Nash Bargaining with Coalitional Threats:

Supplementary Notes

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These notes discuss: (a) the independence of the various axioms considered in the main text, (b) a modification of the proof of Theorem 2 of Kaneko (1980) (in the pure bargaining case) that conforms to our axioms, and (c) an extension of Theorem 2 in the main text to allow for feasible sets that satisfy log subconvexity rather than convexity.

1. Independence of the Axioms Used For Theorem 1

We've shown in the main text that [Exp] is independent of the other axioms. We now consider the independence of each of the other axioms in turn. In what follows, we normalize $d_i = 0$ for all i, which means that [Inv] will only refer to scale invariance.

Independence of [UHC]

Say that a threat configuration $\boldsymbol{\Theta}$ is *fully effective* (FE) if for every i, there is some coalition $S \ni i$ and some $x \in \Theta(S)$ such that $x_i > 0$. In what follows we work with the space $\mathcal{E}$ of all FE threat configurations.

Define two FE threat configurations $\boldsymbol{\Theta}$ and $\boldsymbol{\Theta}'$ in $\mathcal{E}$ to be *connected* if there is $\lambda \gg 0$ such that for every coalition S,

$$\Theta'(S) = \{x \in \mathbb{R}^S | x = \lambda \otimes y \text{ for some } y \in \Theta(S)\}.$$

Connectedness is an equivalence relation that partitions $\mathcal{E}$. For each element H of this partition, we can pick a representative $\boldsymbol{\Theta}_H \in \mathcal{E}$ such that for every i, the maximum payoff (over all coalitions containing i, and all allocations in those coalitions) equals 1. For by the compactness of every threat set, that maximum payoff is finite, and because any threat constellation in $\mathcal{E}$ is fully effective, we can always find a multiplicative transform of every agent's payoff such that that maximum equals 1. In fact, this requirement uniquely pins down $\boldsymbol{\Theta}_H$ from every element or equivalence class H.

Now define for every i and every class H, $\alpha(i, H)$ to be the average payoff to i over all coalitions containing i, and over all allocations in those coalitions within $\boldsymbol{\Theta}_H$.

Finally, define for every game G, a new solution $\phi(G)$ by

$$\phi(G) = \begin{cases} \sigma(G) & \text{if } \boldsymbol{\Theta} \text{ is not fully effective} \\ \arg\max_{x \in U(G)} \prod_{j \in N} x_j^{\alpha(j,H)} & \text{if } \boldsymbol{\Theta} \text{ is fully effective and belongs to class } H \end{cases}$$

It is easy to verify that ϕ satisfies all the axioms except for UHC. It fails UHC for sequences of fully effective threat constellations converging "down" to an ineffective constellation. Continuity is actually maintained for fully effective threat constellations that are converging to some limit threat constellation which is also fully effective. Note also that ϕ satisfies a condition stronger than [Sym] that asks for symmetry whenever F and $\boldsymbol{\Theta}$ are symmetric.

Independence of [Par]

It is well known in the context of pure bargaining problems that a solution that assigns the zero vector to each problem satisfies all of Nash's axioms other than [Par]. In our setting, this may be precluded by the requirement that a solution must lie within the unblocked set, which could therefore rule out the disagreement point. So consider instead the solution

$$\phi(G) = \arg\min_{x \in U(G)} \prod_{j \in N} x_j,$$

which clearly satisfies all our axioms except for [Par].

Independence of [Inv]

In standard bargaining problems, one example of a solution that satisfies all of Nash's axioms except for [Inv] is the Pareto-optimal allocation that assigns equal utilities to all players. Of course, this doesn't apply if the unblocked set is nonconvex or not comprehensive, but the the following modification will suffice. Consider a solution

$$\phi(G) = \arg\max_{x \in U(G)} \{\min_{i \in N} x_i\},$$

that seeks to maximize the utility of the worst-off individual under any payoff allocation. This violates [Inv], but clearly satisfies [Par], [Sym], [IIA] and [UHC]. Additionally, [Exp] is met under any "uniform expansion" λ with $\lambda_i = \lambda_j > 1$ for all $i, j \in N$.

Independence of [IIA]

In the standard bargaining framework, the Kalai-Smorodinsky (1975) solution satisfies all of Nash's axioms except [IIA]. If $U(G)$ were comprehensive we could use the Kalai-Smorodinsky solution with respect to $U(G)$. But we cannot do so because of the possible presence of coalitional threats. So we proceed as follows. For $U(G) \neq \emptyset$, let

$$b_i(U(G)) = \max\{x_i | (x_i, x_{-i}) \in U(G) \text{ for some } x_{-i}\}$$

and define

$$\phi(G) = \arg\max_{x \in U(G)} \left\{ \min_{i \in N} \frac{x_i}{b_i(U(G))} \right\}.$$

It's easy to verify that this solution satisfies all our axioms except for [IIA].

Independence of [Sym]

As in standard bargaining problems,

$$\phi(G) = \arg\max_{x \in U(G)} \prod_{j \in N} x_j^{c_j},$$

for $c_j > 0$ for all $j \in N$ and $c_i \neq c_j$ for some $i, j \in N$, serves as an example of a solution that satisfies all our axioms except for [Sym].

2. Modifying the Proof of Theorem 2 in Kaneko (1980)

For a bargaining problem (F, d) and a solution $s(F, d) \in F$, Nash defines the Pareto optimality axiom in terms of weak Pareto optimality:

[**WPO**] There does not exist $y \in F$ such that $y \gg s(F, d)$.

Note that our Pareto optimality axiom [Par] reduces precisely to [WPO] in a bargaining problem. A stronger assumption that is often made in the literature[1] is:

[**PO**] There does not exist $y \in F$ such that $y > s(F, d)$.

We insist on [WPO] rather than [PO] because this is the Pareto axiom that corresponds to a no-blocking condition for the grand coalition that aligns with our blocking relation for all subcoalitions: blocking requires a coalition to make *all* its members strictly better off.

Since Nash assumes that there is a feasible allocation $x \gg d$, the solution to maximizing the Nash product must yield a solution that satisfies [PO]. This means that that his axioms taken together imply [PO]. In fact, as Roth (1977) showed, it's possible to go quite far without directly assuming [WPO]. He showed that if the solution satisfies *strong individual rationality* [SIR]; that is, if $\phi(F) \gg 0$ by assumption, then (given [Inv] and [IIA]) it must satisfy [PO]. (His arguments are not directly applicable to nonconvex feasible sets.)

[1] See for example Thomson (1994).

Although Theorem 2 of Kaneko (1980) assumes [PO] as well as [SIR], we will now show that his proof can be strengthened to weaken [PO] to [WPO] and drop [SIR] entirely.

Define a binary relation $\succsim$ on $\mathbb{R}^N_+$ by $x \succsim y$ if and only if $x \in \phi(\{x, y\})$. Because Kaneko's (and our) domain includes sets containing 2 points, $\phi(\{x, y\})$ is well defined. Because Kaneko assumes [SIR], he restricts attention to this relation on $\mathbb{R}^N_{++}$. For now, we follow Kaneko and do the same. In his Lemma 2, Kaneko shows (using [IIA]), that $\succsim$ on $\mathbb{R}^N_{++}$ is complete and transitive.

In his Lemma 3, Kaneko observes that $\succsim$ satisfies a strong monotonicity property:

(1) $$x > y \text{ implies } x \succ y.$$

This follows directly from [PO]. If we assume [Par] instead, we can only claim:

(2) $$x \gg y \text{ implies } x \succ y.$$

Kaneko also shows, using his (and our) continuity axiom, that $\succsim$ is continuous. Next, he claims that $\succsim$ can be represented by a continuous utility function $H : \mathbb{R}^N_{++} \to \mathbb{R}$; that is, $H(x) \geq H(y)$ if and only if $x \succsim y$. To do so, he appeals to Lemma 3.3 in Kaneko and Nakamura (1979), making use of (1). But this is by now a familiar representation result that does not require strong monotonicity. In fact, the proof of Proposition 3.C.1 in Mas-Colell et al. (1995) makes use only of "weak" monotonicity as in (2). Thus, Kaneko's representation of the weak ordering by a continuous function remains valid with (2) instead of (1). In his next step, Kaneko applies a Lemma from Osborne (1976) to show that H can be taken to be of the form

(3) $$H(x) = \prod_i x_i^c,$$

where $c \gg 0$. Osborne's Lemma requires that (i) H be non-decreasing in each argument and (ii) H satisfy scale invariance. Clearly, (i) follows from (2) and continuity. So, again, we can proceed with Kaneko's proof using [WPO] instead of [PO]. The rest of Kaneko's proof does not rely on [PO]. In particular, by [Sym] we can choose $c = (1, \ldots, 1)$.

We now verify that [SIR] can be dropped with no additional loss. To do this we rely on the representation (3), which allows us to assert that for any $x \gg 0$, $i \in N$ and positive number a, there exists $y \gg 0$ such that $y_i < x_i$, $y_j > a$ for all $j \neq i$ and $\prod_i y_i < \prod_i x_i$. Because H represents $\succsim$, this means that $x \succ y$, or equivalently that $\phi(\{x, y\}) = \{x\}$.

Now consider any $z \in \mathbb{R}^N_+$. with $z_i = 0$ for some $i \in N$. By [WPO], $\phi(\{y, z\}) = \{y\}$ for any $y \gg z$, and in particular, defining $a = \max_{j \in N} z_j$, this is true of the y identified in the previous paragraph. Putting these arguments together, we have

$$\phi(\{x, y\}) = \{x\} \text{ and } \phi(\{y, z)\} = \{y\}.$$

Observe that if $z \in \phi(\{x, y, z\})$, then by [IIA], $z \in \phi(\{y, z\})$, which contradicts $\phi(\{y, z\}) = \{y\}$. Similarly, by [IIA], $y \in \phi(\{x, y, z\})$ would contradict $\phi(\{x, y\}) = \{x\}$. Therefore $\phi(\{x, y, z\}) = \{x\}$. Another application of [IIA] implies that $\phi(\{x, z\}) = \{x\}$.

In summary, we have shown that for any $x \in \mathbb{R}^N_{++}$ and any z that is not in $\mathbb{R}^N_{++}$, $\phi(\{x, z\}) = \{x\}$. Since our domain assures us that there exists $x \in F$ with $x \gg 0$, it follows from [IIA] that if $z_i = 0$ for any i, then $z \notin \phi(F)$. In other words ϕ satisfies [SIR], and the characterization $\sigma(G) = \arg\max_{x \in U(G)} \prod_{j \in N} x_j$ holds over $\mathbb{R}^N_+$.

3. Log Subconvexity

In the main text we pointed out that Theorem 2 can be generalized by weakening the assumption that all feasible sets are convex. In this Section we elaborate on that claim.

Say that a set $A \subseteq \mathbb{R}^m$ is *subconvex* if for every $x, y \in A$ and $t \in [0, 1]$, there is $z \in A$ such that $z \geq tx + (1 - t)y$, and a set $A \subseteq \mathbb{R}^m_+$ is *log subconvex* if $\ln A$ is subconvex.[2] Log subconvexity is a weak property that could apply to connected sets as well as to sets with isolated elements. (However, in Example 3, $F(\{1, 2, 3\})$ is not log subconvex.) It can be also be verified that a convex set in $\mathbb{R}^m_+$ must be log subconvex.[3]

Theorem 2 holds in the more general case where, for every coalition T, $F(T)$ is assumed to be log subconvex (rather than convex).

In proving Theorem 2, convexity of $F(T)$ was used only in Claim 2, which can be generalized as follows.

Claim 2′. Assume that $F(T)$ is log subconvex for every coalition T. Suppose there exists $x \in \Theta(T)$ such that $x \notin \Psi(T)$. Then there exists $W \subset T$ and $y \in \Psi(W)$ such that $y \geq x_W$.

[2]Note that log subconvexity is only defined for subsets A of $\mathbb{R}^m_+$, so that $\ln x$ is well-defined in the extended reals for all $x \in A$. The vector ordering "$\geq$" is then applied to the extended reals in the obvious way.

[3]Let A be convex. Then for every $x, y \in A$ and $t \in (0, 1)$, $z \equiv tx + (1 - t)y \in A$. But we know that $\ln(z) \geq t \ln x + (1 - t) \ln y$, which proves that A is log subconvex.

Proof. We proceed by induction on coalition size. Clearly the assertion is trivially true for all coalitions T of size 2 or less: $\Theta(T) = \Psi(T)$. Now suppose that for some $k \geq 2$, the lemma is true for any coalition of size k or less. Let T have cardinality $k+1$. Suppose that $x \in \Theta(T)$ and $x \notin \Psi(T)$. Then the Nash product over $F(T)$ exceeds that under x, so

$$\sum_{i \in T} \ln \tilde{x}_i > \sum_{i \in T} \ln x_i. \tag{4}$$

for any $\tilde{x} \in \Psi(T)$. For any $t \in (0,1)$, consider the allocation $t \ln x + (1-t) \ln \tilde{x}$. By the log-subconvexity of $F(T)$, there is a payoff allocation $z(t) \in F(T)$ such that $\ln z(t) \geq t \ln x + (1-t) \ln \tilde{x}$. Combining this information with (4), we must conclude that

$$\sum_{i \in T} \ln z_i(t) \geq \sum_{i \in T} [t \ln x_i + (1-t) \ln \tilde{x}] > \sum_{i \in T} \ln x_i;$$

i.e., the Nash product under $z(t)$ exceeds that under x. So $z(t)$ is blocked by some subcoalition of T using some coalitional Nash solution for that subset. Notice that for any limit point z of $z(t)$ as $t \to 1$, we have $z \geq x$. So there exist $H \subset T$ and $y \in \Theta(H)$ such that

$$y \geq z_H \geq x_H. \tag{5}$$

If $y \in \Psi(H)$ we are done. If $y \notin \Psi(H)$, then because $|H| \leq k$, the induction hypothesis implies there is $W \subset H$ and $z \in \Psi(W)$ such that $z \geq y_W$. Moreover, W can be chosen so that $\Theta(W) = \Psi(W)$. Combining this with (5), we must conclude that $\psi(W) \geq x_W$. ■

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