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SURVIVAL, GROWTH AND TECHNICAL PROGRESS IN A SMALL RESOURCE-IMPORTING ECONOMY*

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1. INTRODUCTION

The problems of planning or of optimal growth in many economies are exacerbated by the fact that production depends, in an essential way, on some resource (input, intermediate good) of which there is little or no domestic supply.[1] Oil is a prime example of such a resource. The difficulties arise for two reasons: (a) the scarce nature of this resource results in its international price (relative to that of domestic output) rising over time, and (b) the resource-importing economy must pay for its imports by exporting an equivalent quantity.

It is intuitive that if the economy is 'productive' enough, it will possess the ability to compensate for the resource price-rise by a sufficient accumulation of capital. This leads rather naturally to a number of interesting questions. These are two examples. First, do there exist price paths of the resource which constrain the economy to maintaining, at best, an ever-diminishing level of consumption? If the answer is in the affirmative, one must study empirical data and consult the relevant economic theory, to ascertain whether such paths are the rule rather than the exception. Secondly, given some (increasing) resource price path, what rates of technical progress will permit the economy to achieve a *growing* consumption level? These issues are clearly important.

The possible lack of richness in the set of feasible alternatives (hinted at above) has been made explicit in the literature on autarkic intertemporal accumulation in the presence of essential exhaustible resources. A seminal paper by Solow [1974] captured the essence of scarcity, or exhaustibility of the resource, by the twin devices of finiteness of the initial resource stock and an infinite horizon to plan

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[1] To make matters worse, such resources may enter directly into consumption. However, the essential features of the problem may be analysed without introducing this complication.

for.[2] The structure of such a model necessitated an analysis of the question: when is 'survival' possible? The definition of this concept took the broad view that survival obtains whenever there exists a feasible program with consumptions bounded away from zero. Thus, in an aggregative framework, the economy must be capable of maintaining some positive level of consumption forever.

A somewhat more ambitious question is that of growth. If the survival issue has been settled in the affirmative, it is natural to enquire whether 'growth' is possible. In other words, does there exist a feasible program with consumption growing to infinity over time?

In a trading model, the notation of resource scarcity is captured in the phenomenon of ever-increasing prices of resource imports. In Mitra, Majumdar and Ray [1982], a two-sector trading economy was considered, and necessary conditions for survival were presented in terms of the production technology and the path of international resource prices. Sufficient conditions were also provided, though a 'complete characterization' of the conditions under which survival occurs was not obtained. The growth issue was also not considered in their paper.

In this exercise, I use an aggregative model of intertemporal accumulation, with domestic output and an imported resource the only two commodities. There is no domestic stock of the resource; imports of it are nonstorable[3] and the balance of trade must be in equilibrium at every period.[4] The economy takes resource prices as given at every date.

I seek answers to the following questions.

(1) When is an economy capable of survival? In other words, what conditions on the terms of trade and the technology guarantee the existence of a feasible program with consumption bounded away from zero?

(2) When survival does obtain, is it possible to compute the *maximum* stationary level of consumption maintainable forever: i.e., the *survival level* generated by the economy? This would permit a comparison with exogenous 'minimum-needs' data, to determine whether an economy meets some exogenously specified 'standard of living' at every date.

(3) When is an economy capable of *growth*; i.e., what conditions guarantee the existence of a feasible program with consumption growing to infinity over time?

(4) What is the relationship between technical progress and growth? In particular, given some rate of growth of consumption to be maintained, what is the minimum rate of technical progress (given the trend in resource prices) that will guarantee this?

[2] This work has subsequently been generalized. See, for example, Mitra [1978] and Cass and Mitra [1979].

[3] It has been shown that the impact of deteriorating terms of trade persists in a very definite way even when the resource import is storable, provided that either the capital or the resource stock depreciates. For details, see Mitra, Majumdar and Ray [1982].

[4] At the cost of some computational complexity, it is possible to introduce various forms of foreign aid. This is, however, not done in the present model.

In an important sense, an analysis of this kind is logically prior to, and perhaps more relevant than a full-fledged 'optimization' exercise. The logical priority of this exercise arises from a need to recognize (or characterize) feasible programs before a choice among them is actually made.[5] Its somewhat greater relevance stems from the fact that most planning exercises employ precisely such 'targets' (for example, the achievement of some fixed rate of growth) instead of a social welfare function of the kind commonly used in the literature.[6] It is certainly true that all statements of planning objectives are, in a broad sense, equivalent to some 'social welfare function.' However what are explicitly stated as 'primitive objectives' in many planning exercises are simply the achievement of some growth rate in output, or employment. It is precisely objectives of this type[7] which I take as basic to the present analysis.

In Section 3, I consider an aggregative model of the kind previously described, with changing technology and resource prices. At each date, the technology is assumed to be linear homogeneous in capital and resource inputs (concavity of the technology is *not* assumed). Outputs are divided into consumption, investment and exports. Exports pay for resource imports, investment augments the stock of capital. This process repeats itself indefinitely.

The basic result of this section is Theorem 3.1, which provides a complete characterization of feasible and efficient programs in a given economy, for arbitrary resource price paths and technical change. This is used in Theorem 3.2 to provide a complete characterization of those economies capable of survival. The survival level is computed (Theorem 3.3) and examples with Cobb-Douglas technologies discussed. Theorem 3.4 goes on to show that *any* exponential growth of resource prices is inconsistent with survival, in a situation of no technical change. This result is of interest, since the theoretical literature on exhaustible resources following Hotelling [1931] does suggest that resource prices grow exponentially, at least asymptotically. Theorem 3.5 deals with growth, and provides a somewhat surprising result. 'Survival' and 'growth' are equivalent problems, in this model, in the sense that an economy is capable of survival if and only if it is capable of growth. Theorem 3.6 completely characterizes those economies capable of exponential growth of consumption at some rate $g>0$. Theorem 3.7 provides a sufficient condition, which, if satisfied, permits an economy to sustain any exponential growth of consumption (by choosing initial consumption levels suitably). This is applied in Theorem 3.8 to yield an interesting result for a special case when resource prices are growing exponentially. It is demonstrated that if the rate of

[5] By this, I mean the recognition of some *feature* of a feasible program, important to most actual planning exercises, such as the ability to sustain exponentially growing consumption forever.

[6] A classic example of economic analysis using planning 'targets' is the work of Tinbergen. See for example, Tinbergen ([1952], [1955]). Also of relevance here is the approach of Manne [1970] using 'gradualist' consumption paths.

[7] A real-world example is the case of India. One need only glance at the mode of target-setting in her Five-Year Plans.

resource augmenting technical progress exceeds that of the price-rise, *any* exponential growth of consumption can be maintained. If the rate falls short of that of the price-rise, no exponential growth is maintainable. (The case of equal rates is also analyzed). This result illuminates an interesting 'knife-edge' property of technical progress in the present model. In addition, an example is given to show that when capital is 'important enough' in production (in a sense explained below), capital-augmenting technical progress may be 'better' than resource-augmenting technical progress even though resource prices are increasing.

If labor is introduced as a factor of production, the linear homogeneity assumption on capital and resource inputs alone is clearly unsatisfactory. Section 4 indicates briefly how an extension of this kind may be carried out. Some results on survival and growth are provided for a full-employment model, where the available labor force is fully used at every date. Finally, I indicate how these results may be extended to a surplus-labor economy.

2. SOME NOTATION

In the following text, the subscript t refers to time period. The following symbols are also used:

K: capital
R: resource
C: consumption
I: investment
E: exports
L: employment
p: resource price

$\boldsymbol{R}^n_+$ (resp. $\boldsymbol{R}^n_{++}$) denotes the set of all nonnegative n-vectors (resp. strictly positive n-vectors). For a differentiable function f of one variable, denote its derivative by f^1.

3. MODEL WITH NO LABOR

3.1. *The Technology and Environment.* The technology is given by a sequence of *net-output production functions* $\langle G_t \rangle_0^\infty$ where, for each $t \geqslant 0$, G_t: $\boldsymbol{R}^2_+ \rightarrow \boldsymbol{R}_+$. These functions combine capital (K) and the resource (R) to produce output ($G_t(K, R)$). The following assumptions are made on the technology.

(G.1) For all $t \geqslant 0$, $G_t(0, R) = G_t(K, 0) = 0$.
(G.2) For all $t \geqslant 0$, $\lim_{K \to \infty} G_t(K, R) = \infty$ if $R > 0$, and $\lim_{R \to \infty} G_t(K, R) = \infty$ if $K > 0$.
(G.3) For all $t \geqslant 0$, $G_t(\cdot, \cdot)$ is homogeneous of degree one, continuous, and increasing in its arguments.

Capital will be assumed to be nondepreciating.
The environment is given by

(E.1) A sequence $\langle p_t \rangle_0^\infty$ of strictly positive resource prices[8]
(E.2) An initial stock of capital, $\underset{\sim}{K} > 0$.

Remarks. Note that (G.1)–(G.3) place no restriction on the rate of technical change, also that no differentiability or concavity assumptions have been placed on the technology.[9] The feature (E.1) reflects the 'small-country' assumption of international trade theory. It rules out any analysis of the bargaining issue involved in the process of resource price-setting.

However, note that no assumption has been made on the form of the time path of resource prices. The general characterization of feasible alternatives that I seek will require no such restrictions. Of course, particular price paths will be used below in examples, or to make additional points.

3.2. *The Structure of the Economy.* In this section, I describe feasible allocations within the economy. An initial capital stock is given, along with an international price of the resource. A quantity of the resource is imported, and output is produced. Part of this output is exported to pay for resource imports, part is invested, to add to the capital stock, and the remainder consumed. Equipped with the new capital stock, the entire process is repeated (with resource prices possibly at a different level), and so the economy moves over time.

These features are captured in the following system.

$$\underset{\sim}{K} \geqslant K_0 \tag{1}$$

$$G_t(K_t, R_t) \geqslant C_t + I_t + E_t, \qquad t \geqslant 0 \tag{2}$$

$$E_t = p_t R_t, \qquad t \geqslant 0 \tag{3}$$

$$K_t + I_t = K_{t+1}, \qquad t \geqslant 0 \tag{4}$$

$$(K_t, Y_t, C_t, E_t, R_t) \geqslant 0, \qquad t \geqslant 0^{10}. \tag{5}$$

Remarks. Equation (3) is restrictive, in that it requires the balance of trade condition to hold in every period. A more general formulation would include the possibility of external financing at some rate of interest. Realistically, one would stipulate that both fresh loans obtainable at any date, and past accumulated debt, must be bounded. The accomodation of these additional features compli-

[8] These are prices with domestic output as numeraire.

[9] The treatment of technical progress, in particular, its apparent exogeneity in this model, may be a bit misleading. The kind of questions to be asked are: *if* the rate of consumption growth is to be $g > 0$, given the rate of price rise, what must the rate of technical progress be? Viewed from this 'feasibility' viewpoint, it should be clear that technical progress, while formally exogenous, is not regarded as such in the spirit of the exericse.

[10] Note that investment is not constrained to be nonnegative; i.e., that the capital stock may be run down to provide for current consumption needs. Call this a reversibility model. If, in addition, the condition $I_t \geqslant 0$, $t \geqslant 0$ is met, the system may be said to describe an irreversibility model. All the results here may be worked out for such a model; these are omitted for lack of space. The reader is referred to Ray [1981] for details.

cates the workings of the model, but does not alter the qualitative nature of the arguments.

3.3. *Some Definitions.* A program $\langle K, C, I, E, R\rangle \equiv \langle K_t, C_t, I_t, E_t, R_t\rangle_0^\infty$ is *feasible* if it satifies (1)–(5). Denote by $\langle C_t\rangle_0^\infty$ the corresponding feasible consumption program. A feasible consumption program $\langle C_t\rangle_0^\infty$ is *efficient* if there does not exist another feasible consumption program $\langle C_t'\rangle_0^\infty$ with $C_t' \geqslant C_t$, $t \geqslant 0$, and $C_s' > C_s$ for some $s \geqslant 0$.

A *survival program* is a feasible program $\langle K, C, I, E, R\rangle$ with $\inf_{t \geqslant 0} C_t > 0$. An economy is capable of survival if there exists a survival program. The *survival level*, γ, of an economy, is given by $\gamma \equiv \{\inf_{t \geqslant 0} C_t \colon \langle C_t\rangle$ is a feasible consumption program$\}$.

A *growth program* is a feasible program $\langle K, C, I, E, R\rangle$ with $C_{t+1} \geqslant C_t$, $t \geqslant 0$, and $\lim_{t\to\infty} C_t = \infty$. An economy is capable of growth if there exists a growth program.

An *exponential growth program* with rate $g > 0$ is a feasible program $\langle K, C, I, E, R\rangle$ with $C_0 > 0$ and $C_t = C_0(1+g)^t$, $t \geqq 0$. An Economy is capable of growth at rate $g > 0$ if there exists an exponential growth program with rate g.

Remarks. The definition of a growth program may appear to be unduly restrictive. A less ambitious requirement would be the existence of a consumption path, increasing to some pre-specified, possibly finite, consumption value. It turns out that in the present model (with no labor), this makes no difference to the analysis, in the light of the equivalence theorem below (Theorem 3.5). However, the restriction, if any, is a real one in the model of Section IV below.

3.4. *Characterizations of Survival and Growth.* In this section, I restate briefly and then provide some answers to the questions raised earlier.

(1) When is an economy capable of survival? When survival is possible, can one explicitly compute γ, the survival level of the economy? (See Theorems 3.2 and 3.3).
(2) When is an economy capable of growth, or of *exponential* growth at some rate $g > 0$? (See Theorems 3.5 and 3.6).
(3) What is the relationship between the pace of technical progress, the price path of the resource, and maintainable growth rates? (See Theorems 3.7 and 3 8)

All proofs are relegated to the Appendix

First, I state

LEMMA 3.1. *Under* (G.1)–(G.3), *for each* $t \geqslant 0$, *the function* $H_t \colon \boldsymbol{R}_+ \times \boldsymbol{R}_{++} \to \boldsymbol{R}_+$ *defined by* $H_t(K, p) \equiv \max_{R \geqslant 0} [G_t(K, R) - pR]$, *for* $K \geqslant 0$, $p > 0$, *exists and is positive on* $\boldsymbol{R}_{++}^2$.

Denote by $R_t(K, p)$ a maximizer of $G_t(K, R) - pR$.

Lemma 3.2 outlines a useful property of the functions $\langle H_t \rangle_0^\infty$.

LEMMA 3.2. *Under* (G.1)–(G.3), *for each* $t > 0$, *and* $p > 0$

$$H_t(K, p) = f_t(p)K \tag{6}$$

where $f_t\colon \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$ *is decreasing.*

REMARKS. Lemma 3.2 tells us that the 'reduced' production function, obtained by maximizing output net of exports, has a convenient linearity property (in capital stocks). This linearity is obtained through the linear homogeneity assumption on the production function, and, as a result, no concavity assumptions are required to drive any of the results in this section (see, however, Section 4).

If, in addition, one has for each $t \geqslant 0$, G_t twice differentiable and satisfying the well-known Inada conditions, f_t may be exhibited explicitly. Define $g_t(z) \equiv G_t(1, z)$, for $z \geqslant 0$, and $r_t(z) \equiv g_t'^{-1}(z) > 0$. Then

$$f_t(p) = g_t(r_t(p)) - pr_t(p) \tag{7}$$

(which is easily checked to be positive). This may be interpreted as the marginal product of a given stock of capital when the resource is imported to maximize output net of imports.

For a Cobb-Douglas technology of the form $G_t(K, R) = (\lambda_t K)^\alpha (\beta_t R)^{1-\alpha}$, $\alpha \in (0, 1)$, $(\lambda_t, \beta_t) \gg 0$, $t \geqslant 0$, it is easy to check that

$$f_t(p) = \frac{A\lambda_t}{(p_t/\beta_t)^\delta} \tag{8}$$

where $A \equiv \alpha(1-\alpha)^{\frac{1-\alpha}{\alpha}} \in (0, 1)$, and $\delta \equiv \dfrac{1-\alpha}{\alpha}$.

I first provide a complete characterization of feasible and efficient programs in this model. For an economy $\langle G, p, K \rangle$, define $M_t \equiv [\prod_{s=0}^{t} (1 + f_s(p_s))]^{-1}$, $t \geqslant 0$.

THEOREM 3.1. (i) *A consumption program* $\langle \bar{C}_t \rangle_0^\infty$ *is feasible for an economy* $\langle G, p, \underset{\sim}{K} \rangle$ *if and only if*

$$\sum_{t=0}^{\infty} M_t \bar{C}_t \leqslant \underset{\sim}{K} \tag{9}$$

(ii) *It is efficient if and only if equality holds in* (9).

REMARKS. If the resource is imported at every date to maximize the value of output net of imports, Lemmas 3.1 and 3.2 demonstrate that the resulting model has a useful linearity property. The characterization of efficient programs in linear models is well-known: see for example, Majumdar [1974]. In the present context, Theorem 3.1, while of independent interest, is mainly a device to be applied to characterizations of survival and growth.

A useful interpretation of (9) may be obtained using the concept of 'profit-maximizing' or 'efficiency' prices. At any date $t \geqslant 0$, the total output, net of imports, is $G_t(K_t, R_t)+K_t-p_tR_t$. The input is capital, K_t (the resource having been accounted for in the definition of net output). Efficient programs in this model have the following property. There exists a sequence of nonnull, nonnegative efficiency prices $\langle Q_t \rangle_0^\infty$ such that (valuing output of period t at the price of date $t+1$, and input at the price of period t) the profit at date t

$$Q_{t+1}[G_t(K_t, R_t)+K_t-p_tR_t] - Q_tK_t \tag{10}$$

is maximized with respect to that obtainable using any nonnegative pair (K, R). It is easy to check that if $Q_0>0$, and $Q_{t+1}\cdot Q_t^{-1}=[1+f_t(p_t)]$ for all $t \geqslant 0$, the sequence $\langle Q_t \rangle_0^\infty$ is a system of efficiency prices for all efficient programs. In particular, the sequence $Q_0=1$, $Q_t=M_{t-1}$ for all $t \geqslant 1$ is a system of efficiency prices. Restating (9) yields

$$\sum_{t=0}^{\infty} Q_{t+1}\bar{C}_t \leqslant \underset{\sim}{K} \tag{11}$$

for all feasible consumption programs $\langle \bar{C}_t \rangle_0^\infty$, with equality holding for all efficient programs.

Thus, feasibility is captured by the fact that the 'value of consumption,' evaluated at efficiency prices, must not be too 'large', and efficient programs $\langle \bar{C}_t \rangle_0^\infty$ are precisely those which satisfy

$$\sum_{t=0}^{\infty} Q_{t+1}\bar{C}_t \geqslant \sum_{t=0}^{\infty} Q_{t+1}C_t \tag{12}$$

for all feasible $\langle C_t \rangle_0^\infty$, i.e., those which maximize 'consumption value'. In this context, the work of Debreu [1954] is important, see also Cass and Yaari [1971]. A 'direct' characterization of efficient programs (i.e., one not involving comparison with other feasible programs) using efficiency prices may be found in Cass [1972].

While the interpretation using efficiency prices is important and of wide applicability, it turns out that the central feature of these survival-growth problems is the behavior of the *pure accumulation program*. For example, the characterization of survival in the full-employment model of Section IV has no direct interpretation using efficiency prices, whereas the behavior of pure accumulation stocks is crucial. I shall therefore interpret (9) in terms of the pure accumulation program.

Define such a program as a feasible program with associated capital stocks $\langle \hat{K}_t \rangle_0^\infty$ satisfying $\hat{K}_t \geqslant K_t$, $t \geqslant 0$ for all sequences $\langle K_t \rangle$ associated with some feasible program. It is obvious that the pure accumulation capital stocks $\langle \hat{K}_t \rangle_0^\infty$ form a unique sequence; and it is easy to see that a pure accumulation program may be constructed by setting $\hat{K}_0=\underset{\sim}{K}$, $\hat{K}_{t+1}=H_t(\hat{K}_t, p_t)+\hat{K}_t$, $t \geqslant 0$, $C_t=0$, $t \geqslant 0$, $I_t=\hat{K}_{t+1}-\hat{K}_t$, $E_t=p_tR_t(\hat{K}_t, p_t)$, $R_t=R_t(\hat{K}_t, p_t)$.

Thus, for all $t \geqslant 0$, $\hat{K}_{t+1}=[1+f_t(p_t)]\hat{K}_t$, and so $\hat{K}_t=M_{t-1}^{-1}\underset{\sim}{K}$, $t \geqslant 1$, with $\hat{K}_0=\underset{\sim}{K}$. The condition (9) may now be rephrased as

$$\sum_{t=0}^{\infty} \frac{\bar{C}_t}{\hat{K}_{t+1}} \leqslant 1 \tag{13}$$

for all feasible programs $\langle \bar{C}_t \rangle_0^\infty$, with equality holding for all efficient programs.

The set of feasible consumption programs is composed precisely of those consumption sequences which tend to zero relative to pure accumulation stocks 'sufficiently fast,' the exact condition being provided by (13). The prescribed consumption stream must dwindle (over time) relative to the pure accumulation capital stocks. Intuitively, if this is not the case, then there is too little left over to sustain capital accumulation, and the consumption program becomes infeasible. This is the criterion made explicit in (13).

Survival. Theorem 3.1 readily yields a complete characterization of economies capable of survival.

THEOREM 3.2. *An economy $\langle G, p, \underset{\sim}{K} \rangle$ is capable of survival if and only if*

$$\sum_{t=0}^{\infty} M_t < \infty. \tag{14}$$

The following statements are each equivalent to (14).

$$\sum_{t=0}^{\infty} Q_t < \infty \tag{15}$$

$$\sum_{t=0}^{\infty} \frac{1}{\hat{K}_t} < \infty. \tag{16}$$

The next theorem computes the survival level of a given economy.

THEOREM 3.3. *The survival level of an economy $\langle G, p, \underset{\sim}{K} \rangle$, γ, is given by*

$$\gamma = \frac{\underset{\sim}{K}}{\sum_{t=0}^{\infty} M_t} = \frac{\underset{\sim}{K}}{\sum_{t=0}^{\infty} Q_{t+1}} = \frac{1}{\sum_{t=0}^{\infty} \frac{1}{\hat{K}_t}} \tag{17}$$

interpreting γ as zero if the sums in the denominators diverge. Any feasible program with consumption equal to γ for all $t \geqslant 0$ is efficient.

Observe that the feasible program which generates the survival level of consumption every year is also the program that is obtained through an application of a Rawlsian maximum criterion, provided that all generations have identical utility functions, increasing in consumption. For a further discussion of this relationship, see Solow [1974].

Taken together, Theorem 3.2 and 3.3 provide an answer to question (1), posed at the beginning of this section.

Example (3.1). (Cobb-Douglas production function). For a Cobb-Douglas net-output function $G_t(K, R) = (\lambda_t K)^\alpha (\beta_t R)^{1-\alpha}$, $\alpha \in (0, 1)$, $(\lambda_t, \beta_t) \gg 0$, $t \geqslant 0$, recall from (8) that $f_t(p) = \dfrac{A\lambda_t}{(p_t/\beta_t)^\delta}$, and hence, that survival is possible if and only if

$$(18) \qquad \sum_{t=0}^{\infty} \prod_{s=0}^{t} \left[1+\frac{A\lambda_s\beta_s^\delta}{p_s^\delta}\right]^{-1} < \infty,$$

The condition (18) may be used to test for survival. For example, suppose that there is no technical change, so that $\lambda_t=\beta_t=1$ for all $t\geqslant 0$, and let $p_t=(t+2)^k$, $k>0$, $t\geqslant 0$. Consider two cases.

Case 1: $(k\delta>1)$ In this case, $\sum_{t=0}^{\infty}\frac{A}{p_t^\delta}<\infty$, and hence $\sum_{s=0}^{t}\left(1+\frac{A}{p_s^\delta}\right)\leqslant B<\infty$ for all $t\geqslant 0$. (See, for example, Knopp [1956], p. 94). Therefore, $\sum_{t=0}^{T}\left\{\prod_{s=0}^{t}\cdot\left[1+\frac{A}{p_s^\delta}\right]\right\}^{-1}\to\infty$ as $T\to\infty$ and survival is impossible.

Case 2: $(k\delta<1)$ In this case, one can show (see Mitra, Majumdar and Ray [1982, Appendix]) that (putting $\eta=k\delta$),

$$(19) \qquad \prod_{s=0}^{t}\left[1+\frac{A}{p_s^\delta}\right]\geqslant \exp\left[H(t+2)^{1-\eta}-B\right]$$

where H and B are positive constants, and this easily establishes that (18) is satisfied. Hence, survival is possible.

This example illustrates two points (a) survival is possible, in the absence of technical progress, even with unbounded decline in the terms of trade, and (b) survival is more 'likely' the more 'important' capital is in production (the lower the value of δ, in this example).

Consider, now, an exponential price-path of the form $p_t=p_0(1+\rho)^t$, $p_0>0$, $\rho>0$, $t\geqslant 0$. Suppose, further, that the technology is stationary, i.e. $G_t=G$ for all $t\geqslant 0$, and satisfies

(G.4) G is twice differentiable, and $g(z)\equiv G(a, z)$, $z\geqslant 0$ is strictly concave.

(G.5) $\liminf_{z\to 0}\frac{g'(z)z}{g(z)}>0.$

Remark. (G.5) asserts that the resource is not only essential in production, but has some impact at 'small' levels of its use. This assumption is satisfied by the Cobb-Douglas production function, as well as other functional forms.[11]

THEOREM 3.4. *Suppose that an economy* $\langle G, p, \underline{K}\rangle$ *satisfies* (G.4) *and* (G.5), *and that* $p_t=p_0(1+\rho)^t$, $p_0>0$, $\rho>0$, $t\geqslant 0$. *Then it is not capable of survival.*

REMARK. The literature on exhaustible resources following Hotelling [1931] suggests that resource prices may exhibit exponential growth in various theoretical frameworks, at least asymptotically. Such asymptotically exponential price paths would obtain, for example, with constant or bounded marginal costs of extraction and interest rates bounded away from zero. Theorem 3.4, while stated

[11] Consider, for example, $G(K, R)=A(K+R)^{1/2}K^{1/4}R^{1/4}$, $A>0$. This satisfies all our assumptions, with $\frac{zg'(z)}{g(z)}\geqslant\frac{1}{4}$ for all z>0.

for exponential resources price paths, is easily seen to hold for these paths, too.

Growth. The results here characterize those economies capable of growth. In this model, the problems of survival and growth turn out to be equivalent. This is expressed in

THEOREM 3.5. *An economy $\langle G, p, \underset{\sim}{K}\rangle$ is capable of growth if and only if it is capable of survival.*

REMARK. The equivalence of survival and growth is often seen in autarkic exhaustible resource models. See, for example, Cass and Mitra [1979]. This equivalence follows from linear homogeneity of the production functions, and does not, in general, hold in the diminishing returns case (see Section 4). The intuition is as follows. If survival is possible, then by starting with a small enough initial consumption, it is possible to build up capital stocks faster than the capital stocks associated with the survival level. Eventually, this magnifies the scale of the economy, and hence, by linear homogeneity, raises the constant maintainable consumption level proportionately. Consumption can then be adjusted upward. Repeat this process indefinitely to obtain the result.

Example 3.2. (Cobb-Douglas production function). Consider Case 2 of Example 3.1. There, survival is possible, and so by Theorem 3.5, the economy is capable of growth. In addition, the feasibility characterization (9) may be exploited to reveal more about the kind of growth which is possible. Here, I demonstrate that 'polynomial' growth paths of the form $\bar{C}_t = \sum_{i=0}^{n} a_i t^i$ are feasible (for some choice of $a_i > 0$, $i = 0, \ldots, n$), for *any* positive integer n. Observe that for any integer n, $\eta \in (0, 1)$ and $(M, B) \gg 0$, there exists an integer T such that

$$\exp\left[M(t+2)^{1-\eta} - B\right] \geqslant t^{n+2}, \qquad \text{for all} \quad t \geqslant T. \tag{20}$$

Now write the condition (9) with polynomial consumption path as

$$\sum_{t=0}^{\infty} \frac{\sum_{i=0}^{n} a_i t^i}{\prod_{s=0}^{t}\left[1 + \frac{A}{p_s^{\delta}}\right]} \leqslant \underset{\sim}{K}. \tag{21}$$

It is easy to check, using (19) and (20), that for any positive integer n, there exists $(a_i)_{i=0}^{n} \gg 0$ such that (21) holds.

Those economies capable of sustaining exponential growth at a rate $g > 0$ are characterized in

THEOREM 3.6. *An economy $\langle G, p, \underset{\sim}{K}\rangle$ is capable of exponential growth at rate g if and only if*

$$\sum_{t=0}^{\infty} M_t (1+g)^t < \infty \tag{22}$$

Taken together, Theorem 3.5 and 3.6 provide an answer to question (2) posed at the beginning of the section.

Technical Progress. Here, the preceding theorems are used to provide a precise relationship between the rate of technical progress and the rates of maintainable exponential growth. First, I provide a sufficient condition on the price path and the nature of technical progress such that *any* exponential rate of growth can be satisfied.

THEOREM 3.7. *Suppose that an economy* $\langle G, p, \underset{\sim}{K} \rangle$ *satisfies* $\lim_{t\to\infty} f_t(p_t) = \infty$. *Then it is capable of exponential growth at rate* g, *for all* $g > 0$.

Though Theorem 3.7 is obvious (given (22)), it has important consequences. As an illustration, suppose that resources prices follow an exponential path, $p_t = p_0(1+\rho)^t$, $\rho > 0$, and suppose that the economy exhibits resource-augmenting technical progress at some rate $\beta > 0$, i.e., $G_t(K, R) = G(K, (1+\beta)^t R)$ for all $t \geqslant 0$. We then have

THEOREM 3.8. *Suppose that* $\langle G, p, \underset{\sim}{K} \rangle$ *satisfies* $G_t(K, R) = G(K, (1+\beta)^t R)$, $\beta > 0$, *for* $t \geqslant 0$ *and* $p_t = p_0(1+\rho)^t$, $p_0 > 0$, $\rho > 0$, *for* $t \geqslant 0$. *Then*

(i) *if* $\beta > \rho$, *the economy is capable of exponential growth at rate* g *for all* $g > 0$.

(ii) *if* $\beta < \rho$, *the economy is not capable of exponential growth at any rate* $g > 0$.

(iii) *if* $\beta = \rho$, *there exists* $g^* > 0$ *such that the economy is capable of exponential growth at rate* g *if and only if* $g < g^*$.

REMARK. Note that the condition for sustaining a particular growth rate is independent of the growth rate. Thus if some exponential growth rate is feasible, so are all exponential growth rates (in the situation of Theorem 3.8). This feature is a consequence of the linear homogeneity of the technology.[12]

Example 3.3. (Cobb-Douglas production function) Recall the Cobb-Douglas technology of Examples 3.1 and 3.2. Note that if the rates of technical progress are such that $\frac{\lambda_t \beta_t^\delta}{p_t^\delta} \to \infty$ as $t \to \infty$, then by Theorem 3.7, any exponential growth path is sustainable. It is worth reiterating that the growth rate g itself does not enter into this condition.

Suppose now, that $\lambda_t = (1+\lambda)^t$, $\lambda \geqslant 0$, $t \geqslant 0$, $\beta_t = (1+\beta)^t$, $\beta \geqslant 0$, $t \geqslant 0$, and $p_t = p_0(1+\rho)^t$, $\rho > 0$, $t \geqslant 0$. Then a sufficient condition for sustaining any growth rate g is $(1+\lambda)(1+\beta)^\delta > (1+\rho)^\delta$. When $\beta = 0$, this condition reduces to $\lambda > (1+\beta)^\delta - 1$. When $\lambda = 0$, the condition is $\beta > \rho$. This discussion illustrates a point of some significance, regarding the relative efficiencies of resource-augmenting and capital-augmenting technical progress. Suppose that equal rates of technical progress

[12] See Kemp and Long [1982] for an analysis of the diminishing-returns-to-scale case

in capital and resource ($\lambda = \beta$) are associated with equal costs of such progress. Suppose, further, that the 'share' of capital exceeds that of the resource in production, i.e., $\delta < 1$. It follows that the minimum value of λ guaranteeing sustainability of exponential growth is *less* than the corresponding value of β. This seems to indicate that a policy of capital-augmenting technical progress may dominate a policy of resource-augmenting technical progress when capital is important in production (even though it is the price of the *resource* which is rising), since the indirect saving on resource use is higher.

4. A MODEL WITH LABOR

In this section, I extend the framework developed in Section 3 to explicitly include labor as a factor of production. For the sake of brevity, I shall concentrate on the full-employment case (Section 4a), where the available labor force is given at each date, and fully used in productive activity. However, I shall also briefly comment on a surplus-labor version[13] (Section 4b); the reader is referred to Ray [1983a, b] for details.

4.1. *The Full Employment Model.*[14] The technology is given by a sequence of net-output production functions $\langle G_t \rangle_0^\infty$, where, for each $t \geqslant 0$, $\hat{G}_t$: $\boldsymbol{R}_+^3 \to \boldsymbol{R}_+$, providing for each triple of capital (K), resource (R) and labor (L), an output $\hat{G}_t(K, R, L)$. Consider the following assumptions on the technology.

($\hat{\text{G}}$.1) For all $t \geqslant 0$, $\hat{G}_t(K, R, 0) = \hat{G}_t(K, 0, L) = \hat{G}_t(0, R, L) = 0$

($\hat{\text{G}}$.2) For all $t \geqslant 0$, $\hat{G}_t(\cdot, \cdot, \cdot)$ is homogenous of degree one, increasing in its arguments, and concave.

($\hat{\text{G}}$.3) For all $t \geqslant 0$, $G_t(\cdot, \cdot, \cdot)$ is differentiable.

($\hat{\text{G}}$.4) For all $t \geqslant 0$ and each $(K, L) \in \boldsymbol{R}_{++}^2$, $\lim_{R \to 0} G_t^R(K, R, L) = \infty$ and $\lim_{R \to \infty} \cdot G_t^R(K, R, L) = 0$

($\hat{\text{G}}$.5) There exists $t \geqslant 0$ such that for all $t \geqslant 0$ and

$$(K, R, L) \in \boldsymbol{R}_{++}^3, \frac{L\hat{G}_t^L(K, R, L)}{\hat{G}_t(K, R, L)} \geqslant \eta > 0.$$

Capital is assumed to be nondepreciating.

The environment is given by (E.1) and (E.2) (Section 3), and (E.4) a stationary, positive, labor force $\bar{L} > 0$.

Remarks. ($\hat{\text{G}}$.5) states that in a world where labor is paid its marginal product, the share of labor in total output is bounded away from zero.[15] This assumption is satisfied, for example, where $\hat{G}_t$ is Cobb-Douglas for all $t \geqslant 0$. The feature

[13] See, for example, the dual-economy model of Lewis [1954].

[14] The material of Section 4.1 is based on Mitra and Ray [1982].

[15] This assumption may be dropped, but a complete characterization is thereby sacrificed; see Mitra and Ray [1982].

(E.4) postulates a stationary labor force. Nonstationarity presents no technical problems, but is unsatisfactory, because the concept of survival is defined in terms of aggregate consumption. Such a definition is of questionable significance in the face of a changing labor force.

Note that concavity of $\hat{G}_t$ is assumed, in contrast to the previous section. While this may not be necessary for the results here, it is indispensable for the techniques of proof that are used.

Define an *economy* to be $\langle \hat{G}, p, \bar{L}, \underaccent{\tilde}{K} \rangle \equiv \{\langle G_t, p_t \rangle_0^\infty, \bar{L}, \underaccent{\tilde}{K}\}$, with $\langle G_t \rangle_0^\infty$ satisfying ($\hat{\text{G}}$.1)–($\hat{\text{G}}$.5), and $\{\langle p_t \rangle_0^\infty, \bar{L}, \underaccent{\tilde}{K}\}$ given as in (E.1), (E.2) and (E.4).

Define, for a given economy and $t \geqslant 0$, G_t: $\boldsymbol{R}_+^2 \to \boldsymbol{R}_+$ by $G_t(K, R) \equiv \hat{G}_t(K, R, \bar{L})$, $(K, R) \in \boldsymbol{R}_+^2$.

Feasible allocations may be described as in Section 3, with the additional requirement that the available labor force be used up at each date. Feasible allocations are thus captured by the following system.

$$\underaccent{\tilde}{K} \geqslant K_0 \tag{23}$$

$$\hat{G}_t(K_t, R_t, L_t) = \hat{G}_t(K_t, R_t, \bar{L}) = G_t(K_t, R_t) \geqslant C_t + I_t + E_t, \qquad t \geqslant 0 \tag{24}$$

$$E_t = p_t R_t, \qquad t \geqslant 0 \tag{25}$$

$$K_t + I_t = K_{t+1}, \qquad t \geqslant 0 \tag{26}$$

$$(K_t, C_t, E_t, R_t) \geqslant 0, \qquad t \geqslant 0.^{16} \tag{27}$$

By (G.1)–(G.5), the function H_t: $\boldsymbol{R}_+ \times \boldsymbol{R}_{++} \to \boldsymbol{R}_+$ given by $H_t(K, p) \equiv \max_{R>0} \cdot G_t(K, R) - pR$ is well-defined for all $t \geqslant 0$. Denote by $R_t(K, p)$ a maximizer of $G_t(K, R) - pR$, for $K \geqslant 0$, $p > 0$, $t \geqslant 0$.

Define feasible programs in the obvious manner, following Section 3, and similarly define the concepts of survival program, growth program, and pure accumulation program. As in Section 3, the last exhibits capital stocks $\langle \hat{K}_t \rangle_0^\infty$ given by $\hat{K}_0 = \underaccent{\tilde}{K}$, and $\hat{K}_t + H_t(\hat{K}_t, p_t)$, $t \geqslant 0$.

First, I completely characterize those economies capable of survival.

THEOREM 4.1. *An economy $\langle \hat{G}, p, \bar{L}, \underaccent{\tilde}{K} \rangle$ is capable of survival if and only if*

$$\inf_{t>0} \frac{\hat{K}_t}{t+1} > 0. \tag{28}$$

Compare this result with its counterpart in Section 3, stated for the case where $G_t(\cdot, \cdot)$ is linear homogeneous. There, an economy is capable of survival in the reversibility model if and only if

[16] Again, I omit a discussion of the irreversibility model. For the relevant results, see Mitra and Ray [1982].

$$\sum_{t=0}^{\infty} \frac{1}{\hat{K}_t} < \infty. \tag{29}$$

'Linear' pure accumulation in the decreasing returns to scale framework of Section 4 (the decreasing returns being exhibited by the 'reduced' functions $\langle G_t \rangle_0^\infty$) guarantees survival, but does not suffice in the constant returns to scale model of Section 3. This is because in the former case, there are 'proportional gains' to be had by a lowering of scale, and this essentially compensates for the fact that $\sum_{t=0}^{T} \frac{1}{\hat{K}_t}$ may diverge (as $T \to \infty$) by creating enough 'surplus' to maintain survival. In the constant returns case, such 'compensation' is not possible.

Finally, I provide sufficient conditions for the existence of a growth program.

THEOREM 4.2. *An economy* $\langle \hat{G}, p, \bar{L}, \underline{K} \rangle$ *is capable of growth if*

$$\sum_{t=0}^{\infty} \frac{1}{\hat{K}_t} < \infty. \tag{30}$$

Observe, now, that survival and growth are no longer equivalent problems. If pure accumulation stocks grow linearly (equation (28)), a constant amount of this growth can be set aside for consumption, permitting survival. However, if a growing amount is set aside, the capital stocks do not necessarily have enough room to expand, given diminishing returns to scale. To permit growth, therefore, pure accumulation stocks must be capable of growth which is somewhat greater than linear (equation (30)).

4.2. *Comments on a Surplus Labor Model.* Here I very briefly comment on the method of extending these results to a surplus labor model. In such a framework, an unlimited supply of labor at an exogenously given wage rate is postulated. Produced output now has four components: (1) a wage bill, to feed the employed labor at the institutional wage; (2) 'luxury' consumption, over and above wage payments; (3) investment; and (4) exports.

The second of these components may be usefully interpreted as the standard of living of the economy (over and above the consumption of 'wage goods' by the employed population).[17]

A surplus-labor model of the sort described here raises two broad issues. The first pertains to an analysis of feasible and efficient consumption paths; i.e., allocations describing attainable standards of living, with employment generation being granted a secondary place. The second deals with an analysis of feasible and efficient employment paths, with consumption surpluses (standards of living over and above wage rates) being of secondary importance.[18]

The first of these issues can be treated using the same techniques as those

[17] See also the remarks in Gangopadhyay [1982].

[18] The issues are, of course, polar. A general analysis would place positive weights on both a higher standard of living and generation of employment.

developed for the model of Section 3. Results analogous to all the theorems and examples of Section 3 hold, in this case.

Analysis of the second issues requires a different approach (see Ray [1983a]). While the results here are somewhat restrictive, it is possible to provide a complete characterization of all programs that are efficient in employment-generation. Such programs are defined by the property that no other feasible program exists, affording at least as much employment at every date, and strictly more at some date.

5. CONCLUSION

This paper has dealt with some theoretical issues, raised by the problems of an open economy importing an essential resource at progressively higher prices. The general approach is the following. A plan is proposed, e.g., consumption growing at some constant proportional rate. For any such plan, a criterion has been provided, involving the plan itself, the resource price path, and the technological conditions, which will permit the 'planner' to deduce whether this plan is feasible. Particular emphasis has been placed on the problems of survival (does there exist a plan with consumption bounded away from zero?) and of growth (does there exist a plan with consumption growing to infinity?). This 'feasibility' approach is openly nonutilitarian, but is clearly not devoid of any ethic. In particular, in an economy with surplus labor, plans involving consumption generation, and those involving employment generation may be tested for feasibility and efficiency within this framework. Choice among these plans requires, of course, a sharper ethical criterion.

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APPENDIX

PROOF OF LEMMA 3.1. Consider the problem (for fixed $t \geqslant 0$): $\max_{R \geqslant 0} [G_t(K, R) - pR]$ where $K \geqslant 0$ and $p > 0$ are given. Clearly, by setting $R = 0$, $G_t(K, R) - pR = 0$, so that the existence of a maximum is established by demonstrating some $\bar{R} \geqslant 0$ such that $G_t(K, R) - pR < 0$ for all $R > \bar{R}$ (the maximum would then exist by an application of Weierstrass' Theorem). Suppose, on the contrary, that there exists $\langle R_n \rangle_0^\infty$ with $0 < R_n \to \infty$ as $n \to \infty$, such that $G_t(K, R_n) - pR_n \geqslant 0$ for all $n \geqslant 0$. Rearranging, $\dfrac{G_t(K, R_n)}{R_n} \geqslant p$ for all $n \geqslant 0$, and using (G.3), and passing to the limit, as $n \to \infty$, $G_t(0, 1) \geqslant p > 0$ which contradicts (G.1). Hence, $H_t(K, p)$ is welldefined for $K \geqslant 0$, $p > 0$.

To show that H_t is positive on $\boldsymbol{R}^2_{++}$, it suffices to show that for all $(K, p) \gg 0$, there exists $R \geqslant 0$ such that $G_t(K, R) - pR > 0$. Suppose not; then there exists $\langle R_n \rangle_0^\infty$ with $0 < R_n \to 0$ as $n \to \infty$, and $G_t(K, R_n) - pR_n \leqslant 0$ for all $n \geqslant 0$. Rearranging, $\frac{G_t(K, R_n)}{R_n} \leqslant p$. Using (G.3) and passing to the limit as $n \to \infty$, one obtains $\limsup_{z \to \infty} G_t(z, 1) \leqslant p$, contradicting (G.2). Q.E.D.

PROOF OF LEMMA 3.2. For all $\lambda > 0$, $H_t(\lambda K, p) = \max_{R>0} [G_t(\lambda K, R) - pR] = \lambda \max_{R>0} \left[G_t\left(K, \frac{R}{\lambda}\right) - p\frac{R}{\lambda} \right] = \lambda \max_{R>0} [G_t(K, R) - pR] = \lambda H_t(K, p)$. Together with $H(0, p) = 0$ (using G.1), it follows that $H_t(K, p) = f_t(p)K$ for some f_t: $\boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$ (that f_t maps into $\boldsymbol{R}_{++}$ follows from Lemma 3.1).

To show that for each $t \geqslant 0$, $f_t(p)$ is decreasing, it suffices to show that $H_t(K, p)$ is decreasing in p for all $K > 0$. For $p_1 > p_2 > 0$, $H_t(K, p_1) = G_t(K, R_t(K, p_1)) - p_1 R_t(K, p_1) < G_t(K, R_t(K, p_1)) - p_2 R_t(K, p_1) \leqslant H_t(K, p_2)$ (the strict inequality following from G.1 and Lemma 3.1).

PROOF OF THEOREM 3.1. (i) (Necessity) Suppose $\langle \bar{C}_t \rangle_0^\infty$ is feasible. Then, for some feasible program with $\langle \bar{C}_t \rangle_0^\infty$ as its consumption program, using (2)–(4), $C_t \leqslant K_t - K_{t+1} + G_t(K_t, R_t) - p_t R_t \leqslant K_t - K_{t+1} + H_t(K_t, p_t) = K_t - K_{t+1} + f_t(p_t)K_t = [1 + f_t(p_t)]K_t - K_{t+1}$, $t \geqslant 0$. Multiplying both sides of this inequality by M_t and summing, one obtains, for $T \geqslant 0$,

$$\sum_{t=0}^{T} M_t \bar{C}_t \leqslant \underset{\sim}{K} - M_T K_{T+1}. \tag{A.1}$$

From (A.1), the left-hand side is monotone nondecreasing in T, and is bounded above by $\underset{\sim}{K}$, hence $\sum_{t=0}^{\infty} M_t \bar{C}_t$ exists and satisfies (9).

(Sufficiency) Define a program $\langle K, C, I, E, R \rangle$ by $K_0 = \underset{\sim}{K}$, $K_t = M_{t-1} K_0 - [\bar{C}_{t-1} + \sum_{s=1}^{t-1} \bar{C}_{s-1} \prod_{\tau=s}^{t-1} [1 + f_\tau(p_\tau)]$, $t \geqslant 1$, $C_t = \bar{C}_t$, $t \geqslant 0$, $I_t = K_{t+1} - K_t$, $t \geqslant 0$, $E_t = p_t R_t(K_t, p_t)$, $t \geqslant 0$, $R_t = R_t(K_t, p_t)$, $t \geqslant 0$. To verify its feasibility, the condition that $K_t \geqslant 0$ for all $t \geqslant 0$, and (12) must be checked. That $K_t \geqslant 0$ for all $t \geqslant 0$ follows by rearranging (9). To verify (2), note that for all $t \geqslant 1$, $G_t(K_t, R_t) - p_t R_t = H_t(K_t, p_t) = f_t(p_t)K_t = [1 + f_t(p_t)]K_t - K_t = K_{t+1} - K_t + \bar{C}_t$. A similar verification is is easy for $t = 0$.

(ii) (Necessity) Suppose, on the contrary, that $\langle \bar{C}_t \rangle_0^\infty$ is efficient, but $\sum_{t=0}^{\infty} \cdot M_t \bar{C}_t < \underset{\sim}{K}$. Let $\underset{\sim}{K} - \sum_{t=0}^{\infty} M_t \bar{C}_t = \delta > 0$. For some $t_0 \geqslant 0$, define $\varepsilon = M_{t_0}^{-1} \delta > 0$, and a sequence $\langle C'_t \rangle_0^\infty$ by $C'_t = \bar{C}_t$ for $t \neq t_0$, $C'_{t_0} = \bar{C}_{t_0} + \varepsilon > \bar{C}_{t_0}$. Also $\sum_{t=0}^{\infty} M_t C'_t = \sum_{t=0}^{\infty} M_t \bar{C}_t + \delta = \underset{\sim}{K}$. Therefore, $\langle C'_t \rangle_0^\infty$ is feasible, contradicting the efficiency of $\langle \bar{C}_t \rangle_0^\infty$. Hence $\sum_{t=0}^{\infty} M_t \bar{C}_t = \underset{\sim}{K}$.

(Sufficiency) Suppose, on the contrary, that $\sum_{t=0}^{\infty} M_t \bar{C}_t = \underset{\sim}{K}$, but that $\langle \bar{C}_t \rangle_0^\infty$ is inefficient. Then there exists feasible $\langle C'_t \rangle_0^\infty$ with $C'_t \geqslant \bar{C}_t$ for all $t \geqslant 0$, $C'_s > \bar{C}_s$ for some $s \geqslant 0$. But then $\sum_{t=0}^{\infty} M_t C'_t > \underset{\sim}{K}$, contradicting the feasibility of $\langle C'_t \rangle_0^\infty$. Hence $\langle C_t \rangle_0^\infty$ is efficient. Q.E.D.

PROOF OF THEOREM 3.2. The equivalence of (14)–(16) is immediate from the definitions of $\langle Q_t \rangle_0^\infty$ and construction of $\langle \hat{K}_t \rangle_0^\infty$.

(Necessity) Suppose that $\langle G, p, K \rangle$ is capable of survival. Then there exists a survival program $\langle \bar{C}_t \rangle$ and $\theta > 0$ with $\bar{C}_t \geqslant \theta > 0$ for all $t \geqslant 0$. Using (9), for each $T \geqslant 0$,

$$\sum_{t=0}^{T} M_t = \frac{1}{\theta} \sum_{t=0}^{T} M_t \theta \leqslant \frac{1}{\theta} \sum_{t=0}^{T} M_t \bar{C}_t \leqslant \frac{\underset{\sim}{K}}{\theta}.$$

Since $M_t \geqslant 0$ for all $t \geqslant 0$, and $\sum_{t=0}^{T} M_t$ is bounded above, (14) is satisfied.

(Sufficiency) If (14) is met, there exists $\eta > 0$ such that $\sum_{t=0}^{T} M_t \eta \geqslant K$. Thus there exists a feasible program with consumption equal to η for all $t \geqslant 0$, i.e., a survival program exists. Q.E.D.

PROOF OF THEOREM 3.3. Obvious from Theorem 3.1.

PROOF OF THEOREM 3.4. It suffices to prove that $\sum_{t=0}^{\infty} f(p_t) < \infty$. For then, $\prod_{t=0}^{\infty} [1 + f(p_t)] < \infty$, and hence $\sum_{t=0}^{\infty} M_t = \infty$. The convergence of $\sum_{t=0}^{\infty} f(p_t)$ will be established using the ratio test. First, observe that $f(p_t)$ may be given the explicit form

$$f(p_t) = g(r(p_t)) - p_t r(p_t)$$

where $r \equiv g'^{-1}$ (recall (7)). By (G.4), $f(p_t)$ is differentiable, and it is easily seen that $f'(p_t) = -r(p_t)$. Now, using the mean value theorem,

$$f(p_t) - f(p_{t+1}) = (p_t - p_{t+1}) f'(p)$$

where $p_{t+1} > p > p_t$, and so,

$$f(p_t) - f(p_{t+1}) = -p_t \rho f'(p).$$

This yields $\dfrac{f(p_t)}{f(p_{t+1})} - 1 = -\dfrac{p_{t+1} \rho f'(p)}{(1+\rho) f(p_{t+1})} \geqslant -\dfrac{p_{t+1} \rho f'(p)}{(1+\rho) f(p)} \geqslant -\dfrac{\rho}{1+\rho} \dfrac{f'(p) p}{f(p)}$

$$= \frac{\rho}{1+\rho} \frac{r(p) g'(r(p))}{g(r(p)) - g'(r(p)) r(p)} = \frac{\rho}{1+\rho} \cdot \frac{1}{\dfrac{a(r(p))}{g'(r(p)) r(p)} - 1}.$$

Now observe that $r(p) \to 0$ as $p \to \infty$ (this is easily checked), and that $p \to \infty$ as $p_t \to \infty$. Applying (G.5),

$$\liminf_{t \to \infty} \frac{f(p_t)}{f(p_{t+1})} > 1$$

which suffices to establish that $\sum_{t=0}^{\infty} f(p_t) < \infty$. Q.E.D.

PROOF OF THEOREM 3.5. (Necessity). Trivial.

(Sufficiency). Suppose that $\langle G, p, \underset{\sim}{K} \rangle$ is capable of survival. Then, by

Theorem 3.2, $\sum_{t=0}^{\infty} M_t < \infty$, and so there exists an increasing, positive sequence $\langle \theta_t \rangle_0^\infty$ with $\lim_{t\to\infty} \theta_t = \infty$ such that $\sum_{t=0}^{\infty} M_t\theta_t = B < \infty$ (see for example, Knopp, [1956], p. 127, Theorem 4). Using Theorem 3.1, it is easy to check that a growth program with $C_t = \frac{\theta_t \tilde{K}}{B}$, $t \geqslant 0$, exists.

Proof of Theorem 3.6. Follows from Theorem 3.1, using the same line of reasoning as in the proof of Theorem 3.2.

Proof of Theorem 3.7. This is an immediate consequence of the ratio test applied to the left hand side of (22), recalling that $M_t = [\prod_{s=0}^{t} [1 + f_s(p_s)]]^{-1}$ for all $t \geqslant 0$. Q.E.D.

Proof of Theorem 3.8. First I claim that for all $t \geqslant 0$, $f_t(p) = f\left(\frac{p}{(1+\beta)^t}\right)$, where $f: \boldsymbol{R}_{++}^2 \to \boldsymbol{R}_{++}$ is decreasing. Let $H(K, p) \equiv \max_{z>0} \{G(K, z) - pz\}$, then, by Lemma 3.2, $H(K, p) = f(p)K$ for some $f: \boldsymbol{R}_{++}^2 \to \boldsymbol{R}_{++}$, decreasing. Now, for each $t \geqslant 0$, $K > 0$, $f_t(p)K = H_t(K, p) = \max_{R>0} \{G_t(K, R) - pR\} = \max_{R>0} \{G(K, (1+\beta)^t R) - pR\}$ $= \max_{R=0} \{G(K, (1+\beta)^t R) - \frac{p}{(1+\beta)^t}(1+\beta)^t R\} = \max_{z>0} \left\{G(K, z) - \frac{p}{(1+\beta)^t} z\right\} = H\left(K, \frac{p}{(1+\beta)^t}\right) = f\left(\frac{p}{(1+\beta)^t}\right)$. K, establishing the claim.

Now, $f(z) \to 0$ as $z \to \infty$. To show this, it suffices to verify that for each $K > 0$ and $0 < z_n \to \infty$, and any sequence of maximizers $R(K, z_n)$, $G(K, R(K, z_n)) - z_n R(K, z_n) \to 0$ as $n \to \infty$. Suppose, on the contrary, that there exists $K > 0$, $0 < z_n \uparrow \infty$, a corresponding sequence of maximizers $R(K, z_n)$, and $\varepsilon > 0$ such that

$$\text{(A.2)} \qquad G(K, R(K, z_n)) - z_n R(K, z_n) \geqslant \varepsilon \qquad \text{for all} \quad n.$$

Hence $G(K, R(K, z_n)) \geqslant \varepsilon$ for all n, and so (since G is continuous, increasing, with $G(K, 0) = 0$), there exists $\delta > 0$ with $R(K, z_n) \geqslant \delta$ for all n. Pick $z > 0$ and N such that $z_n > z$ for all $n \geqslant N$. Using (A.2), $G(K, R(K, z)) - zR(K, z) \geqslant G(K, R(K, z_n)) - zR(K, z_n) \geqslant (z_n - z) R(K, z_n) + \varepsilon \geqslant (z_n - z)\delta \to \infty$ as $n \to \infty$, and this contradicts Lemma 3.1.

Also, $f(z) \to \infty$ as $z \to 0$. To show this, it suffices to verify that for each $K > 0$ and $z_n \downarrow 0$, and any sequence of maximizers $R(K, z_n)$, $G(K, R(K, z_n)) - z_n R(K, z_n) \to \infty$ as $n \to \infty$. Suppose, on the contrary, that there exists $K > 0$, $z_n \downarrow 0$, a corresponding sequence of maximizers $R(K, z_n)$ and $B < \infty$ such that

$$\text{(A.7)} \qquad G(K, R(K, z_n)) - z_n R(K, z_n) \leqslant B \qquad \text{for all} \quad n.$$

Using (G.2), for some $\varepsilon > 0$, pick $R \geqslant 0$ such that $G(K, R) > B + \varepsilon$, and n such that $z_n R < \varepsilon$. Then, for that n, $G(K, R(K, z_n)) - z_n R(K, z_n) \geqslant G(K, R) - z_n R \geqslant B$, contradicting (A.3).

Now consider (i) of the theorem. If $\beta > \rho$, then $\frac{p_t}{(1+\beta)^t} \to 0$ as $t \to \infty$, and so by the preceding argument $\lim_{t\to\infty} f_t(p_t) = \infty$. Hence the condition of Theorem 3.7 is satisfied, so that the economy is capable of exponential growth at any rate $g > 0$.

Consider (ii). If $\beta < \rho$, then $\frac{p_t}{(1+\beta)^t} \to \infty$ as $t \to \infty$, and by the preceding argument, $\lim_{t\to\infty} f_t(p_t) = 0$. It is then easily verified that $\sum_{t=0}^{\infty} M_t(1+g)^t = \infty$ for any $g > 0$, and using Theorem 3.6, this proves (ii).

In part (iii), if $\beta = \rho$, then $\frac{p_t}{(1+\beta)^t} = p^* > 0$ for all $t \geqslant 0$. Define $g^* = f(p^*)$, then $f_t(p_t) = g^*$ for all $t \geqslant 0$. It is easily checked that $\sum_{t=0}^{\infty} M_t(1+g)^t < \infty$ for all $g < g^*$, while $\sum_{t=0}^{\infty} M_t(1+g)^t = \infty$ for all $g \geqslant g^*$. This establishes (iii). Q. E. D.

PROOF OF THEOREMS 4.1 and 4.2. See Mitra and Ray (1982).

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