Notes on "Notes on the Gini Coefficient(s)"

Debraj Ray[†]

September 23 2021

A point of historical interest — undoubtedly arcane to some — is that the worthy Corrado Gini produced no fewer than 13 Gini coefficients for our perusal. I'm not entirely sure what all thirteen are but two of them seem to have survived the test of time and are widely used. The first is the formula

$$G = \frac{1}{2\mu n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} |y_i - y_j|,$$

where the n stands for population, the y's are incomes, and μ is their mean. The second is given by the formula

$$G' = \frac{1}{2\mu n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} |y_i - y_j|,$$

a seemingly minor difference, dividing as it does by $n(n-1)$ instead of n^2. So G and G' agree ordinally on all comparisons *within* any given population, while they could disagree across comparisons with varying populations.

Is this worth a sleepless night or two? Not really, especially if you're not devoted to teaching the stuff with the care and precision which my erstwhile student (and now friend) Julia Schwenkenberg brings to it. Julia was teaching her Rutgers students the fundamentals of inequality measurement and came across the following example in the CORE Open Access textbook:[*]

> "There are two people in the society and one has all the income . . . [This is] perfect inequality, as you would expect,"

going on to observe that the Gini (or perhaps I should say, *the* Gini in their view) was therefore equal to its maximum value of 1: a foregone conclusion, seemingly.[†]

But was it? Julia was well aware of the "population neutrality" principle underlying inequality measurement, which stated as an axiom that population cloning of all individuals — while keeping their incomes unchanged — "should not" change any Lorenz-consistent measure of economic inequality. So the configuration in the text should exhibit no change in inequality if, instead, two people had one unit each of income and the other two had none. But the latter configuration is surely *less* unequal than one in which one person had two units of income and the other three had none. Ergo, the very

---

[†]New York University and University of Warwick, debraj.ray@nyu.edu. Thanks to Sam Bowles, Wendy Carlin, Julia Schwenkenberg and Rajiv Sethi for their useful comments. The present iterated draft is written side by side with Sethi (2021). The reader should also consult Bowles and Carlin (2021) for their perspective.

[*]For more detail on the CoreEcon project, see https://www.core-econ.org/.

[†]See https://bit.ly/3kB8KNN.

first situation should be less unequal than one in which one person out of four (as opposed to one of out of two) had all the income.

And yet: if we agree with that, should there not be room for the Gini coefficient to rise some more from its two-person value of 1? What is it doing at 1 already?

This was intriguing enough (and confusing enough for all concerned) that Julia wrote to me about it, especially since my textbook (Ray 1998) and landmark monographs such as Sen (1973) use the formula for G, whereas the CORE text seemed to be using another formula altogether. After discussions with Rajiv Sethi, and later Sam Bowles and Wendy Carlin, it soon emerged that "the" Gini in the CORE text was G', that it was a version favored by two of the CORE authors (Bowles and Carlin 2020), and indeed, that it hits its maximum value in the example above. The other one, G, does not: it equals 1/2, well below its maximum value of 1.

Rajiv Sethi's excellent notes on the subject (Sethi 2021), to which this is a response, lay bare the difference. As already noted, both measures (and eleven others to boot) had been proposed by the prolific Gini, so no claim to true inheritance could be advanced on that somewhat legalistic but otherwise useless basis, "what did Gini really say?" — fortunately so, for many truths have been trampled underfoot by such absurd excuses. Rather, we need to truly evaluate the measures from first principles to advance the discussion, which is what Rajiv does in large part.

The measure G satisfies all the axioms that underly the Lorenz partial order; namely, population and income invariance, as well as the transfers principle. The Lorenz order, in turn, is at the very heart of inequality measurement and forms a welfare basis for it — see, among others, the work of Atkinson (1983) and Dasgupta, Sen and Starrett (1973).[‡] G completes this partial order — see Thon (1982) for an axiomatization. It is, to be sure, not the *only* complete order that completes the Lorenz: other examples include the coefficient of variation or the Theil index — see Ray (1998) for more on these matters. But G is one of them; it is *Lorenz-consistent* in the sense of satisfying the axioms that I've just mentioned.

On the other hand, as Rajiv explains, G' would like to highlight the fact that in the two-person example given above, inequality has been stretched to its limit: how can things be any more unequal *in that two-person society*? This brings us to what Rajiv, following Thon (1982), calls strong comparability: "the range spanned by the index depends neither on total income nor on population." In particular, within any society with a fixed population, the index should be able to move from its minimum value (0) to its maximum (1), so that the index recognizes clearly the upper limits to inequality *for that fixed population size*. Clearly, G does not do this — it varies only from 0 to 1/2 in a two-person society. Indeed, by the argument given five paragraphs ago, we can already see that *no measure satisfying*

[‡]Rothschild and Stiglitz's (1970) characterization of "increasing risk" is also relevant here, despite its focus on risk and uncertainty; the two have parallel features.

the population principle and the transfers principle can be strongly comparable. In a gesture of inclusiveness, Rajiv concludes:

> "It is probably best if students are at least made aware of the existence and historical origins of both, and presented with the arguments in favor of each. There is no uniquely correct Gini coefficient."

Given a choice between two incompatible desiderata, I can sympathize with Rajiv's assertion that there is no "uniquely correct" Gini. Indeed, given the enormous number of Lorenz completions at our disposal, *there is no uniquely correct measure of inequality even under the Lorenz axioms*, let alone a uniquely correct Gini. Every measure must be evaluated by the core ethical axioms they satisfy, and we then need to reach into our own ethical system to see which set of axioms fit the best. So in this sense, I agree with Rajiv.

That said, there are axioms and there are axioms. I have already mentioned the long and venerable history of the Lorenz curve, which goes back to Lorenz (1905). Its foundation is laid by a fundamental set of axioms: population neutrality, income neutrality, as well as the transfers principle of Pigou and Dalton. These axioms can and have been questioned; for instance, my work with Joan Esteban on the measurement of polarization (Esteban and Ray 1994) comes from dropping the transfers principle. But as contributions to a *welfare economics* foundation for inequality measurement, these are the key axioms, and all further explorations begin from them — or should.

The fact that G satisfies all the three axioms, while G' as already noted fails population neutrality, is a priori (though not yet definitive) cause for suspecting the credentials of G'. A noteworthy example comes from James Foster (1983), who writes down all the axioms (or "properties") that underpin Lorenz *except for population neutrality*, and then observes of the rest:

> "In fact, since each property is a restriction . . . in isolation without reference to crosspopulation comparisons, [they] admit even more measures than Fields and Fei (1978) indicate. Consider the measure that takes the Gini coefficient index at even sized populations, and the coefficient of variation index at odd. *This absurd measure* quite clearly satisfies all [the] properties.
>
> The above example serves to point out the desirability of a property which would coordinate the indices *into one cohesive measure*. Another property [the population principle] suggested by Dalton does this in a particularly natural way" (emphases mine).

As Foster correctly notes, the population principle is imposed to make the reader aware that we cannot have potentially unrelated and therefore "absurd" measures on different population layers: they need to be connected in a "cohesive" way. Foster goes on to introduce the very population neutrality principle — due to Dalton and embedded in the Lorenz curve — that G satisfies and G' does not, the latter because it refuses to entertain the requirement that its comparisons within a population must be contextualized in the space of all populations.

To see directly the failure in the cohesion of G', consider again the CORE example, in which one person gets all the income in a two-person society. According to G' (and CORE), this situation *is just as unequal* as one in which one person gets all in the income in a million-person society. To me, this is indeed an example of the absurdity that Foster refers to when different population layers are not connected in any coherent way. Taking the example a step further, we must conclude that under G', a situation in which two persons share all the income in a million-person society is *strictly more equal* than the two-person example in the text. This even more absurd consequence comes from the additional application of the transfers principle, which G' does satisfy.

Of course, G exhibits none of these strange behaviors. It would rank the one-in-two example as more equal than the one-in-a-million example, and ditto with the two-in-a-million example.

Rajiv would respond that G fails strong comparability: it does not hit 1 when the two-person society is stretched to its unequal limit. Well, I don't see why strong comparability makes sense. Why must a measure declare perfect inequality just because feasibility constrains a particular situation from exhibiting still greater inequality? The latter is a property of the feasible set and should not influence evaluation, just as a utility function is not affected by the budget set on which it operates. To explain why G' fails this desideratum, consider a two-person society in which either person can asexually clone itself into two, with further income transfers possible among or across clones. Such cloning is prohibited, and so all we can do in this society is transfer income across two individuals without cloning either of them. Strong comparability then requires an inequality measure to hit its maximum possible value when one person has all the income. But now suppose that a divine decree permits cloning. Then the range of inequality values in the earlier two-person range would need to artificially contract so as to accommodate the newly unequal possibilities that arise. In short, this is a case of our measurement indicator responding to the feasible set.

In all of this, I am aware that my arguments do not constitute a logical attack on G' in favor of G, in the sense of claiming that there is some failure of Aristotelian logic in the very fabric of G'. There is no such attack. But it is an appeal to the reader's intuitive sensibilities via the discussion of axioms. In fact, without an axiomatic system, anything goes: it's a veritable free-for-all. We would be taking all too literally Sen's beautiful dedication to his daughters, at the start of *On Economic Inequality*:

> "In the hope that they will find less of it, no matter how they choose to measure it."

In a free-for-all, that utopian dream cannot happen, but once constrained by the spirit of a reasonable axiomatic system, it can. That is the spirit in which I reject G' in favor of G.

References

Atkinson, Anthony (1983). *The Economics of Inequality.* New York: Oxford University Press.

Bowles, Samuel and Wendy Carlin (2020). "Inequality as Experienced Difference: A Reformulation of the Gini Coefficient." *Economics Letters* **186**, 1087-1089.

Bowles, Samuel and Wendy Carlin (2021)."The Gini Coefficient as a Measure of Experienced Inequality." mimeo.,

Dasgupta, Partha, Sen, Amartya, and David Starrett (1973). "Notes on the Measurement of Inequality." *Journal of Economic Theory* **6**, 180-187.

Esteban, Joan and Debraj Ray (1994). "On the Measurement of Polarization." *Econometrica* **62**, 819-851.

Fields, Gary S. and John Fei (1978). "On Inequality Comparisons." *Econometrica* **46**, 303-16.

Foster, James (1983). "An Axiomatic Characterization of the Theil Measure of Income Inequality." *Journal of Economic Theory* **31**, 105-121.

Lorenz, Max (1905). "Methods of Measuring the Concentration of Wealth." *Publications of the American Statistical Association* **9**, 209-219.

Ray, Debraj (1998). *Development Economics*. New Jersey: Princeton University Press.

Rothschild, Michael and Joseph Stiglitz (1970). "Increasing Risk: I. A Definition." *Journal of Economic Theory* **2** 225-243.

Sen, Amartya (1973). *On Economic Inequality.* Oxford: Clarendon Press.

Sethi, Rajiv (2021). "Notes on the Gini Coefficients." mimeo., Barnard College.

Thon, Dominique (1982). "An Axiomatization of the Gini Coefficient." *Mathematical Social Sciences* **2**, 131–143.