0 = A(1 - \gamma) \sum_{n=0}^{\infty} \frac{c_n}{A^{n+1}} = A(1 - \gamma) \left(k_0 - \lim_{n \rightarrow \infty} \frac{k_n}{A^n} \right) = A(1 - \gamma) k_0 = (1 - \gamma)y.$$

To conclude the proof, notice that $\mathbf{c}(g, y) = \{(1 - \gamma)(A\gamma)^n y\}_{n=0}^{\infty}$ and $\mathbf{k}(g, y) = \{A^{n-1} \gamma^n y\}_{n=1}^{\infty}$. Therefore, $\mathbf{c}(g, y)$ satisfies the Euler equation, $\mathbf{k}(g, y)$ satisfies $\lim_{n \rightarrow \infty} A^{-n} k_n(g, y) = 0$, and $c_0(g, y) = (1 - \gamma)y$. Since the objective is concave, the Euler equation and the transversality condition are sufficient to establish maximization.

For $\sigma > 1$, Pareto optimality already follows from Theorem 2; no auxiliary sum of felicity levels is needed. $\square$

The supporting-weight representation for $\sigma \leq 1$ suggests a re-framing of the sophisticated planner's problem. Under commitment, the planner maximizes the social welfare function ($W(\mathbf{c})$) weighted by the *true* Pareto coefficients $(\beta^n)_{n=0}^{\infty}$. Under discretion, she faces a dynamically inconsistent recursive problem: as it turns out, this problem is equivalent to the maximization of a social welfare function weighted by the *implied* Pareto coefficients $(\pi_n)_{n=0}^{\infty}$. We can plot the true and implied Pareto coefficients to see which generations benefit from commitment and which generations suffer from it, because the true Pareto coefficient of an early generation exceeds its implied coefficient, while for a late generation the reverse holds. Under discretion, some of the early generations' welfare must be sacrificed in order to design a self-enforcing consumption plan.

The Pareto coefficients capture how much the planner “cares” about the utility of each generation. We now study “material coefficients,” which capture how much the planner cares about

the felicity of each generation. These coefficients are obtained by re-writing the objective function as a weighted sum of $(u(c_n))_{n=0}^{\infty}$ rather than $(U(c^n))_{n=0}^{\infty}$. The true material coefficients are $(\sum_{i=0}^n \beta^i \delta^{n-i})_{n=0}^{\infty}$, and the implied material coefficients are $(\sum_{i=0}^n \pi_i \delta^{n-i})_{n=0}^{\infty}$. Notice that (for $\beta \neq \delta$):

$$\sum_{i=0}^n \beta^i \delta^{n-i} = \frac{\delta^{n+1} - \beta^{n+1}}{\delta - \beta} \quad \text{and} \quad \sum_{i=0}^n \pi_i \delta^{n-i} = (A^{\sigma-1} \gamma^{\sigma})^n.$$

The implied material coefficients form a geometric sequence, which means that the *implied* problem is dynamically consistent: even though the agent faces a dynamically inconsistent problem, it is “as if” she were facing a dynamically consistent one, but with a higher discount factor (since $A^{\sigma-1} \gamma^{\sigma} > \delta$). Figure 1 suggests (and a simple computation with the corresponding parameters confirms) that commitment increases the felicity of early generations and decreases that of late ones. Perhaps of interest, there is a set of “intermediate” generations ($n = 4, 5, 6$) for which, under commitment, felicity is higher but utility is lower. This happens because commitment lowers the felicity of later generations to such a degree that the intermediate generations suffer a loss of altruistic utility larger than their gain of felicity.

7. Bibliographical Notes

7.1. Hedonistic versus Obligatory Altruism. The distinction relates to Kant’s separation of duty (*Pflicht*) from inclination (*Neigung*): help given because the moral law demands it, as opposed to help given out of genuine pleasure in the welfare of another. Kant’s “friend of humanity” earns social and moral approbation precisely when, his pleasures extinguished by some private grief, he helps nonetheless from duty alone (Kant, 1785, 4: 397–399 in the Akademie pagination). The connection to our problem can be drawn tightly. Faced with a parent who is a friend of humanity, a planner of Kantian persuasion would see no reason to add to the social accounting: the child’s claim has been honored already, by the parent, on society’s behalf. The hedonistic parent’s pleasure, on the other hand, earns no moral credit, but it *is* a component of her well-being. A planner who counts well-being cannot ignore it. In economics, the most direct parallel is the distinction drawn by Sen (1977) between *sympathy* and *commitment*, the former sentiment activated when another’s welfare directly affects your own, and the latter when one acts on another’s behalf even though doing so does not increase one’s own welfare. See also Harsanyi (1955), who makes a related separation between “subjective preferences,” which encompass personal sympathies and tastes, and “ethical preferences,” which embody judgments about social welfare.

7.2. Welfare Implications of Hedonistic Altruism. The welfare implications of hedonistic altruism are noted in Ray (2013). The double-counting argument itself already appears in Bernheim (1989), though without the explicit contrast between the two forms of altruism, and in Caplin and Leahy (2004), who are concerned with the weighted sum of the utilities of all temporal selves, rather than the utility of the first. They do not take up the planner's own inconsistency, which is our subject. Ray and Wang (2000) and Ray et al. (2024) study individuals who discount the past as well as the future, and observe that caring for such future selves can generate greater patience as well as time inconsistency. Their equilibrium analysis concerns the individual, however, and not the planner. Farhi and Werning (2007) study long-run inequality when the social discount factor exceeds the private one, and Galperti and Strulovici (2017) provide axiomatic foundations for altruism defined on the total utility of future generations.

7.3. Efficient Capital Accumulation. The distinction between consumption efficiency and welfare optimality connects our analysis to the literature on capital accumulation beginning with Malinvaud (1953). The Phelps–Koopmans theorem, conjectured by Phelps (1962) and proved by Koopmans in Phelps (1965), supplies the overaccumulation criterion used in our constant-elasticity production example. Cass (1972) gives a sharper characterization of efficiency, also used there.

The supporting-price result used in Theorem 1 is due to Cass and Yaari (1971). Their Section 5 characterizes consumption efficiency through a finite-sum $\liminf$ comparison at technological present-value prices. Their corollary gives ordinary value maximization when the reference consumption value is finite. The distinction is essential here: the golden-rule case can have infinite consumption value, and the sum of weighted felicity levels need not converge even when consumption value is finite. Our argument uses finite summation by parts, nonnegative generational weights, and strict concavity to pass from the price comparison to Pareto optimality.

Brock (1971, Corollary 3) derives maximality from competitiveness and componentwise efficiency using the Cass–Yaari comparison. This identifies the supporting-price route, but the Pareto comparison in our model additionally requires control of the terminal utility-difference term. Mitra (1979b) studies existence and stability with variable discount rates and distinguishes weak maximality from the stronger overtaking criterion. His assumptions include strict concavity and a limiting marginal product below one; those results therefore cannot be applied directly to our full technology class. See also Mitra (1979a) for further efficiency criteria.

Benveniste and Gale (1975) extend Cass's analysis to unbounded programs using capital value and conditions on production elasticities. Their uniform negative second-elasticity condition does not include linear production. The directly applicable result for our strongly productive case is instead Benveniste (1976, Section 1): under stationary, increasing, concave production with $\inf f' > 1$

and nonnegative consumption, capital value converges, and a program is efficient exactly when that limit is zero. Theorem 2 combines this technological characterization with regularity, the equilibrium envelope equation, and the representation of value as actual discounted welfare to rule out a positive limiting capital value.

Neither our dichotomy nor its strongly productive strengthening establishes existence of a regular equilibrium. Both conclusions retain regularity as a maintained restriction on the equilibrium.

7.4. The Efficiency of Consistent Plans. Models of intergenerational altruism and their attendant inconsistencies can all be seen as descendants of the seminal paper by Strotz (1955). The literature on consistent planning that followed, including Phelps and Pollak (1968), Goldman (1979), Pearce (1983) and Laibson (1997), typically studies an individual, or a sequence of generations, with present bias, and interrogates the efficiency properties of equilibrium. For instance, Goldman (1979) and Pearce (1983) study cake-eating economies with finitely many generations. In Goldman's model generations care about one another's consumptions; in Pearce's, about one another's utilities. Both show that every interior perfect equilibrium is Pareto dominated by a program with slower consumption. The source is an externality: when a late generation makes its choice, the utilities of its predecessors are sunk, and so it ignores the pleasure they would take in a better-provided future. Ray (1987) studies an infinite-horizon model with production, in which each generation cares about the utilities of its successors, and shows by example that equilibria may be consumption inefficient once utility depends on at least two successors. Lane and Mitra (1981) and Bernheim and Ray (1987) also study the efficiency of plans in which each generation cares about its own consumption and the *consumption* of its successor.¹¹

Our setting reverses the roles. Individuals are perfectly consistent; it is the planner, with her future bias, who is not. And Theorem 1 does not conflict with the earlier results just discussed, for three reasons. First, that literature judges efficiency by the decision-makers themselves. Here the players are the planner's successive selves, but efficiency is judged by the generations on whose behalf she acts. Second, those papers assume that each player's utility is increasing in the utilities of others, whereas the planner's recursion (5) effectively involves two successor selves, where the second enters with a *negative* sign. Third, the generations' own preferences in (2) represent an (important) boundary case that is excluded from the strategic interactions studied in this literature: each generation cares directly about its immediate successor alone. An economy run by the generations themselves would then be Ramsey-like, and Blackwell's unimprovability argument would yield Pareto optimality.

¹¹Interdependent utilities among agents who are not necessarily located on a timeline are studied by Pearce (1983), Bergstrom (1999), Ray and Vohra (2020), and Vásquez and Weretka (2021), among others.

The comparison is instructive nonetheless. In Goldman and Pearce, multiple counting is an externality that the players ignore, and so they consume too quickly. Our planner internalizes it fully, and can err only in the opposite direction: the one inefficiency permitted by Theorem 1 is over-accumulation, as in the example of Ray (1987). We do not ask whether the planner's equilibrium is efficient from the standpoint of her own successive selves. Pareto domination in the utilities of all generations implies domination in the values of all the planner's selves, but not conversely, so Theorem 1 is silent on that question.

Finally, it might seem unusual to evaluate efficiency not from the standpoint of the active players, but from that of the inactive generations on whose behalf they play. Yet this distinction is natural whenever economic agents delegate their choices to separate decision-makers, such as governments, whose objectives may differ from those of the agents they represent. This issue will likely become increasingly relevant as individuals delegate their decisions to artificial intelligence; the objectives guiding the artificial agents' behavior need not coincide with the preferences against which their decisions should be evaluated.

8. Concluding Remarks

We close with a word on what this paper does *not* do. The model invites several applied questions. Because the planner is more forward-looking than every generation, she wishes to “distort” private intertemporal choices even when $\beta = \delta$, and the wedges that appear in the proof of Theorem 1 measure exactly the gap between the saving that her equilibrium calls for and the saving that individual generations would choose for themselves. Those wedges suggest a foundation for savings subsidies, and possibly a role for expenditure taxation in the tradition of Kaldor (1955). The same gap offers a narrative for social paternalism, or for resistance to populist demands for present consumption, that owes nothing to mistakes or behavioral biases on the part of citizens. It also yields an argument for retiring public debt faster than the public would like. Then there are questions of implementation. Social planners do not make consumption and investment decisions; individuals do, and in our model they have no say at all. A fuller account would give them one, and would ask which fiscal instruments or political institutions could sustain the planner's equilibrium. Such an account would turn a familiar question on its head: it would study individuals who must deal with an inconsistent planner, rather than a planner who must deal with inconsistent individuals.

We regard all these lines as reasonable directions of research, but we have not pursued them here.

Appendix

We show that W (and therefore U) is bounded above under [Sum]. We actually assume a slightly weaker condition than [Sum] in the Proposition below.

Proposition 1. *Let $\lambda = \lim_{k \rightarrow \infty} f'(k)$, and let $e(c) = u'(c)c/u(c)$ be the elasticity of felicity relative to consumption. Then, W is bounded above and upper semicontinuous in the space of feasible consumption plans from y_0 if either of the following holds:*

- (1) $\lambda < 1$.
- (2) $\lambda \geq 1$ with $\limsup_{c \rightarrow \infty} e(c) < \infty$ and $\limsup_{c \rightarrow \infty} \max\{\beta, \delta\} \lambda^{e(c)} < 1$.

Proof. To prove our claim, it suffices to make sure that we can apply the (positive) one-sided version of the Weierstrass M-test, i.e., find a sequence $\{M_n\}$ such that $\sum_{t=0}^n \beta^t \delta^{n-t} u(c_n) \leq M_n$ for all n and $\sum_{n=0}^{\infty} M_n < \infty$. First, suppose that $\lambda < 1$. Then there exists a finite steady state $k^\circ > 0$ such that $f(k^\circ) = k^\circ$ and $f(k) < k$ for all $k > k^\circ$. Let $c^\circ = \max\{k^\circ, y_0\}$ and $u(c^\circ)^+ = \max\{u(c^\circ), 0\}$. Then the Weierstrass M-test holds for $M_n = (n+1) \max\{\beta, \delta\}^n u(c^\circ)^+$. Suppose instead that condition (2) holds. If $\limsup_{c \rightarrow \infty} e(c) < 0$, then u is bounded above by 0, so just take $M_n = 0$. If $\limsup_{c \rightarrow \infty} e(c) \geq 0$, fix some $e^* > \limsup_{c \rightarrow \infty} e(c)$ and $\lambda_\varepsilon > \lambda$ such that $\max\{\beta, \delta\} \lambda_\varepsilon^{e^*} < 1$. Since in the limit u grows at finite elasticity, it can be dominated by a positive function that in the limit grows at higher constant elasticity. Formally, there exists some $p > 0$ such that $u(c) \leq p + pc^{e^*}$ for all $c \geq 0$. Since in the limit f grows at finite rate, it can be dominated by an affine function that grows at higher rate. Formally, there exists some $b > 0$ such that $f(k) \leq \lambda_\varepsilon k + b$ for all $k \geq 0$. Thus, along a feasible consumption plan we have:

$$k_{n+1} \leq f(k_n) \leq \lambda_\varepsilon k_n + b \leq \left(k_0 + \frac{b}{\lambda_\varepsilon - 1}\right) \lambda_\varepsilon^{n+1} - \frac{b}{\lambda_\varepsilon - 1} \leq \left(k_0 + \frac{b}{\lambda_\varepsilon - 1}\right) \lambda_\varepsilon^{n+1}.$$

Let $q = k_0 + b/(\lambda_\varepsilon - 1)$. Then:

$$c_n \leq f(k_n) \leq q \lambda_\varepsilon^{n+1}.$$

But then, $u(c_n) \leq p + pc_n^{e^*} \leq p + p(q \lambda_\varepsilon)^{e^*} [\lambda_\varepsilon^{e^*}]^n$. Thus, the Weierstrass M-test holds by choosing:

$$M_n = p(n+1) \max\{\beta, \delta\}^n + p(q \lambda_\varepsilon)^{e^*} (n+1) [\max\{\beta, \delta\} \lambda_\varepsilon^{e^*}]^n.$$

□

Proof of Observation 1. To simplify notation, let $H(k) = (\beta + \delta)V^g(f(k)) - \beta\delta V^g(f(g(f(k))))$. By optimality of g , for any $y > y' > 0$ we can write, provided $g(y)$ and $g(y')$ are feasible for both

y and y' :

$$\begin{aligned} V^g(y) &= u(y - g(y)) + H(g(y)) \geq u(y - g(y')) + H(g(y')) \\ V^g(y') &= u(y' - g(y')) + H(g(y')) \geq u(y' - g(y)) + H(g(y)). \end{aligned}$$

We can combine the two inequalities to obtain:

$$(48) \quad u(y' - g(y)) - u(y - g(y)) \leq V^g(y') - V^g(y) \leq u(y' - g(y')) - u(y - g(y')).$$

(i) First we prove monotonicity of g . Suppose by contradiction that $g(y') > g(y)$. Since $y > y' > g(y') > g(y)$, $g(y)$ and $g(y')$ are feasible for both y and y' , and strict concavity of u yields the following contradiction to (48):

$$u(y - g(y)) + u(y' - g(y')) < u(y' - g(y)) + u(y - g(y')).$$

To prove Lipschitz continuity of g , simply note that monotonicity of g and regularity imply $0 \leq |g(y) - g(y')| \leq |y - y'|$ for all $y, y' > 0$.

(ii) Monotonicity of $c(g, y)$ and $k(g, y)$ follows easily. Indeed, suppose that $f(k_1(g, y)) \geq y$. Then by monotonicity of g , $g(f(k_1(g, y))) \geq g(y)$, so $k_2(g, y) \geq k_1(g, y)$. By regularity, $c_1(g, y) \geq c_0(g, y)$. Finally, by monotonicity of f we have $f(k_2(g, y)) \geq f(k_1(g, y))$, which allows us to iterate the same reasoning and conclude $k_{t+1}(g, y) \geq k_t(g, y)$ and $c_{t+1}(g, y) \geq c_t(g, y)$ for all t . The case with $f(k_1(g, y)) \leq y$ follows symmetrically by swapping all inequalities.

(iii) Finally, we prove differentiability of V^g . Since g is continuous and $g(y) < y$, $g(y)$ and $g(y')$ are feasible for both y and y' whenever y' is sufficiently close to y . For $y' > y$, we can rewrite (48) as:

$$\frac{u(y' - g(y)) - u(y - g(y))}{y' - y} \leq \frac{V^g(y') - V^g(y)}{y' - y} \leq \frac{u(y' - g(y')) - u(y - g(y'))}{y' - y}.$$

For $y' < y$, both inequality signs in this display are reversed, because division is by $y' - y < 0$. Now the continuity of g implies that:

$$\lim_{y' \rightarrow y} \frac{u(y' - g(y)) - u(y - g(y))}{y' - y} = \lim_{y' \rightarrow y} \frac{u(y' - g(y')) - u(y - g(y'))}{y' - y} = u'(y - g(y)).$$

Therefore, V^g is differentiable and $V^{g'}(y) = u'(y - g(y))$. □

Details for part (i) of the proof of Theorem 3. We continue the contradiction argument in the main text, using its sequence g^n , limiting policy g , value differences ΔV^n and ΔV , and the Euler gap E defined in (42). We justify the inherited properties used there and then recover the required slope restriction from the left, without assuming that g is differentiable. The final contradiction is the same as in the main text.

Passing the equilibrium conditions to the limit. Let $I := \{y \in [0, K] : y - g(y) > 0\}$. The limit g is 1-Lipschitz, so $y - g(y)$ is continuous and nondecreasing. Thus I is an interval containing a neighborhood of $\tilde{y}$, since $g(\tilde{y}) = \tilde{k} < \tilde{y}$. For y sufficiently close to $\tilde{y}$, both y and $f(g(y))$ lie in I . Uniform convergence of g^n then gives convergence of current and next-period consumption to strictly positive limits. Passing to the limit in Lemma 1 proves (42); the equality (43) follows as in the main text.

The envelope identity in Observation 1(iii) gives

$$\Delta V^n(y) = \int_{\bar{y}(n)}^y u'(s - g^n(s)) ds.$$

On every closed subinterval of I , limiting consumption is bounded away from zero. Hence $(\Delta V^n)'$ converges uniformly there to $u'(y - g(y))$. Using $\bar{y}(n) \rightarrow \tilde{y}$ and integrating on a closed subinterval of I containing the normalization points as well, we obtain the convergence asserted in the main text, with

$$\Delta V(y) = \int_{\tilde{y}}^y u'(s - g(s)) ds.$$

In particular, $\Delta V(\tilde{y}) = 0$ and $\Delta V'(\tilde{y}) = u'(\tilde{c})$.

For convenience, denote the continuation terms in the normalized objectives by

$$\begin{aligned} H^n(k) &:= (\beta + \delta(n))\Delta V^n(f(k)) - \beta\delta(n)\Delta V^n(f(g^n(f(k)))), \\ H(k) &:= (1 + \beta)\Delta V(f(k)) - \beta\Delta V(f(g(f(k)))). \end{aligned}$$

The domain of H consists of those $k \in (0, K]$ for which both $f(k)$ and $f(g(f(k)))$ belong to I . It is an interval containing a neighborhood of $\tilde{k}$: both expressions are continuous and nondecreasing in k , and both equal $\tilde{y}$ at $k = \tilde{k}$. On every closed subinterval of this domain, $H^n \rightarrow H$ uniformly, and $H(\tilde{k}) = 0$. For $y \in I$ such that $g(y)$ belongs to this domain, passing to the limit in equilibrium optimality gives, for every k in the domain of H with $0 < k < y$,

$$(49) \quad u(y - g(y)) + H(g(y)) \geq u(y - k) + H(k).$$

Indeed, $g^n(y) \rightarrow g(y) < y$, and the convergence of H^n is uniform on a closed interval containing these choices and k . This justifies the local optimality property invoked in the main text.

We also need concavity of H below $\tilde{k}$. Each H^n is locally Lipschitz on $(0, \bar{k}(n))$, hence absolutely continuous on its closed subintervals. At any differentiability point k in this interval, continuity of g^n and $g^n(\bar{y}(n)) = \bar{k}(n)$ provide an income $y \in (0, \bar{y}(n))$ with $g^n(y) = k$. Optimality gives

$$(H^n)'(k) = u'(y - k).$$

If $k_1 < k_2$ are two such points, their corresponding incomes satisfy $y_1 < y_2$ by monotonicity of g^n , and $y_1 - k_1 \leq y_2 - k_2$ by regularity. Thus $(H^n)'(k_1) \geq (H^n)'(k_2)$. Absolute continuity now implies that H^n is concave on $(0, \bar{k}(n))$. Since $\bar{k}(n) \rightarrow \tilde{k}$, the limit H is concave on the portion of its domain below $\tilde{k}$, and by continuity also at the endpoint $\tilde{k}$.

Recovering the slope restriction from optimality. The main text obtains $g'(\tilde{y}) = 1$ from a first-order condition. To recover this restriction without presupposing differentiability, we first exclude a plateau ending at $\tilde{y}$. Such a plateau would prevent choices below $\tilde{k}$ from being reached as income approaches $\tilde{y}$ from below.

Suppose, then, that $g(y) = \tilde{k}$ on a left neighborhood of $\tilde{y}$, and let a be the smallest income with $g(a) = \tilde{k}$. Continuity and monotonicity give $\tilde{k} \leq a < \tilde{y}$ and $g(y) = \tilde{k}$ for $y \in [a, \tilde{y}]$. Whenever $f(k) \in (a, \tilde{y})$, we have $f(g(f(k))) = \tilde{y}$ and therefore

$$(50) \quad H(k) = (1 + \beta)\Delta V(f(k)), \quad H'_-(\tilde{k}) = (1 + \beta)u'(\tilde{c}).$$

For each $y \in (a, \tilde{y})$, apply (49) with the alternative $k = \tilde{k} - h$ and let $h \downarrow 0$. This gives

$$(51) \quad (1 + \beta)u'(\tilde{c}) = H'_-(\tilde{k}) \geq u'(y - \tilde{k}).$$

If $a = \tilde{k}$, the right-hand side diverges as $y \downarrow a$, a contradiction. Hence $a > \tilde{k}$, so $[a, \tilde{y}] \subseteq I$. At $q := f^{-1}(a) < \tilde{k}$, the two income arguments defining H are a and $\tilde{y}$. Thus q is in the interior of the domain of H , where H is concave.

For $y < a$ sufficiently close to a , continuity gives $q < g(y) < \tilde{k}$ and $f(g(y)) \in (a, \tilde{y})$. Also $y - g(y) > 0$. By (49), $g(y)$ locally maximizes $u(y - k) + H(k)$. Here H is differentiable by (50), so its first-order condition is

$$(52) \quad u'(y - g(y)) = (1 + \beta)f'(g(y))u'(f(g(y)) - \tilde{k}).$$

To extract the left derivative at a , write

$$F(y, k) := u'(y - k) - (1 + \beta)f'(k)u'(f(k) - \tilde{k}).$$

Equation (52) gives $F(y, g(y)) = 0$ to the left of a , and continuity gives $F(a, \tilde{k}) = 0$. At this point, twice differentiability of u and f yields

$$\begin{aligned} F_y(a, \tilde{k}) &= u''(a - \tilde{k}) < 0, \\ F_k(a, \tilde{k}) &= -u''(a - \tilde{k}) - (1 + \beta)[f''(\tilde{k})u'(\tilde{c}) + u''(\tilde{c})] > 0. \end{aligned}$$

Since $|g(y) - \tilde{k}| \leq |y - a|$, the first-order expansion of $F(y, g(y)) = 0$ at $(a, \tilde{k})$ gives

$$0 = F_y(a, \tilde{k})(y - a) + F_k(a, \tilde{k})(g(y) - \tilde{k}) + o(|y - a|).$$

Consequently, $g'_-(a) = -F_y(a, \tilde{k})/F_k(a, \tilde{k}) > 0$, whereas $g'_+(a) = 0$ on the plateau. Taking the one-sided derivatives of H at q , and using $f'(\tilde{k}) = 1$, we obtain

$$\begin{aligned} H'_+(q) &= (1 + \beta)f'(q)\Delta V'(a), \\ H'_-(q) &= (1 + \beta)f'(q)\Delta V'(a) - \beta f'(q)g'_-(a)\Delta V'(\tilde{y}). \end{aligned}$$

Thus $H'_+(q) > H'_-(q)$, contrary to concavity. This excludes the plateau. Since g is nondecreasing and $g(\tilde{y}) = \tilde{k}$, it follows that $g(y) < \tilde{k}$ for every $y < \tilde{y}$.

We can now recover the slope restriction at the golden rule. For $y < \tilde{y}$ sufficiently close to $\tilde{y}$, apply (49) at incomes y and $\tilde{y}$, comparing the choices $g(y)$ and $\tilde{k}$. Rearranging gives

$$u(\tilde{y} - g(y)) - u(\tilde{y} - \tilde{k}) \leq H(\tilde{k}) - H(g(y)) \leq u(y - g(y)) - u(y - \tilde{k}).$$

Divide by $\tilde{k} - g(y) > 0$ and let $y \uparrow \tilde{y}$. Continuity and the absence of a plateau imply that $g(y)$ takes every value sufficiently close to $\tilde{k}$ from below. The two bounds therefore give

$$(53) \quad H'_-(\tilde{k}) = u'(\tilde{c}).$$

For $h > 0$ small, put $k_h = f^{-1}(\tilde{y} - h)$ and $B_h = \tilde{k} - g(\tilde{y} - h)$. Since $f'(\tilde{k}) = 1$, we have $\tilde{k} - k_h = h + o(h)$; and the 1-Lipschitz property gives $0 \leq B_h \leq h$. Expanding the defining expression for H at k_h , and using (53), gives

$$u'(\tilde{c})h + o(h) = H(\tilde{k}) - H(k_h) = (1 + \beta)u'(\tilde{c})h - \beta u'(\tilde{c})B_h + o(h).$$

It follows that $B_h = h + o(h)$, or equivalently,

$$(54) \quad g(\tilde{y} - h) = \tilde{k} - h + o(h), \quad g'_-(\tilde{y}) = 1.$$

This is the one-sided counterpart of the slope restriction derived in the main text.

Contradiction with the Euler inequality. We now repeat the main text's final calculation from the left. Equation (54) and $f'(\tilde{k}) = 1$ imply

$$c_0(g, \tilde{y} - h) = \tilde{c} + o(h), \quad f(g(\tilde{y} - h)) = \tilde{y} - h + o(h), \quad c_1(g, \tilde{y} - h) = \tilde{c} + o(h).$$

Also $f'(g(\tilde{y} - h)) = 1 - f''(\tilde{k})h + o(h)$. Substituting in the expression for E in (42), we obtain

$$E(\tilde{y} - h) = u'(\tilde{c})f''(\tilde{k})h + o(h) < 0$$

for all sufficiently small $h > 0$. This contradicts (42), exactly as the strictly positive derivative in the main text does. The contradiction excludes the sequence with $\ell(n) \geq 1$ and completes the justification of part (i). $\square$

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