Information Aggregation in a Financial Market with General Signal Structure

Online Appendix

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In this online appendix, we prove Proposition 6 when there is no idiosyncratic noise (i.e., $\boldsymbol{\epsilon}_i(j)) = \mathbf{0}$ for all i and j). All notation has the same meaning as in the main text. We consider two cases:

CASE 1. $\boldsymbol{\mu}$ contains at least two nonzero sub-vectors ($\boldsymbol{\mu}_i \neq \mathbf{0}$ for at least two indices i). For this case, it is without loss of generality to assume that $\mathbf{Var}(\boldsymbol{y})$ is positive definite. Otherwise, re-write $\boldsymbol{\mu}_t'\boldsymbol{y}$ as $\bar{\boldsymbol{\mu}}_t'\bar{\boldsymbol{y}}$ ($\bar{\boldsymbol{\mu}}_t \neq \mathbf{0}$), where $\bar{\boldsymbol{y}}$ is a maximal linearly independent subset of $\boldsymbol{y}$, and apply the same technique by substituting $\boldsymbol{\mu}_t'\boldsymbol{y}$ with $\bar{\boldsymbol{\mu}}_t'\bar{\boldsymbol{y}}/|\bar{\boldsymbol{\mu}}_t|$, and ι_t with $\iota_t/|\bar{\boldsymbol{\mu}}_t|$.

All the proofs are the same as that in the proof for the case that there is idiosyncratic noise in the main text except for a separate argument to show that (A.23) in the main text cannot hold, reproduced here for convenience as:

$$\mathrm{Var}(\boldsymbol{\mu})\,\mathrm{Var}(\boldsymbol{\mu}_i'\boldsymbol{y}_i) - \mathrm{Cov}(\boldsymbol{\mu}, \boldsymbol{\mu}_i'\boldsymbol{y}_i)^2 = 0, \tag{1}$$

with $\boldsymbol{z}_i$ replaced by $\boldsymbol{y}_i$ in the absence of idiosyncratic noise. Suppose, on the contrary, that (1) holds for every i. Because $\boldsymbol{\mu}'\boldsymbol{y} \neq \mathbf{0}$ and $\boldsymbol{\mu}$ contains two nonzero sub-vectors, it follows from (1) that for every i, $\sum_j \boldsymbol{\mu}_j'\boldsymbol{y}_j = \boldsymbol{\mu}_i'\boldsymbol{y}_i$ with probability one. That means that for every i, $\boldsymbol{\mu}_i'\boldsymbol{y}_i = 0$ with probability one. But this is impossible, given $\boldsymbol{\mu}_k \neq \mathbf{0}$ for some k and the assumption of positive definiteness of $\mathbf{Var}(\boldsymbol{y}_k)$ in Assumption 1.

CASE 2. $\boldsymbol{\mu}$ contains only one nonzero sub-vector.

Without loss of generality, we assume $\boldsymbol{\mu}_1 \neq \mathbf{0}$ and $\boldsymbol{\mu}_i = \mathbf{0}$ for $i \neq 1$. Because $\boldsymbol{Q}_t = \boldsymbol{\pi}_t/\gamma_t$, $\boldsymbol{Q}_t/|\boldsymbol{Q}_t| \to \boldsymbol{\mu}$. Observe that

$$\frac{\mathrm{Cov}(\boldsymbol{Q}_t, \theta)}{\mathrm{Var}(\boldsymbol{Q}_t) + \mathrm{Var}_t(u)}\mathbf{Cov}(\boldsymbol{Q}_t, \boldsymbol{y}_i) = \frac{\mathrm{Cov}(\boldsymbol{Q}_t/|\boldsymbol{Q}_t|, \theta)}{\mathrm{Var}(\boldsymbol{Q}_t/|\boldsymbol{Q}_t|) + \mathrm{Var}_t(u)/|\boldsymbol{Q}_t|^2}\mathbf{Cov}(\boldsymbol{Q}_t/|\boldsymbol{Q}_t|, \boldsymbol{y}_i). \tag{2}$$

Following the same arguments in the proof of Claim 1 in the main text, we have $\mathrm{Var}_t(u)/|\boldsymbol{Q}_t|^2 \to 0$. Similar to the proof in Proposition 4, we know that $\{\boldsymbol{Q}_t\}$ is bounded. For easy reference, we repeat equations (14) and (15) from the main text below:

$$\boldsymbol{Q}_i = \frac{\left[\mathbf{Var}(\boldsymbol{z}_i) - \frac{\mathbf{Cov}(\boldsymbol{Q},\boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{Q},\boldsymbol{y}_i)'}{\mathrm{Var}(\boldsymbol{Q})+\mathrm{Var}(u)}\right]^{-1}\left[\mathbf{Cov}(\theta,\boldsymbol{y}_i) - \frac{\mathrm{Cov}(\boldsymbol{Q},\theta)}{\mathrm{Var}(\boldsymbol{Q})+\mathrm{Var}(u)}\,\mathbf{Cov}(\boldsymbol{Q},\boldsymbol{y}_i)\right]}{\Delta_i\,\mathrm{Var}_{\boldsymbol{Q}}(\theta|i)}, \tag{3}$$

$$\gamma = \frac{1 + \sum_{i=1}^n \frac{\mathrm{Cov}(\boldsymbol{Q},\theta) - \mathbf{Cov}(\theta,\boldsymbol{y}_i)'\,\mathbf{Var}^{-1}(\boldsymbol{z}_i)\,\mathbf{Cov}(\boldsymbol{Q},\boldsymbol{y}_i)}{\Delta_i\,\mathrm{Var}(\theta|i)[\mathrm{Var}(\boldsymbol{Q})+\mathrm{Var}(u) - \mathbf{Cov}(\boldsymbol{Q},\boldsymbol{y}_i)'\,\mathbf{Var}^{-1}(\boldsymbol{z}_i)\,\mathbf{Cov}(\boldsymbol{Q},\boldsymbol{y}_i)]}}{\sum_{i=1}^n \frac{1}{\Delta_i\,\mathrm{Var}(\theta|i)}}. \tag{4}$$

Because $\boldsymbol{Q}_{it} \to \mathbf{0}$ for each $i \neq 1$ (and with (2) and $\mathrm{Var}_t(u)/|\boldsymbol{Q}_t|^2 \to 0$ in mind), we can pass to the limit in (3) to obtain

$$\mathrm{Var}(\boldsymbol{\mu}_1'\boldsymbol{y}_1)\,\mathbf{Cov}(\theta,\boldsymbol{y}_i) = \mathrm{Cov}(\theta,\boldsymbol{\mu}_1'\boldsymbol{y}_1)\,\mathbf{Cov}(\boldsymbol{\mu}_1'\boldsymbol{y}_1,\boldsymbol{y}_i) \tag{5}$$

for each $i = 2,\ldots,n$. But (5) also holds for $i = 1$. To see this, multiply both sides of (3) by the positive definite matrix $\mathbf{Var}(\boldsymbol{y}_1) - \frac{\mathbf{Cov}(\boldsymbol{Q}_t,\boldsymbol{y}_1)\,\mathbf{Cov}(\boldsymbol{Q}_t,\boldsymbol{y}_1)'}{\mathrm{Var}(\boldsymbol{Q}_t)+\mathrm{Var}(u)}$, then by $\boldsymbol{Q}_{1t}'$, and finally pass to the limit as $t \to \infty$. Then:

$$\boldsymbol{Q}_{1t}'\left[\mathbf{Var}(\boldsymbol{y}_1) - \frac{\mathbf{Cov}(\boldsymbol{Q}_t,\boldsymbol{y}_1)\,\mathbf{Cov}(\boldsymbol{Q}_t,\boldsymbol{y}_1)'}{\mathrm{Var}(\boldsymbol{Q}_t)+\mathrm{Var}_t(u)}\right]\boldsymbol{Q}_{1t} = \frac{\boldsymbol{Q}_{1t}'\left[\mathbf{Cov}(\theta,\boldsymbol{y}_1) - \frac{\mathrm{Cov}(\boldsymbol{Q}_t,\theta)}{\mathrm{Var}(\boldsymbol{Q}_t)+\mathrm{Var}_t(u)}\,\mathbf{Cov}(\boldsymbol{Q}_t,\boldsymbol{y}_1)\right]}{\Delta_1\,\mathrm{Var}_t(\theta|1)}$$
$$= |\boldsymbol{Q}_t|\left(\frac{\frac{\boldsymbol{Q}_{1t}'}{|\boldsymbol{Q}_t|}\left[\mathbf{Cov}(\theta,\boldsymbol{y}_1) - \frac{\mathrm{Cov}(\boldsymbol{Q}_t/|\boldsymbol{Q}_t|,\theta)}{\mathrm{Var}(\boldsymbol{Q}_t/|\boldsymbol{Q}_t|)+\mathrm{Var}_t(u)/|\boldsymbol{Q}_t|^2}\,\mathbf{Cov}(\boldsymbol{Q}_t/|\boldsymbol{Q}_t|,\boldsymbol{y}_1)\right]}{\Delta_1\,\mathrm{Var}_t(\theta|1)}\right) \to 0,$$

where the second equality follows from (2) and the limit follows from the boundedness of $\{\boldsymbol{Q}_t\}$, $\boldsymbol{Q}_{1t}/|\boldsymbol{Q}_t| \to \boldsymbol{\mu}_1$, $\boldsymbol{Q}_t/|\boldsymbol{Q}_t| \to \boldsymbol{\mu}$ ($\boldsymbol{\mu}_i = \mathbf{0}$ for every $i \geq 2$), and $\mathrm{Var}_t(u)/|\boldsymbol{Q}_t|^2 \to 0$. So

$$\left[\mathbf{Var}(\boldsymbol{y}_1) - \frac{\mathbf{Cov}(\boldsymbol{Q}_t,\boldsymbol{y}_1)\,\mathbf{Cov}(\boldsymbol{Q}_t,\boldsymbol{y}_1)'}{\mathrm{Var}(\boldsymbol{Q}_t)+\mathrm{Var}_t(u)}\right]\boldsymbol{Q}_{1t} \to \mathbf{0}. \tag{6}$$

Combining (2), (3) and (6) along with $\mathrm{Var}_t(u)/|\boldsymbol{Q}_t|^2 \to 0$, we must conclude that

$$\mathbf{Cov}(\theta,\boldsymbol{y}_1) - \frac{\mathrm{Cov}(\boldsymbol{\mu}_1'\boldsymbol{y}_1,\theta)}{\mathrm{Var}(\boldsymbol{\mu}_1'\boldsymbol{y}_1)}\,\mathbf{Cov}(\boldsymbol{\mu}_1'\boldsymbol{y}_1,\boldsymbol{y}_1) = \mathbf{0} \tag{7}$$

so that (5) also holds for $i = 1$.

Now, (7) along with $\mathbf{Cov}(\theta, \boldsymbol{y}_1) \neq \mathbf{0}$[*] also implies that

$$\boldsymbol{\mu}_1 = \frac{\mathbf{Var}^{-1}(\boldsymbol{y}_1)\,\mathbf{Cov}(\theta, \boldsymbol{y}_1)}{|\,\mathbf{Var}^{-1}(\boldsymbol{y}_1)\,\mathbf{Cov}(\theta, \boldsymbol{y}_1)|} \tag{8}$$

and

$$\frac{\mathrm{Cov}(\theta, \boldsymbol{\mu}_1'\boldsymbol{y}_1)}{\mathrm{Var}(\boldsymbol{\mu}_1'\boldsymbol{y}_1)} = |\,\mathbf{Var}^{-1}(\boldsymbol{y}_1)\,\mathbf{Cov}(\theta, \boldsymbol{y}_1)|. \tag{9}$$

Multiplying both sides of (5) by $\boldsymbol{\mu}_{it}$ and adding over all i, we have

$$\mathrm{Var}(\boldsymbol{\mu}_1'\boldsymbol{y}_1)\,\mathrm{Cov}(\boldsymbol{\mu}_t, \theta) - \mathrm{Cov}(\theta, \boldsymbol{\mu}_1'\boldsymbol{y}_1)\,\mathrm{Cov}(\boldsymbol{\mu}_t, \boldsymbol{\mu}_1'\boldsymbol{y}_1) = 0 \tag{10}$$

for every t, while for $i = 2, \ldots, n$,

$$\begin{aligned}
&\mathrm{Cov}(\boldsymbol{\mu}_t, \theta) - \mathbf{Cov}(\theta, \boldsymbol{y}_i)'\,\mathbf{Var}^{-1}(\boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{\mu}_t, \boldsymbol{y}_i)\\
&\to \mathrm{Cov}(\theta, \boldsymbol{\mu}_1'\boldsymbol{y}_1) - \mathbf{Cov}(\theta, \boldsymbol{y}_i)'\,\mathbf{Var}^{-1}(\boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{\mu}_1'\boldsymbol{y}_1, \boldsymbol{y}_i)\\
&= \frac{\mathrm{Cov}(\theta, \boldsymbol{\mu}_1'\boldsymbol{y}_1)}{\mathrm{Var}(\boldsymbol{\mu}_1'\boldsymbol{y}_1)}\left[\mathrm{Var}(\boldsymbol{\mu}_1'\boldsymbol{y}_1) - \mathbf{Cov}(\boldsymbol{\mu}_1'\boldsymbol{y}_1, \boldsymbol{y}_i)'\,\mathbf{Var}^{-1}(\boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{\mu}_1'\boldsymbol{y}_1, \boldsymbol{y}_i)\right],
\end{aligned} \tag{11}$$

where the limit follows from the fact that $\boldsymbol{\mu}_i = \mathbf{0}$ for every $i \geq 2$, and the equality again makes use of (5). By (9) and (10), $\mathrm{Cov}(\boldsymbol{\mu}_t, \theta) = |\,\mathbf{Var}^{-1}(\boldsymbol{y}_1)\,\mathbf{Cov}(\theta, \boldsymbol{y}_1)|\,\mathrm{Cov}(\boldsymbol{\mu}_t, \boldsymbol{\mu}_1'\boldsymbol{y}_1)$ for every t. Consequently, for every t,

$$\begin{aligned}
&\mathrm{Cov}(\boldsymbol{\mu}_t, \theta) - \mathbf{Cov}(\theta, \boldsymbol{y}_1)'\,\mathbf{Var}^{-1}(\boldsymbol{y}_1)\,\mathbf{Cov}(\boldsymbol{\mu}_t, \boldsymbol{y}_1)\\
&= \mathrm{Cov}(\boldsymbol{\mu}_t, \theta) - |\,\mathbf{Var}^{-1}(\boldsymbol{y}_1)\,\mathbf{Cov}(\theta, \boldsymbol{y}_1)|\,\mathrm{Cov}(\boldsymbol{\mu}_t, \boldsymbol{\mu}_1'\boldsymbol{y}_1) = 0,
\end{aligned} \tag{12}$$

where the first equality uses (8). (3) and (12) together let us conclude that for every t,

$$\boldsymbol{Q}_{1t} = \frac{\mathbf{Var}^{-1}(\boldsymbol{y}_1)\,\mathbf{Cov}(\theta, \boldsymbol{y}_1)}{\Delta_1\,\mathrm{Var}_t(\theta|1)}. \tag{13}$$

[*] If $\mathbf{Cov}(\theta, \boldsymbol{y}_1) = \mathbf{0}$, then $\mathbf{Cov}(\theta, \boldsymbol{y}_i) = \mathbf{0}$ for all i by (5), which contradicts the hypothesis that $\mathbf{Cov}(\theta, \boldsymbol{y}_i) \neq \mathbf{0}$ for at least one i.

To see this, use (3) to observe that (13) is equivalent to

$$\left[\mathbf{Var}(\boldsymbol{y}_1) - \frac{\mathbf{Cov}(\boldsymbol{Q}_t, \boldsymbol{y}_1)\,\mathbf{Cov}(\boldsymbol{Q}_t, \boldsymbol{y}_1)'}{\mathrm{Var}(\boldsymbol{Q}_t) + \mathrm{Var}_t(u)}\right]^{-1} \left[\mathbf{Cov}(\theta, \boldsymbol{y}_1) - \frac{\mathrm{Cov}(\boldsymbol{Q}_t, \theta)}{\mathrm{Var}(\boldsymbol{Q}_t) + \mathrm{Var}_t(u)}\,\mathbf{Cov}(\boldsymbol{Q}_t, \boldsymbol{y}_1)\right]$$
$$= \mathbf{Var}^{-1}(\boldsymbol{y}_1)\,\mathbf{Cov}(\theta, \boldsymbol{y}_1).$$

Therefore, multiplying by $\mathbf{Var}(\boldsymbol{y}_1) - \frac{\mathbf{Cov}(\boldsymbol{Q}_t,\boldsymbol{y}_1)\,\mathbf{Cov}(\boldsymbol{Q}_t,\boldsymbol{y}_1)'}{\mathrm{Var}(\boldsymbol{Q}_t)+\mathrm{Var}_t(u)}$ on both sides of this equality, we see that to establish (13), it suffices to show that

$$\mathbf{Cov}(\theta, \boldsymbol{y}_1) - \frac{\mathrm{Cov}(\boldsymbol{Q}_t, \theta)}{\mathrm{Var}(\boldsymbol{Q}_t) + \mathrm{Var}_t(u)}\,\mathbf{Cov}(\boldsymbol{Q}_t, \boldsymbol{y}_1)$$
$$= \left[\mathbf{Var}(\boldsymbol{y}_1) - \frac{\mathbf{Cov}(\boldsymbol{Q}_t, \boldsymbol{y}_1)\,\mathbf{Cov}(\boldsymbol{Q}_t, \boldsymbol{y}_1)'}{\mathrm{Var}(\boldsymbol{Q}_t) + \mathrm{Var}_t(u)}\right] \mathbf{Var}^{-1}(\boldsymbol{y}_1)\,\mathbf{Cov}(\theta, \boldsymbol{y}_1).$$

The above equality is further equivalent to

$$\frac{\mathrm{Cov}(\boldsymbol{Q}_t, \theta)}{\mathrm{Var}(\boldsymbol{Q}_t) + \mathrm{Var}_t(u)}\,\mathbf{Cov}(\boldsymbol{Q}_t, \boldsymbol{y}_1) = \frac{\mathbf{Cov}(\boldsymbol{Q}_t, \boldsymbol{y}_1)\,\mathbf{Cov}(\boldsymbol{Q}_t, \boldsymbol{y}_1)'}{\mathrm{Var}(\boldsymbol{Q}_t) + \mathrm{Var}_t(u)}\,\mathbf{Var}^{-1}(\boldsymbol{y}_1)\,\mathbf{Cov}(\theta, \boldsymbol{y}_1),$$

which is indeed true due to (12).

We have

$$\begin{aligned}
\gamma_t &= \frac{1 + \sum_{i=1}^n \frac{\mathrm{Cov}(\boldsymbol{Q}_t,\theta) - \mathbf{Cov}(\theta,\boldsymbol{y}_i)'\,\mathbf{Var}^{-1}(\boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{Q}_t,\boldsymbol{y}_i)}{\Delta_i \mathrm{Var}_t(\theta|i)[\mathrm{Var}(\boldsymbol{Q}_t) + \iota_t^2 \mathrm{Var}_t(u) - \mathbf{Cov}(\boldsymbol{Q}_t,\boldsymbol{y}_i)'\,\mathbf{Var}^{-1}(\boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{Q}_t,\boldsymbol{y}_i)]}}{\sum_{i=1}^n \frac{1}{\Delta_i \mathrm{Var}_t(\theta|i)}} \\
&= \frac{1 + \frac{1}{|\boldsymbol{Q}_t|} \sum_{i=1}^n \frac{\mathrm{Cov}(\boldsymbol{\mu}_t,\theta) - \mathbf{Cov}(\theta,\boldsymbol{y}_i)'\,\mathbf{Var}^{-1}(\boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{\mu}_t,\boldsymbol{y}_i)}{\Delta_i \mathrm{Var}_t(\theta|i)[\mathrm{Var}(\boldsymbol{\mu}_t) + \iota_t^2 \mathrm{Var}_t(u) - \mathbf{Cov}(\boldsymbol{\mu}_t,\boldsymbol{y}_i)'\,\mathbf{Var}^{-1}(\boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{\mu}_t,\boldsymbol{y}_i)]}}{\sum_{i=1}^n \frac{1}{\Delta_i \mathrm{Var}_t(\theta|i)}} \\
&= \frac{1 + \frac{1}{|\boldsymbol{Q}_t|} \sum_{i=2}^n \frac{\mathrm{Cov}(\boldsymbol{\mu}_t,\theta) - \mathbf{Cov}(\theta,\boldsymbol{y}_i)'\,\mathbf{Var}^{-1}(\boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{\mu}_t,\boldsymbol{y}_i)}{\Delta_i \mathrm{Var}_t(\theta|i)[\mathrm{Var}(\boldsymbol{\mu}_t) + \iota_t^2 \mathrm{Var}_t(u) - \mathbf{Cov}(\boldsymbol{\mu}_t,\boldsymbol{y}_i)'\,\mathbf{Var}^{-1}(\boldsymbol{y}_i)\,\mathbf{Cov}(\boldsymbol{\mu}_t,\boldsymbol{y}_i)]}}{\sum_{i=1}^n \frac{1}{\Delta_i \mathrm{Var}_t(\theta|i)}},
\end{aligned} \tag{14}$$

where the first equality follows from (4) (note that here there is no idiosyncratic noise, so $\boldsymbol{z}_i = \boldsymbol{y}_i$), the second equality uses the fact that $\boldsymbol{\mu}_t = \boldsymbol{Q}_t / |\boldsymbol{Q}_t|$, and the third equality follows from (12). Consequently, from (14) and the fact that $|\boldsymbol{Q}_t| - |\boldsymbol{Q}_{1t}| \to 0$ (because $\boldsymbol{Q}_{it} \to \mathbf{0}$ for every $i \geq 2$), we have

$$\gamma_t - \frac{1 + \frac{\Delta_1 \mathrm{Var}_t(\theta|1)}{|\mathbf{Var}^{-1}(\boldsymbol{y}_1)\,\mathbf{Cov}(\theta,\boldsymbol{y}_1)|} \sum_{i=2}^n \frac{\mathrm{Cov}(\theta, \boldsymbol{\mu}_1' \boldsymbol{y}_1)}{\Delta_i \mathrm{Var}_t(\theta|i)\,\mathrm{Var}(\boldsymbol{\mu}_1' \boldsymbol{y}_1)}}{\sum_{i=1}^n \frac{1}{\Delta_i \mathrm{Var}_t(\theta|i)}} \to 0. \tag{15}$$

From (9) we also have

$$\frac{1+\frac{\Delta_1 \operatorname{Var}_t(\theta|1)}{|\mathbf{Var}^{-1}(\boldsymbol{y}_1)\,\mathbf{Cov}(\theta,\boldsymbol{y}_1)|}\sum_{i=2}^{n}\frac{\operatorname{Cov}(\theta,\boldsymbol{\mu}_1'\boldsymbol{y}_1)}{\Delta_i \operatorname{Var}_t(\theta|i)\operatorname{Var}(\boldsymbol{\mu}_1'\boldsymbol{y}_1)}}{\sum_{i=1}^{n}\frac{1}{\Delta_i \operatorname{Var}_t(\theta|i)}} = \frac{1+\Delta_1 \operatorname{Var}_t(\theta|1)\sum_{i=2}^{n}\frac{1}{\Delta_i \operatorname{Var}_t(\theta|i)}}{\sum_{i=1}^{n}\frac{1}{\Delta_i \operatorname{Var}_t(\theta|i)}}$$
$$= \Delta_1 \operatorname{Var}_t(\theta|1). \tag{16}$$

Combining (15) and (16), we obtain

$$\gamma_t - \Delta_1 \operatorname{Var}_t(\theta|1) \to 0. \tag{17}$$

From (13) and (17), we can derive the two limits:

$$\boldsymbol{\pi}_1 \to \mathbf{Var}^{-1}(\boldsymbol{y}_1)\,\mathbf{Cov}(\theta,\boldsymbol{y}_1),\ \boldsymbol{\pi}_i \to \mathbf{0}, i=2,...,n, \text{ and } \gamma_t^2 \operatorname{Var}(u_t) \to 0.$$

Thus, $\mathbf{Cov}(\theta,\boldsymbol{y}_1) = \mathbf{Cov}(\boldsymbol{\pi},\boldsymbol{y}_1)$. Multiplying by $\boldsymbol{\pi}_1$ on both sides, we obtain $\operatorname{Cov}(\theta,\boldsymbol{\pi}) = \operatorname{Var}(\boldsymbol{\pi})$. Combining this with (5) leads to $\mathbf{Cov}(\theta,\boldsymbol{y}_i) = \mathbf{Cov}(\boldsymbol{\pi},\boldsymbol{y}_i)$ for every $i \geq 2$.

In a similar way to (13), we can show that $\boldsymbol{\alpha}_{1t} = \mathbf{Var}^{-1}(\boldsymbol{y}_1)\,\mathbf{Cov}(\theta,\boldsymbol{y}_1)$ for every t. By (12), we have $\beta_{1t} = 0$ for every t. If follows from (5) that $\boldsymbol{\alpha}_{it} \to \mathbf{0}$ for any $i \geq 2$. By (11) and (9), we have $\beta_{it}|\boldsymbol{\pi}| \to |\,\mathbf{Var}^{-1}(\boldsymbol{y}_1)\,\mathbf{Cov}(\theta,\boldsymbol{y}_1)|$, i.e., $\beta_{it} \to 1$ for any $i \geq 2$. Then the limit on $\{c_t\}$ follows from the equality (8) in the main text, and the proof is now complete. □