Partially Additive Utility Representations[1]

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General classes of utility representations are introduced which are partially additive. Preferences that admit such representations are characterized.

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1. Introduction

Consider a preference $\succsim$ over a product set $X \times Y$ where X is A^n for some interval A in in $\mathbb{R}$, Y is B^n for some topological space B, and $n \geq 3$. When $\succsim$ admits the utility representation

$$V(\mathbf{x}, \mathbf{y}) = f\Big(\sum_{i=1}^{n} h_i(x_i, \mathbf{y}), \mathbf{y}\Big) \text{ for all } (\mathbf{x}, \mathbf{y}) \in X \times Y,$$

where $\mathbf{x} = (x_1, \ldots, x_n)$, we say that it is *weakly partially separable.* When each y_i is thought of as a "characteristic" associated with the dimension i, and x_i is the observed outcome along that dimension, we can ask for additional separability in the form:

$$V(\mathbf{x}, \mathbf{y}) = f\Big(\sum_{i=1}^{n} h_i^*(x_i, y_i), \mathbf{y}\Big) \text{ for all } (\mathbf{x}, \mathbf{y}) \in X \times Y,$$

where $\mathbf{y} = (y_1, \ldots, y_n)$. That is, the characteristics y_k for $k \neq i, j$ do not influence the pairwise ranking of situations which only differ in the ith and jth dimensions. We refer to this representation as *partially separable.* A further natural subclass has the form:

$$V(\mathbf{x}, \mathbf{y}) = f\Big(\sum_{i=1}^{n} h^{**}(x_i, y_i), \mathbf{y}\Big) \text{ for all } (\mathbf{x}, \mathbf{y}) \in X \times Y;$$

that is, controlling for the "characteristics" $\mathbf{y}$, the partially separable representation is *anonymous.* We characterize preferences which admit such partially additive (utility) representations.

If Y is a singleton, then the above three forms reduce to additive utilities as in Debreu [1960], Gorman [1968] and Wakker [1988].

The rest of the article is organized as follows. Section 2 discusses some applications. Section 3 presents the axioms and the representations. Section 4 presents the main results and Section 5 outlines the proof strategy. Section 6 comments on "continuity" aspects of the representations.

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2. Applications

Let there be n states of the world, with x_i the scalar prize in state i, and suppose that y_i is the probability of state i. The ordering $\succsim$ is a preference ordering of lotteries of the form $(\mathbf{x}, \mathbf{y})$. Then a weakly partially separable representation is one in which there is separability across states, but not one that is necessarily linear or even separable in probability. That corresponds to a literature on non–expected utility theory starting from Allais [1953], Kahneman & Tversky [1979], and Machina [1982]; for a survey, see Starmer [2000].

Let $1, \ldots, n$ refer to individuals, x_i to their income or educational attainment (or any outcome describable by a scalar) and y_i to some individual–specific characteristics that might need to be invoked to evaluate the social implications of $\mathbf{x}$, such as age, gender, race or disability. The ordering $\succsim$ might now refer to a social planner's welfare ordering over different outcome distributions. While those distributions may be separable across individual achievements, the compositional balance of society might matter in non-separable ways.

Let $1, \ldots, n$ refer to commodities, x_i to their production levels, and y_i to some good-specific characteristic such as market value at international prices and social value with respect to pollution or need or cultural importance. The ordering $\succsim$ might now refer to a social planner's welfare ordering over different production composition, taking into account both market and social values. This is a common feature of social GDP accounting, expanded to take note of pollution or non-market household production; see, e.g., Dasgupta & Mäler [2000].

Let $1, \ldots, n$ refer to individuals, y_i to their baseline incomes, and y_i to the subsequent *growth rate* of that income. Then $\succsim$ may be viewed as a ranking of economic mobility over pairs of situations described by baseline income and income-specific growth rates. Genicot & Ray [2021] use our results to axiomatize a mobility measure of the form

$$V(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} \frac{y_i^{\alpha}}{\sum_{j=1}^{n} y_j^{\alpha}} x_i \text{ for some } \alpha > 0,$$

which is partially separable and anonymous, but is not separable in $\mathbf{y}$.

3. Setting

Notation. Let N be the set of indices $\{1, \ldots, n\}$. For any pair of n–dimensional vectors $\mathbf{z}$ and $\mathbf{z}'$, and $I \subseteq N$, $(\mathbf{z}_I, \mathbf{z}'_{N \setminus I})$ is the vector whose jth component is z_j if $j \in I$, and z'_j if $j \notin I$. For distinct $i, j \in N$, we write ij to denote $\{i, j\}$. For i and j in N, we use $-i$ and $-ij$ to denote $N \setminus \{i\}$ and $N \setminus \{i, j\}$ respectively, so vectors such as $\mathbf{z}_{-i}$ or $(\mathbf{z}_{ij}, \mathbf{z}'_{-ij})$ have the obvious interpretation.

For any n–dimensional vector z and permutation σ of N,[5] $\mathbf{z}_\sigma$ is the vector whose ith component is $z_{\sigma(i)}$ for each $i \in N$. For any $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$, write $\mathbf{x} \geq \mathbf{x}'$ iff $x_i \geq x'_i$ for every $i \in N$, and $\mathbf{x} > \mathbf{x}'$ if $\mathbf{x} \geq \mathbf{x}'$ and $\mathbf{x} \neq \mathbf{x}'$.

Setting. For $n \geq 3$, let X be A^n where A is any interval in $\mathbb{R}$ and Y be a topological space B, with typical elements $\mathbf{x} \equiv (x_1, \ldots, x_n)$ and $\mathbf{y} \equiv (y_1, \ldots, y_n)$. A *preference* is a complete and transitive binary relation $\succsim$ on $X \times Y$, with $\succ$ and $\sim$ as its strict and

[5] A *permutation* of N is a bijection $\sigma : N \to N$.

indifference components. Some axioms on $\succsim$ follow. All free variables that appear below are universally quantified over their respective ranges.

A1. *Any strict upper (lower) contour set of* $(\mathbf{x},\mathbf{y})$ *is open in* $X \times Y$.

A2. $\mathbf{x} > \mathbf{x}' \implies (\mathbf{x},\mathbf{y}) \succ (\mathbf{x}',\mathbf{y})$.

A3. $(\mathbf{x}_I, \mathbf{x}_{N\setminus I}\,;\,\mathbf{y}) \succsim (\mathbf{x}'_I, \mathbf{x}_{N\setminus I}\,;\,\mathbf{y})$ *iff* $(\mathbf{x}_I, \mathbf{x}'_{N\setminus I}\,;\,\mathbf{y}) \succsim (\mathbf{x}'_I, \mathbf{x}'_{N\setminus I}\,;\,\mathbf{y})$.

A4. $(\mathbf{x}_{ij}, \mathbf{x}_{-ij}\,;\,\mathbf{y}_{ij}, \mathbf{y}_{-ij}) \succsim (\mathbf{x}'_{ij}, \mathbf{x}_{-ij}\,;\,\mathbf{y}_{ij}, \mathbf{y}_{-ij})$ *if and only if*:

$$(\mathbf{x}_{ij}, \mathbf{x}_{-ij}\,;\,\mathbf{y}_{ij}, \mathbf{y}'_{-ij}) \succsim (\mathbf{x}'_{ij}, \mathbf{x}_{-ij}\,;\,\mathbf{y}_{ij}, \mathbf{y}'_{-ij})$$

A5. $(\mathbf{x},\mathbf{y}) \sim (\mathbf{x}_\sigma,\mathbf{y}_\sigma)$

$V : X \times Y \to \mathbb{R}$ is a *representation* of $\succsim$ if or every $(\mathbf{x},\mathbf{y})$ and $(\mathbf{x}',\mathbf{y}')$ in $X \times Y$:

$$(\mathbf{x},\mathbf{y}) \succsim (\mathbf{x}',\mathbf{y}') \iff V(\mathbf{x},\mathbf{y}) \geq u(\mathbf{x}',\mathbf{y}')$$

Definition 1: A *weakly partially additive representation for* $\succsim$ is a tuple $(V, f, \{h_i\}_{i\in N})$ such that (1) $\theta \mapsto h_i(\theta,\mathbf{y})$ is strictly increasing for $\theta \in A$, for any $\mathbf{y} \in Y$, (2) $\theta \mapsto f(\theta,\mathbf{y})$ is strictly increasing over all θ in the range of $\sum_{i\in N} h_i(x_i,\mathbf{y})$, for each $\mathbf{y} \in Y$, and (3) the map $V : X \times Y \to \mathbb{R}$ defined by:

$$V(\mathbf{x},\mathbf{y}) := f\Big(\sum_{i\in N} h_i(x_i,\mathbf{y})\,,\,\mathbf{y}\Big)$$

for all $(\mathbf{x},\mathbf{y}) \in X \times Y$, is a continuous utility representation of $\succsim$.

Definition 2: A partially additive representation for $\succsim$ is a *weakly partially additive representation* for $\succsim$, $(V, f, \{h^*_i\}_{i\in N})$, in which each function h^*_i can be written as $h^*_i(x_i, y_i)$ (*i.e.*, it does not depend on $\mathbf{y}_{-i}$).

Definition 3: A partially additive representation for $\succsim$, $(V, f, \{h^*_i\}_{i\in N})$, is *anonymous* there is a function h^{**} with $h^*_i = h^{**}$ for every $i \in N$.

4. Main Results

With the formalism in place, three representation theorems follow.

Theorem 1: *A preference* $\succsim$ *satisfies axioms A1 to A3 if and only if* $\succsim$ *admits a weakly partially additive representation.*

Theorem 2: *A preference* $\succsim$ *satisfies axioms A1 to A4 if and only if* $\succsim$ *admits a partially additive representation.*

Theorem 3: *A preference* $\succsim$ *satisfies axioms A1 to A5 if and only if* $\succsim$ *admits a partially additive representation that is anonymous.*

5. Proof Strategy

This section has two objectives. First, an overview of the proofs of existence of representations. The body of lemmas critical to this are stated and it is argued that they prove the existence claims. Second is the necessity of the axioms. The following lemma, established using Debreu's results on existence of additive representations, is the heart of theorems 1 to 3.

Lemma 1: *If $\succsim$ satisfies axioms $A1$ to $A3$, then there exists maps $h_1, \ldots, h_n : \mathbb{R} \times Y \to \mathbb{R}$ such that*:

(a) *For each $i \in N$ and any $\mathbf{y} \in Y$, the map $\theta \in \mathbb{R} \mapsto h_i(\theta, \mathbf{y}) \in \mathbb{R}$ has full range and is strictly increasing, and*

(b) *For each $\mathbf{y} \in Y$, $\mathbf{x} \in X \mapsto \sum_{i \in N} h_i(x_i, \mathbf{y})$ represents $\succsim|_{X \times \{\mathbf{y}\}}$.*

From 1a and 1b it is immediate that $A2$ is necessary for $\succsim$ to have type 0 representation because the map f in a type 0 representation is strictly increasing in its first argument. Further, 1b implies that $A3$ holds for $\succsim$ if it has a type 0 representation. As type 0 representations are continuous, the necessity of $A1$ follows.

Thus, axioms $A1$ to $A3$ are necessary conditions on $\succsim$ to admit type 0 representations. These axioms are also necessary for $\succsim$ to admit type 1 or type 2 representations. This is so because both type 1 and type 2 representations are type 0 representations by definition.

The ith map, in the n–tuple of maps whose existence is asserted by lemma 1, has $\mathbf{y}$ as its second argument. These maps possess more structure under additional axioms on the preference.

Lemma 2: *Suppose $\succsim$ satisfies $A4$ and $h_1, \ldots, h_n : \mathbb{R} \times Y \to \mathbb{R}$ satisfy*:

(1) *For each $i \in N$ and any $\mathbf{y} \in Y$, the map $\theta \in \mathbb{R} \mapsto h_i(\theta, \mathbf{y}) \in \mathbb{R}$ has full range and is strictly increasing, and*

(2) *For each $\mathbf{y} \in Y$, $\mathbf{x} \in X \mapsto \sum_{i \in N} h_i(x_i, \mathbf{y})$ represents $\succsim|_{X \times \{\mathbf{y}\}}$.*

Then, there exists $h_1^*, \ldots, h_n^* : \mathbb{R} \times \mathbb{R}_{++} \to \mathbb{R}$ such that:

(a) *For each $i \in N$ and any $y_i \in \mathbb{R}_{++}$, the map $\theta \in \mathbb{R} \mapsto h_i^*(\theta, y_i) \in \mathbb{R}$ has full range and is strictly increasing, and*

(b) *For each $\mathbf{y} \in Y$, $\mathbf{x} \in X \mapsto \sum_{i \in N} h_i^*(x_i, y_i)$ represents $\succsim|_{X \times \{\mathbf{y}\}}$.*

The above lemma asserts that the ith map, in the n–tuple of maps in lemma 1, can be taken to be such that its dependence on $\mathbf{y}$ is only through the ith component y_i. This is ensured by assuming that $\succsim$ additionally satisfies axiom $A4$. However, the ith map may perhaps depend on the index i. The next lemma asserts that this dependency may be assumed to be absent if $\succsim$ also satisfies axioms $A5$.

Lemma 3: *If $\succsim$ satisfies $A5$ and $h_1^*, \ldots, h_n^* : \mathbb{R} \times \mathbb{R}_{++} \to \mathbb{R}$ satisfy*:

(1) *For each $i \in N$ and any $y_i \in \mathbb{R}_{++}$, the map $\theta \in \mathbb{R} \mapsto h_i^*(\theta, y_i) \in \mathbb{R}$ has full range and is strictly increasing, and*

(2) *For each* $\mathbf{y} \in Y$, $\mathbf{x} \in X \mapsto \sum_{i\in N} h_i^*(x_i, y_i)$ *represents* $\succsim|_{X\times\{\mathbf{y}\}}$.

Then, there exists $h^{**} : \mathbb{R} \times \mathbb{R}_{++} \to \mathbb{R}$ *such that*:

(a) $\theta \in \mathbb{R} \mapsto h^{**}(\theta, \rho) \in \mathbb{R}$ *has full range and is strictly increasing, and*

(b) *For each* $\mathbf{y} \in Y$, $\mathbf{x} \in X \mapsto \sum_{i\in N} h^{**}(x_i, y_i)$ *represents* $\succsim|_{X\times\{\mathbf{y}\}}$.

The necessity of $A4$ for type 1 and type 2 representations is clear as h_i^* depends on $\mathbf{y}$ only through y_i allowing "cancellations". Further, $A5$ is obviously necessary for type 2 representations.

Lemmas 1 to 3 assert the existence of additive representations for $\succsim|_{X\times\{\mathbf{y}\}}$. However, to arrive at type m representations, it is required to assert the existence of an "aggregator" f such that the composition, say $V : X \times Y \to \mathbb{R}$, yields an utility representation of $\succsim$ over *all* of $X \times Y$. Further, this should be accomplished in such a way that V is continuous. The following lemma addresses these concerns.

Lemma 4: *Suppose the maps* $\bar{h}_1, \ldots, \bar{h}_n : \mathbb{R} \times Y \to \mathbb{R}$ *are such that*:

(1) *For each* $i \in N$ *and any* $\mathbf{y} \in Y$, *the map* $\theta \in \mathbb{R} \mapsto \bar{h}_i(\theta, \mathbf{y}) \in \mathbb{R}$ *has full range and is strictly increasing, and*

(2) *For each* $\mathbf{y} \in Y$, $\mathbf{x} \in X \mapsto \sum_{i\in N} \bar{h}_i(x_i, \mathbf{y})$ *represents* $\succsim|_{X\times\{\mathbf{y}\}}$.

Then, there exists a map $f : \mathbb{R} \times Y \to \mathbb{R}$ *such that*:

(a) *For each* $\mathbf{y} \in Y$, $\theta \in \mathbb{R} \mapsto f(\theta, \mathbf{y}) \in \mathbb{R}$ *is strictly increasing, and*

(b) *The map* $V : X \times Y \to \mathbb{R}$ *defined by*:

$$V(\mathbf{x}, \mathbf{y}) := f\Big(\sum_{i\in N} \bar{h}_i(x_i, \mathbf{y})\,,\, \mathbf{y}\Big) \text{ for all } (\mathbf{x}, \mathbf{y}) \in X \times Y,$$

is a continuous utility representation of $\succsim$.

Theorem 1 follows by invoking lemmas 1 and 4. Further, theorem 2 follows by invoking lemmas 1, 2 and 4. Lastly, theorem 3 follows by invoking lemmas 1 to 4. Thus, the theorems stand proven if the lemmas are established. This is done in the Appendix.

6. Further Comments

In definitions 1 to 3, only the composite map V is demanded to be continuous. However, if the constituent maps $h_1, \ldots, h_n$ and f are also required to be continuous by the definitions of the representations, then theorems 1 to 3 continue to hold. The key idea is as follows.

Consider $\mathbf{y}^1$ and $\mathbf{y}^2$ in Y which are close. Then, each "indifference surface" of $\succsim|_{X\times\{\mathbf{y}^2\}}$ on the section $X \times \{\mathbf{y}^2\}$ is arbitrarily close to each corresponding "indifference surface", from a system of such surfaces, of $\succsim|_{X\times\{\mathbf{y}^1\}}$ on the section $X \times \{\mathbf{y}^1\}$. This is because the preference $\succsim$ is continuous over the *whole* of $X \times Y$. This uniform convergence of the "indifference surfaces" must be utilized while extending Debreu's construction. Then, lemma 4 achieves an obvious upgrade.

Appendix

Proof of Lemma 1: Fix an arbitrary $\mathbf{y} \in Y$. Define $\geq_{\mathbf{y}}$ over X as:

$$\mathbf{x} \geq_{\mathbf{y}} \mathbf{x}' \iff (\mathbf{x}, \mathbf{y}) \succsim (\mathbf{x}', \mathbf{y})$$

With $S_{\mathbf{y}} := X \times \{\mathbf{y}\}$, note that $\{(\mathbf{x}', \mathbf{y}) \in S_{\mathbf{y}} : (\mathbf{x}', \mathbf{y}) \succ (\mathbf{x}, \mathbf{y})\}$ is equal to $S_{\mathbf{y}} \cap \{(\mathbf{x}', \mathbf{y}') \in X \times Y : (\mathbf{x}', \mathbf{y}') \succ (\mathbf{x}, \mathbf{y})\}$ for any $\mathbf{x} \in X$. Since $A1$ holds for $\succsim$, it follows that $\{(\mathbf{x}', \mathbf{y}) \in S_{\mathbf{y}} : (\mathbf{x}', \mathbf{y}) \succ (\mathbf{x}, \mathbf{y})\}$ is open in $S_{\mathbf{y}}$ for each $\mathbf{x} \in X$. Since the projection map $(\mathbf{x}', \mathbf{y}) \in S_{\mathbf{y}} \mapsto \mathbf{x}' \in X$ is an open map, it follows that $\{\mathbf{x}' \in X : (\mathbf{x}', \mathbf{y}) \succ (\mathbf{x}, \mathbf{y})\}$ is an open set in X. That is, the set $\{\mathbf{x}' \in X : \mathbf{x}' >_{\mathbf{y}} \mathbf{x}\}$ is open in X by the definition of $\geq_{\mathbf{y}}$. Similarly, the set $\{\mathbf{x}' \in X : \mathbf{x} >_{\mathbf{y}} \mathbf{x}'\}$ is open in X. Thus, $\geq_{\mathbf{y}}$ is continuous. Further, $A3$ holds $\succsim$. Thus, we have:

$$(\mathbf{x}_I, \mathbf{x}_{N\setminus I}) \geq_{\mathbf{y}} (\mathbf{x}'_I, \mathbf{x}_{N\setminus I}) \iff (\mathbf{x}_I, \mathbf{x}'_{N\setminus I}) \geq_{\mathbf{y}} (\mathbf{x}'_I, \mathbf{x}'_{N\setminus I})$$

Hence, by Debreu's theorem on existence of additively separable utility representations (theorem 5.5 of page 71 in Fishburn [1970]), there exists maps $h_1^{\mathbf{y}}, \ldots, h_n^{\mathbf{y}} : \mathbb{R} \to \mathbb{R}$ such that $\mathbf{x} \in X \mapsto \sum_{i\in N} h_i^{\mathbf{y}}(x_i)$ represents $\geq_{\mathbf{y}}$. Further, the construction in the proof shows that $h_i^{\mathbf{y}}$ has full range (see step 6 of theorem 5.4 in Fishburn [1970]). Since $\succsim$ satisfies $A2$, it follows that $h_i^{\mathbf{y}}$ is strictly increasing. Now, for each $i \in N$, define $h_i : \mathbb{R} \times Y \to \mathbb{R}$ by:

$$h_i(x_i, \mathbf{y}) := h^{\mathbf{y}}(x_i) \text{ for every } (x_i, \mathbf{y}) \in \mathbb{R} \times Y.$$

Thus, by the definition of $\geq_{\mathbf{y}}$ and the properties of $(h_1^{\mathbf{y}}, \ldots, h_n^{\mathbf{y}})$, the proof of the lemma is complete. ■

Proof of Lemma 2: Fix $\mathbf{y}^* \in Y$ and let $\mathbf{y} \in Y$ be arbitrary. Let i be 1 and let $j, k \in N \setminus \{1\}$ be distinct. Let $\mathbf{y}^1 := \mathbf{y}$ and $\mathbf{y}^2 := (\mathbf{y}_{1j}, \mathbf{y}^*_{-1j})$. Fix $\mathbf{x}^*_{-1j} \in \mathbb{R}^{n-2}$. Since $\mathbf{x} \in X \mapsto \sum_{i\in N} h_i(x_i, \mathbf{y})$ represents $\succsim|_{X\times\{\mathbf{y}\}}$ and $\succsim$ satisfies $A4$, both the following maps:

$$(x_1, x_j) \in \mathbb{R}^2 \mapsto h_1(x_1, \mathbf{y}^1) + h_j(x_j, \mathbf{y}^1) + \sum_{l \in N\setminus\{1,j\}} h_l(x_l^*, \mathbf{y}^1), \text{ and}$$
$$(x_1, x_j) \in \mathbb{R}^2 \mapsto h_1(x_1, \mathbf{y}^2) + h_j(x_j, \mathbf{y}^2) + \sum_{l \in N\setminus\{1,j\}} h_l(x_l^*, \mathbf{y}^2)$$

represent the same (weak) order over $\mathbb{R}^2$. That is, there exists a (weak) order over $\mathbb{R}^2$ such that it admits the following two maps as its additive utility representations:

$$(x_1, x_j) \in \mathbb{R}^2 \mapsto h_1(x_1, \mathbf{y}^1) + h_j(x_j, \mathbf{y}^1), \text{ and}$$
$$(x_1, x_j) \in \mathbb{R}^2 \mapsto h_1(x_1, \mathbf{y}^2) + h_j(x_j, \mathbf{y}^2)$$

By theorem 5.4 on page 65 of Fishburn [1970], there exists $\alpha > 0$ and $\beta, \beta' \in \mathbb{R}$

such that the following hold:

$$h_1(x_1, \mathbf{y}^1) = \alpha h_1(x_1, \mathbf{y}^2) + \beta \text{ for all } x_1 \in \mathbb{R}, \text{ and}$$
$$h_j(x_j, \mathbf{y}^1) = \alpha h_j(x_j, \mathbf{y}^2) + \beta' \text{ for all } x_j \in \mathbb{R}.$$

Fix $x^*, x^{**} \in \mathbb{R}$ such that $x^{**} > x^*$. Now, define $\alpha^j : Y \to \mathbb{R}_{++}$ and $\beta_1^j, \beta_j^j : Y \to \mathbb{R}$ as follows:

$$\alpha^j(\mathbf{y}) := \frac{h_1(x^{**}, \mathbf{y}^1) - h_1(x^*, \mathbf{y}^1)}{h_1(x^{**}, \mathbf{y}^2) - h_1(x^*, \mathbf{y}^2)}, \text{ and}$$
$$\beta_1^j(\mathbf{y}) := h_1(x^*, \mathbf{y}^1) - \alpha^j(\mathbf{y}) h_1(x^*, \mathbf{y}^2),$$
$$\beta_j^j(\mathbf{y}) := h_j(x^*, \mathbf{y}^1) - \alpha^j(\mathbf{y}) h_j(x^*, \mathbf{y}^2).$$

for all $\mathbf{y} \in Y$. It follows that $\alpha = \alpha^j(\mathbf{y})$, $\beta = \beta_1^j(\mathbf{y})$ and $\beta' = \beta_j^j(\mathbf{y})$. Thus, for any $\mathbf{y} \in Y$, we have the following:

$$h_1(x_1, \mathbf{y}) = \alpha^j(\mathbf{y}) h_1(x_1; \mathbf{y}_{1j}, \mathbf{y}^*_{-1j}) + \beta_1^j(\mathbf{y}) \text{ for all } x_1 \in \mathbb{R} \tag{1}$$
$$h_j(x_j, \mathbf{y}) = \alpha^j(\mathbf{y}) h_j(x_j; \mathbf{y}_{1j}, \mathbf{y}^*_{-1j}) + \beta_j^j(\mathbf{y}) \text{ for all } x_j \in \mathbb{R} \tag{2}$$

Similarly, for any $\mathbf{y} \in Y$, the following holds:

$$h_1(x_1, \mathbf{y}) = \alpha^k(\mathbf{y}) h_1(x_1; \mathbf{y}_{1k}, \mathbf{y}^*_{-1k}) + \beta_1^k(\mathbf{y}) \text{ for all } x_1 \in \mathbb{R} \tag{3}$$
$$h_k(x_k, \mathbf{y}) = \alpha^k(\mathbf{y}) h_k(x_k; \mathbf{y}_{1k}, \mathbf{y}^*_{-1k}) + \beta_2^k(\mathbf{y}) \text{ for all } x_k \in \mathbb{R} \tag{4}$$

for some maps $\alpha^k : Y \to \mathbb{R}_{++}$ and $\beta_1^k, \beta_2^k : Y \to \mathbb{R}$. Now, (3) implies:

$$h_1(x_1; \mathbf{y}_{1j}, \mathbf{y}^*_{1j}) = \alpha^k(\mathbf{y}_{1j}, \mathbf{y}^*_{1j}) h_1(x_1; y_1, \mathbf{y}^*_{-1}) + \beta_1^k(\mathbf{y}_{1j}, \mathbf{y}^*_{1j})$$

for all $x_1 \in \mathbb{R}$. Substituting this in (1), we have:

$$h_1(x_1, \mathbf{y}) = \alpha h_1(x_1; y_1, \mathbf{y}^*_{-1}) + \beta \text{ for all } x_1 \in \mathbb{R}, \tag{5}$$

for some $\alpha > 0$ and $\beta \in \mathbb{R}$. Recall, $x^*, x^{**} \in \mathbb{R}$ are such that $x^{**} > x^*$. Now, define the maps $\alpha_1 : Y \to \mathbb{R}_{++}$ and $\beta_1 : Y \to \mathbb{R}$ as follows:

$$\alpha_1(\mathbf{y}) := \frac{h_1(x^{**}, \mathbf{y}) - h_1(x^*, \mathbf{y})}{h_1(x^{**}; y_1, \mathbf{y}^*_{-1}) - h_1(x^*; y_1, \mathbf{y}^*_{-1})}, \text{ and}$$
$$\beta_1(\mathbf{y}) := h_1(x^*, \mathbf{y}) - \alpha_1(\mathbf{y}) h_1(x^*; y_1, \mathbf{y}^*_{-1})$$

for all $\mathbf{y} \in Y$. Thus, $\alpha = \alpha_1(\mathbf{y})$ and $\beta = \beta_1(\mathbf{y})$. Then, (5) implies:

$$h_1(x_1, \mathbf{y}) = \alpha_1(\mathbf{y}) h_1(x_1; y_1, \mathbf{y}^*_{-1}) + \beta_1(\mathbf{y}) \text{ for all } x_1 \in \mathbb{R} \tag{6}$$

for all $\mathbf{y} \in Y$. Similarly, for any $j \geq 2$, there are maps $\alpha_j : Y \to \mathbb{R}_{++}$ and $\beta_j : Y \to \mathbb{R}$ such that, for any $\mathbf{y} \in Y$, the following holds:

$$h_j(x_j, \mathbf{y}) = \alpha_j(\mathbf{y}) h_j(x_j; y_j, \mathbf{y}^*_{-j}) + \beta_j(\mathbf{y}) \text{ for all } x_j \in \mathbb{R}. \tag{7}$$

Fix an arbitrary $j \in N \setminus \{1\}$. From (1) and (6), we have:

$$\begin{aligned}&\alpha_1(\mathbf{y})h_1(x_1; y_1, \mathbf{y}^*_{-1}) + \beta_1(\mathbf{y}) \\ &\quad = \alpha^j(\mathbf{y})[\alpha_1(\mathbf{y}_{1j}; \mathbf{y}^*_{-1j})h_1(x_1; y_1, \mathbf{y}^*_{-1}) + \beta_1(\mathbf{y}_{1j}, \mathbf{y}^*_{-1j})] + \beta^j_1(\mathbf{y}),\end{aligned}$$

which simplifies to the following:

$$\begin{aligned}&[\alpha_1(\mathbf{y}) - \alpha^j(\mathbf{y})\alpha_1(\mathbf{y}_{1j}; \mathbf{y}^*_{-1j})]h_1(x_1; y_1, \mathbf{y}^*_{-1}) && (8)\\ &\quad = \beta^j_1(\mathbf{y}) - [\beta_1(\mathbf{y}) - \alpha^j(\mathbf{y})\beta_1(\mathbf{y}_{1j}, \mathbf{y}^*_{-1j})] && (9)\end{aligned}$$

for all $x_1 \in \mathbb{R}$. Since (9) is independent of x_1 but (8) is not, we have:

$$\alpha_1(\mathbf{y}) = \alpha^j(\mathbf{y})\alpha_1(\mathbf{y}_{1j}; \mathbf{y}^*_{-1j}) \tag{10}$$

Similarly, (2) and (7) imply the following identities:

$$\alpha_j(\mathbf{y}) = \alpha^j(\mathbf{y})\alpha_j(\mathbf{y}_{1j}; \mathbf{y}^*_{-1j}) \tag{11}$$

From (10) and (11), for every $j \geq 2$: $\alpha_j(\mathbf{y}) = \mu^j(\mathbf{y}_{1j})\alpha_1(\mathbf{y})$ where $\mu^j(\mathbf{y}_{1j}) := \alpha_j(\mathbf{y}_{1j}, \mathbf{y}^*_{-1j})/\alpha_1(\mathbf{y}_{1j}, \mathbf{y}^*_{-1j})$. Thus, (6) and (7) imply:

$$\sum_{i \in N} h_i(x_i, \mathbf{y}) = \alpha_1(\mathbf{y})[h_1(x_1; y_1, \mathbf{y}^*_{-1}) + \sum_{i=2}^{n} \mu^i(\mathbf{y}_{1i})h_i(x_i; y_i, \mathbf{y}^*_{-i})] + \beta(\mathbf{y})$$

where $\beta(\mathbf{y}) := \sum_{i \in N} \beta_i(\mathbf{y})$. Since $\mathbf{x} \in X \mapsto \sum_{i \in N} h_i(x_i, \mathbf{y})$ is a utility representation of $\succsim|_{X \times \{\mathbf{y}\}}$, the following map:

$$\mathbf{x} \in X \mapsto h_1(x_1; y_1, \mathbf{y}^*_{-1}) + \sum_{i=2}^{n} \mu^i(\mathbf{y}_{1i})h_i(x_i; y_i, \mathbf{y}^*_{-i}) \tag{12}$$

is a representation of $\succsim|_{X \times \{\mathbf{y}\}}$. Now, fix an arbitrary $j \geq 3$. By $A4$, each of the following maps:

$$\begin{aligned}&(x_2, x_j) \in \mathbb{R}^2 \mapsto \mu^2(\mathbf{y}_{12})h_2(x_2; y_2, \mathbf{y}^*_{-2}) + \mu^j(\mathbf{y}_{1j})h_j(x_j; y_j, \mathbf{y}^*_{-j}), \text{ and}\\ &(x_2, x_j) \in \mathbb{R}^2 \mapsto \mu^2(y^*_1, y^2)h_2(x_2; y_2, \mathbf{y}^*_{-2}) + \mu^j(y^*_1, y_j)h_j(x_j; y_j, \mathbf{y}^*_{-j})\end{aligned}$$

represents the same (weak) order over $\mathbb{R}^2$. Further, each is additive. Thus, there exists maps $\rho^{2j} : Y \to \mathbb{R}_{++}$ and $\xi^{2j}_2, \xi^{2j}_j : Y \to \mathbb{R}$ such that:

$$\begin{aligned}\mu^2(\mathbf{y}_{12})h_2(x_2; y_2, \mathbf{y}^*_{-2}) &= \rho^{2j}(\mathbf{y})\mu^2(y^*_1, y_2)h_2(x_2; y_2, \mathbf{y}^*_{-2}) + \xi^{2j}_2(\mathbf{y}) && (13)\\ \mu^j(\mathbf{y}_{1j})h_j(x_j; y_j, \mathbf{y}^*_{-j}) &= \rho^{2j}(\mathbf{y})\mu^2(y^*_1, y_j)h_j(x_j; y_j, \mathbf{y}^*_{-j}) + \xi^{2j}_j(\mathbf{y}) && (14)\end{aligned}$$

for all $x_2, x_j \in \mathbb{R}$. Since $x_2 \mapsto h_2(x_2; y_2, \mathbf{y}^*_{-2})$ is strictly increasing, by (13) we obtain the following:

$$\mu^2(\mathbf{y}_{12}) = \rho^{2j}(\mathbf{y})\mu^2(y^*_1, y_2) \quad \text{and} \quad \xi^{2j}_2(\mathbf{y}) = 0. \tag{15}$$

Similarly, (14) leads to the following identity:

$$\mu^j(\mathbf{y}_{1j}) = \rho^{2j}(\mathbf{y})\mu^j(y_1^*, y_j) \quad \text{and} \quad \xi_j^{2j}(\mathbf{y}) = 0. \tag{16}$$

Now, (15) and (16) imply: $\mu^j(\mathbf{y}_{1j}) = [\mu^j(y_1^*, y_j)/\mu^2(y_1^*, y_2)]\mu^2(\mathbf{y}_{12})$ for every $j \geq 3$. Thus, we have:

$$\begin{aligned}
& h_1(x_1; y_1, \mathbf{y}_{-1}^*) + \sum_{i=2}^n \mu^i(\mathbf{y}_{1i}) h_i(x_i; y_i, \mathbf{y}_{-i}^*) \\
&= h_1(x_1; y_1, \mathbf{y}_{-1}^*) + \frac{\mu^2(\mathbf{y}_{12})}{\mu^2(y_1^*, y_2)} \sum_{i=2}^n \mu^i(y_1^*, y_i) h_i(x_i; y_i, \mathbf{y}_{-i}^*) \\
&= \frac{\mu^2(\mathbf{y}_{12})}{\mu^2(y_1^*, y_2)} \Big[\frac{\mu^2(y_1^*, y_2)}{\mu^2(\mathbf{y}_{12})} h_1(x_1; y_1, \mathbf{y}_{-1}^*) + \sum_{i=2}^n \mu^i(y_1^*, y_i) h_i(x_i; y_i, \mathbf{y}_{-i}^*) \Big] \\
&= \frac{\mu^2(\mathbf{y}_{12})}{\mu^2(y_1^*, y_2)} \Big[\frac{\mu^2(y_1^*, y_2)}{\mu^2(\mathbf{y}_{12})} h_1(x_1; y_1, \mathbf{y}_{-1}^*) + \sum_{i=2}^n h_i^*(x_i, y_i) \Big],
\end{aligned}$$

where, for each $i \geq 2$, the map $h_i^* : \mathbb{R} \times \mathbb{R}_{++} \to \mathbb{R}$ is defined by:

$$h_i^*(x_i, y_i) := \mu^i(y_1^*, y_i) h_i(x_i; y_i, \mathbf{y}_{-i}^*) \text{ for all } (x_i, y_i) \in \mathbb{R} \times \mathbb{R}_{++}.$$

Since (12) is a representation of $\succsim|_{X \times \{\mathbf{y}\}}$, so is the following map:

$$\mathbf{x} \in X \mapsto \frac{\mu^2(y_1^*, y_2)}{\mu^2(\mathbf{y}_{12})} h_1(x_1; y_1, \mathbf{y}_{-1}^*) + \sum_{i=2}^n h_i^*(x_i, y_i). \tag{17}$$

Then, as $\succsim$ satisfies $A4$, each of the following maps:

$$\begin{aligned}
&(x_1, x_3) \in \mathbb{R}^2 \mapsto \frac{\mu^2(y_1^*, y_2)}{\mu^2(\mathbf{y}_{12})} h_1(x_1; y_1, \mathbf{y}_{-1}^*) + h_3^*(x_3, y_3), \text{ and} \\
&(x_1, x_3) \in \mathbb{R}^2 \mapsto \frac{\mu^2(\mathbf{y}_{12}^*)}{\mu^2(y_1, y_2^*)} h_1(x_1; y_1, \mathbf{y}_{-1}^*) + h_3^*(x_3, y_3)
\end{aligned}$$

represents the same (weak) order over $\mathbb{R}^2$ and are additive. Thus, there exists maps $\psi : Y \to \mathbb{R}_{++}$ and $\eta_1, \eta_3 : Y \to \mathbb{R}$ such that:

$$\frac{\mu^2(y_1^*, y_2)}{\mu^2(\mathbf{y}_{12})} h_1(x_1; y_1, \mathbf{y}_{-1}^*) = \psi(\mathbf{y}) \frac{\mu^2(\mathbf{y}_{12}^*)}{\mu^2(y_1, y_2^*)} h_1(x_1; y_1, \mathbf{y}_{-1}^*) + \eta_1(\mathbf{y}), \tag{18}$$

$$h_3^*(x_3, y_3) = \psi(\mathbf{y}) h_3^*(x_3, y_3) + \eta_3(\mathbf{y}) \tag{19}$$

for all $x_1, x_3 \in \mathbb{R}$. Since $x_3 \mapsto h_3^*(x_3, y_3)$ is strictly increasing, (19) implies: $\psi(\mathbf{y}) = 1$ and $\eta_3(\mathbf{y}) = 0$. Thus, (18) reduces to:

$$\Big[\frac{\mu^2(y_1^*, y_2)}{\mu^2(\mathbf{y}_{12})} - \frac{\mu^2(\mathbf{y}_{12}^*)}{\mu^2(y_1, y_2^*)} \Big] h_1(x_1; y_1, \mathbf{y}_{-1}^*) = \eta_1(\mathbf{y}) \text{ for all } x_1 \in \mathbb{R}.$$

As $x_1 \mapsto h_1(x_1; y_1, \mathbf{y}^*_{-1})$ is strictly increasing, we have: $\eta_1(\mathbf{y}) = 0$, and

$$\mu^2(y_1^*, y_2)/\mu^2(\mathbf{y}_{12}) = \mu^2(\mathbf{y}^*_{12})/\mu^2(y_1, y_2^*) \tag{20}$$

Define the map $h_1^* : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as follows:

$$h_1^*(x_1, y_1) \coloneqq [\mu^2(\mathbf{y}^*_{12})/\mu^2(y_1, y_2^*)]h_1(x_1; y_1, \mathbf{y}^*_{-1})$$

for all $(x_1, y_1) \in \mathbb{R} \times \mathbb{R}_{++}$. By (17) and (20), $\mathbf{x} \in X \mapsto \sum_{i \in N} h_i^*(x_i, y_i)$ represents $\succsim |_{X \times \{\mathbf{y}\}}$. Note, for all $i \in N$, the map $\theta \in \mathbb{R} \mapsto h_i^*(\theta, \mathbf{y}) \in \mathbb{R}$ is full range and strictly increasing. This follows from the definition of $(h_1^*, \ldots, h_n^*)$ and the properties of $(h_1, \ldots, h_n)$. ■

Proof of Lemma 3: Fix an arbitrary $\mathbf{y} \in Y$. By $A5$, the maps:

$$(x_1, x_2) \in \mathbb{R}^2 \mapsto h_1^*(x_1, y_1) + h_2^*(x_2, y_2), \text{ and}$$
$$(x_1, x_2) \in \mathbb{R}^2 \mapsto h_2^*(x_1, y_1) + h_1^*(x_2, y_2)$$

represent the same (weak) order over $\mathbb{R}^2$ and are additive. Thus, there exists $\alpha > 0$ and $\beta, \beta' \in \mathbb{R}$ such that:

$$h_2^*(\rho, \theta) = \alpha h_1^*(\rho, \theta) + \beta_1 \text{ for all } \rho \in \mathbb{R}, \tag{21}$$
$$h_1^*(\rho, \theta) = \alpha h_2^*(\rho, \theta) + \beta_2 \text{ for all } \rho \in \mathbb{R} \tag{22}$$

for all $\theta \in \mathbb{R}_{++}$. Fix $\rho_*, \rho_{**} \in \mathbb{R}$ such that $\rho_{**} > \rho_*$. Now, define the maps $\alpha : \mathbb{R}_{++} \to \mathbb{R}_{++}$ and $\beta_1, \beta_2 : \mathbb{R}_{++} \to \mathbb{R}$ as follows:

$$\begin{aligned} \alpha(\theta) &\coloneqq \frac{h_2^*(\rho_{**}, \theta) - h_2^*(\rho_*, \theta)}{h_1^*(\rho_{**}, \theta) - h_1^*(\rho_*, \theta)} \text{ for all } \theta \in \mathbb{R}_{++}, \\ \beta_1(\theta) &\coloneqq h_2^*(\rho_*, \theta) - \alpha(\theta) h_1^*(\rho_*, \theta) \text{ for all } \theta \in \mathbb{R}_{++}, \\ \beta_2(\theta) &\coloneqq h_1^*(\rho_*, \theta) - \alpha(\theta) h_2^*(\rho_*, \theta) \text{ for all } \theta \in \mathbb{R}_{++}, . \end{aligned}$$

From (21), we have: $\alpha = \alpha(\theta)$. Thus, (21) and (22) imply: $\beta_1 = \beta_1(\theta)$ and $\beta_2 = \beta_2(\theta)$. Hence, we obtain the following:

$$h_2^*(\rho, \theta) = \alpha(\theta) h_1^*(\rho, \theta) + \beta_1(\theta) \text{ for all } \rho \in \mathbb{R}, \tag{23}$$
$$h_1^*(\rho, \theta) = \alpha(\theta) h_2^*(\rho, \theta) + \beta_2(\theta) \text{ for all } \rho \in \mathbb{R} \tag{24}$$

Thus, (23) and (24) imply the following:

$$\left(1 - [\alpha(\theta)]^2\right) h_2^*(\rho, \theta) = \alpha(\theta)\beta_2(\theta) + \beta_1(\theta) \text{ for all } \rho \in \mathbb{R}. \tag{25}$$

Since $\rho \mapsto h_2^*(\rho, \theta)$ is strictly increasing, (25) implies: $[\alpha(\theta)]^2 = 1$ Also, $\alpha(\theta) > 0$. Thus, we have: $\alpha(\theta) = 1$. Substiuting in (25), we obtain: $\beta_1(\theta) + \beta_2(\theta) = 0$. The argument thus far has been for indices 1 and 2. Clearly, this could have been done for any arbitrary pair of distinct indices i and j in N. Thus, there exist maps $h^{**} : \mathbb{R} \times \mathbb{R}_{++} \to \mathbb{R}$ and $\beta_1, \ldots, \beta_n : Y \to \mathbb{R}$ such that:

$$h_i^*(x_i, y_i) = h^*(x_i, y_i) + \beta_i(\mathbf{y}) \text{ for all } (x_i, \mathbf{y}) \in \mathbb{R} \times Y.$$

Hence, $\sum_{i\in N} h_i^*(x_i,y_i) = \sum_{i\in N} h^{**}(x_i,y_i) + \sum_{i\in N} \beta_i(\mathbf{y})$. That is, the map $\mathbf{x}\in X \mapsto \sum_{i\in N} h^{**}(x_i,y_i)$ represents $\succsim|_{X\times\{\mathbf{y}\}}$ for all $\mathbf{y}\in Y$. ■

PROOF OF LEMMA 4: Since $\succsim$ satisfies $A1$, there exists a coninuous map $V: X\times Y \to \mathbb{R}$ such that V represents $\succsim$. In particular, the map $\mathbf{x}\in X \mapsto V(\mathbf{x},\mathbf{y})$ represents $\succsim|_{X\times\{\mathbf{y}\}}$. Further, the function $\mathbf{x}\in X\mapsto \sum_{i\in N}\bar{h}_i(x_i,\mathbf{y})$ represents $\succsim|_{X\times\{\mathbf{y}\}}$. Hence, there exists a strictly increasing map $\phi^{\mathbf{y}}:\mathbb{R}\to\mathbb{R}$ such that:

$$V(\mathbf{x},\mathbf{y}) = \phi^{\mathbf{y}}\Big(\sum_{i\in N}\bar{h}_i(x_i,\mathbf{y})\Big) \text{ for all } \mathbf{x}\in X. \tag{26}$$

Fix an arbitrary $\mathbf{y}\in Y$. Choose a map $\theta\in\mathbb{R}\mapsto \mathbf{x}^\theta\in X$ such that: $\theta = \sum_{i\in N}\bar{h}_i(x_i^\theta,\mathbf{y})$ for all $\theta\in\mathbb{R}$. Also, define $f(\theta,\mathbf{y}) := \phi^{\mathbf{y}}(\theta)$. Clearly, $\theta\mapsto f(\theta,\mathbf{y})$ is strictly increasing. Thus, (26) implies:

$$V(\mathbf{x}^\theta,\mathbf{y}) = f\Big(\sum_{i\in N}\bar{h}_i(x_i^\theta,\mathbf{y}),\mathbf{y}\Big) \text{ for all } \theta\in\mathbb{R}. \tag{27}$$

Since $\mathbf{y}\in Y$ is arbitrary, (27) holds for all $\mathbf{y}\in Y$.

Fix an arbitrary $\mathbf{x}\in X$. Define $\theta := \sum_{i\in N}\bar{h}_i(x_i,\mathbf{y})$. Also, we know: $\theta = \sum_{i\in N}\bar{h}_i(x_i^\theta,\mathbf{y})$. Thus, $\sum_{i\in N}\bar{h}_i(x_i,\mathbf{y}) = \sum_{i\in N}\bar{h}_i(x_i^\theta,\mathbf{y})$ which gives,

$$f\Big(\sum_{i\in N}\bar{h}_i(x_i^\theta,\mathbf{y}),\mathbf{y}\Big) = f\Big(\sum_{i\in N}\bar{h}_i(x_i,\mathbf{y}),\mathbf{y}\Big). \tag{28}$$

Further, since $\mathbf{x}'\in X\mapsto\sum_{i\in N}\bar{h}_i(x_i',\mathbf{y})$ represents $\succsim|_{X\times\{\mathbf{y}\}}$, we have: $(\mathbf{x}^\theta,\mathbf{y})\sim(\mathbf{x},\mathbf{y})$. That is, $V(\mathbf{x}^\theta,\mathbf{y}) = V(\mathbf{x},\mathbf{y})$. Therefore, (27) and (28) complete the proof of the claim. ■

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