TOO GOOD TO BE TRUE?

Retention Rules for Noisy Agents

Online Appendix [Not For Publication]

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November 2019

ABSTRACT. In this not-for-publication Online Appendix we fill some gaps left in the proof of our main result, Proposition 2, and provide omitted proofs of Propositions 4, 5, 6, 7, and 11. We also provide numerical examples of bounded replacement equilibria and monotone equilibria in the costly noise extension (Section 6.6), together with some missing details, such as the shape of the optimal choice of noise for a generic type, depicted in Figure 3 of the paper.

All numbered references for figures, equations, lemmas, etc. refer to the main text. References that start with “a” refer to corresponding objects in this Online Appendix.

1. Missing Details in the Proof of Proposition 2

First, we rewrite conditions (11) and (12) in terms of the parameter \( \beta \), as this is easier to work with. Define

\[
\alpha := \frac{\theta_y - \theta_b}{2\sigma} > 0,
\]

and then let

\[
\beta_l := \frac{1}{\alpha + \sqrt{1 + \alpha^2}} \exp \left[ -\frac{\alpha}{\alpha + \sqrt{1 + \alpha^2}} \right] < 1,
\]

and

\[
\beta_h := \exp \left[ 2\alpha^2 \right] > 1.
\]

Lemma A.1. At \( \beta = \beta_l \) (resp. \( \beta = \beta_h \)), condition (11) (resp. (12)) holds with equality. Furthermore, (12) is equivalent to \( \beta < \beta_h \), and (11) is equivalent to \( \beta > \beta_l \).

Proof: The only non-immediate assertion of this lemma is the very last: that (11) is equivalent to \( \beta > \beta_l \). To this end, multiplying both sides of (10) by \( \alpha(\beta) \) and defining

\[
y(\beta) := \frac{\alpha(\beta)}{\alpha(\beta) + \sqrt{1 + \alpha(\beta)^2}} \in [0, 1),
\]

we have that (10) is equivalent to

\[
\alpha(\beta)\beta = y(\beta) \exp \{-y(\beta)\}.
\]

for all \( \beta \in (0, 1) \). We claim that \( \alpha(\beta) \) is strictly decreasing in \( \beta \). If the assertion is false, then there is \( \beta \) such that \( \alpha(\beta) \) is locally nondecreasing in \( \beta \). But that means that \( \alpha(\beta)\beta \) is strictly
locally increasing at the very same $\beta$. Because (a.5) holds throughout and $y \exp \{-y\}$ is strictly increasing in $y$ when $y \in [0, 1]$,
\footnote{Note that $dy \exp(-y)/dy = \exp(-y)(1 - y) > 0$ for $y \in [0, 1)$.} it follows that $y(\beta)$ is also locally strictly increasing at that $\beta$. But from (a.4), it is easy to see that $d\alpha/dy < 0$. These last two observations contradict our presumption that $\alpha(\beta)$ is locally nondecreasing in $\beta$. The last assertion of the lemma follows immediately.

For the sake of convenience, we reproduce here the expressions for the principal's thresholds $x_-(\sigma)$ and $x_+(\sigma)$, when type-$b$ plays $\sigma_b = \sigma > \overline{\sigma}$ and type-$g$ plays $\sigma_g = \underline{\sigma}$:

$$ x_-(\sigma) := \frac{\sigma^2 \theta_g - \sigma^2 \theta_b - \sigma \sigma R(\sigma)}{\sigma^2 - \overline{\sigma}^2} \quad \text{and} \quad x_+(\sigma) := \frac{\sigma^2 \theta_g - \sigma^2 \theta_b + \sigma \sigma R(\sigma)}{\sigma^2 - \overline{\sigma}^2}, $$

where

$$ R(\sigma) := + \sqrt{\left(\theta_g - \theta_b\right)^2 + \left(\sigma^2 - \overline{\sigma}^2\right) 2 \ln \left(\frac{\beta}{\overline{\sigma}}\right)}. $$

**Proof of Lemma 5.** Consider the set of pairs $(\sigma, \beta)$ such that $R = 0$. These pairs satisfy

$$ \beta = \frac{\sigma}{\overline{\sigma}} \exp \left[ - \frac{\Delta^2}{2 \left(\sigma^2 - \overline{\sigma}^2\right)} \right]. $$

Let us interpret (a.8) as $\beta$ being a function of $\sigma$, depicted in Figure A.1. Any pair $(\sigma, \beta)$ strictly below the $R = 0$ locus (the curve in the diagram) implies that the argument inside the square root in (a.7) is strictly negative, and therefore the functions $x_-(\sigma)$ and $x_+(\sigma)$ are not well-defined for such a pair: there are no real roots to $\beta \frac{1}{\sigma} \phi \left( \frac{x - \theta_g}{\sigma} \right) = \frac{1}{\sigma} \phi \left( \frac{x - \theta_b}{\sigma} \right)$. On the other hand, when the pair $(\sigma, \beta)$ is strictly above the locus, $R > 0$ and, therefore, two distinct real roots exist.
Now let us analyze the shape of the \( R = 0 \) locus. We have that \( \beta \to 0 \) as \( \sigma \to \sigma^- \) and as \( \sigma \to \infty \). By computing the derivative with respect to \( \sigma \), we find that \( \beta \) in (a.8) strictly increases with \( \sigma \) if and only if \(-\sigma^2 + \sigma \Delta + \sigma^2 > 0\). The two roots to this quadratic polynomial are 
\[
\sigma^* := \sigma \left( \alpha + \sqrt{\alpha^2 + 1} \right),
\]
where \( \alpha \) is defined in (a.1). Since the first one is negative, we have that \( \beta \) is strictly increasing in \( \sigma \) for \( \sigma \in [\sigma, \sigma^*] \), and it is strictly decreasing for \( \sigma > \sigma^* \). At \( \sigma = \sigma^* \) the derivative is zero, and therefore a global maximum is attained. By evaluating (a.8) at \( \sigma = \sigma^* \) we find that this maximum value is equal to \( l \), as defined in (a.2).

This means that, if \( \beta > \beta_1 \) (i.e. if (11) holds, as per Lemma A.1), then \( x_- (\sigma) \) and \( x_+ (\sigma) \) are well-defined for any value of \( \sigma \) and, moreover, these two real roots are always distinct. The converse is clearly true as well: if the roots are not well-defined for some \( \sigma \) or if they are not distinct for all \( \sigma \), this means that the pair \( (\beta, \sigma) \) is at or below the \( R = 0 \) locus, and therefore \( \beta \leq \beta_1 \), so (11) does not hold.

**Proof of Lemma 7.** (i) Inspection of (a.6) immediately reveals that \( \lim_{\sigma \to \sigma^+} x_+ (\sigma) = \infty \). For the corresponding limit of \( x_- (\sigma) \), apply L'Hôpital’s rule to see that
\[
\lim_{\sigma \to \sigma^+} x_- (\sigma) = \lim_{\sigma \to \sigma^+} \frac{2\sigma \theta_g - \sigma R (\sigma) - \sigma \frac{2\sigma \ln (\beta \sigma^{\frac{\sigma}{2}} - (\sigma^2 - \sigma^2)^{\frac{1}{2}})}{R(\sigma)}}{2\sigma} \nonumber
\]
\[
= \frac{\theta_g + \theta_b}{2} - \frac{\sigma^2}{\theta_g - \theta_b} \ln (\beta) \nonumber
\]
\[
= x^* (\sigma). \nonumber
\]

(ii) Notice that
\[
\lim_{\sigma \to \infty} \left( \frac{R (\sigma)}{\sigma} \right)^2 = \lim_{\sigma \to \infty} \frac{(\theta_g - \theta_b)^2}{\sigma^2} + \left(1 - \frac{\sigma^2}{\sigma^2}\right) 2 \ln \left(\frac{\beta \sigma}{\sigma}\right) \nonumber
\]
Then, since \( x_- (\sigma) \) and \( x_+ (\sigma) \) can be written, respectively, as
\[
x_- (\sigma) = \frac{\theta_g - \frac{\sigma^2}{\sigma^2} \theta_b - \frac{\sigma^2}{\sigma^2} R(\sigma)}{1 - \frac{\sigma^2}{\sigma^2}} \nonumber \text{ and } \nonumber
\]
\[
x_+ (\sigma) = \frac{\theta_g - \frac{\sigma^2}{\sigma^2} \theta_b + \frac{\sigma^2}{\sigma^2} R(\sigma)}{1 - \frac{\sigma^2}{\sigma^2}}, \nonumber
\]
it is clear that \( x_- (\sigma) \to -\infty \) and \( x_+ (\sigma) \to \infty \) as \( \sigma \to \infty \).

(iii) Suppose that (12) fails. Using (a.6), we see that
\[
x_- (\sigma, \beta) - \theta_b = \frac{\sigma^2 (\theta_g - \theta_b) - \sigma \sigma R (\sigma)}{\sigma^2 - \sigma^2}. \nonumber
\]
So the claim is established if the right-hand side in (a.10) is non-positive. But that will be true if
\[ \sigma^4(\theta_0 - \theta_b)^2 \leq \sigma^2 \sigma^2 R(\sigma)^2, \]
or equivalently, using (a.7), if
\[ \sigma^2(\theta_0 - \theta_b)^2 \leq \sigma^2 (\theta_0 - \theta_b)^2 + 2 \sigma^2 (\sigma^2 - \sigma^2) \ln \left( \frac{\beta_0}{\sigma} \right). \]
Rearranging terms, this is equivalent to
\[ (\theta_0 - \theta_b)^2 \leq 2 \sigma^2 \ln \left( \frac{\beta_0}{\sigma} \right). \]
But this inequality is implied by the failure of (12), because \( \sigma \geq \sigma \).

Proof of Lemma 8. Here we show that the derivative of \( \Psi(\sigma) \) at any fixed-point is negative. This, together with the end-point conditions verified in the proof of the Lemma in the main text, determines the uniqueness of such point.

First, consider the derivative of the left-hand side of (42) with respect to \( \sigma \); this can be expressed as
\[
\phi \left( \frac{x_+ (\sigma) - \theta_b}{\Psi (\sigma)} \right) [x_+ (\sigma) - \theta_b] \left( \frac{x_+ (\sigma) - \theta_b}{\Psi (\sigma)} - \frac{x_+ (\sigma) - \theta_b}{\Psi (\sigma)} \right) \Psi' (\sigma) - \Psi (\sigma) - (x_+ (\sigma) - \theta_b) \Psi' (\sigma),
\]
where the manipulations used to obtain this expression use the fact that \( \phi \) is the normal density.

For the right-hand side we obtain the same expression but with \( x_- (\sigma) \) instead of \( x_+ (\sigma) \). Now, the first two terms on each side will cancel each other, because \( \Psi(\sigma) \) satisfies (42). Rearranging terms, we obtain
\[
(a.11) \quad \Psi' (\sigma) = \Psi (\sigma) \phi \left( \frac{x_+ (\sigma) - \theta_b}{\Psi (\sigma)} \right) \left( 1 - \frac{(x_+ (\sigma) - \theta_b)^2}{\Psi (\sigma)^2} \right) + \frac{x_+ (\sigma) - \theta_b}{\Psi (\sigma)} \left( \frac{(x_+ (\sigma) - \theta_b)^2}{\Psi (\sigma)^2} - 1 \right),
\]
By differentiating \( \frac{1}{\sigma} \phi \left( \frac{x_+ (\sigma) - \theta_b}{\sigma} \right) \) with respect to \( \sigma \) we find that
\[
(a.12) \quad x_+ (\sigma) = \frac{\sigma}{R (\sigma)} \left[ 1 - \left( \frac{x_+ (\sigma) - \theta_b}{\sigma} \right)^2 \right] \quad \text{and} \quad x_- (\sigma) = \frac{\sigma}{R (\sigma)} \left[ \left( \frac{x_- (\sigma) - \theta_b}{\sigma} \right)^2 - 1 \right],
\]
where \( R(\sigma) \) is defined in (a.7). Substituting (a.12) into (a.11) and evaluating the resulting derivative at \( \sigma = \Psi (\sigma) \), we see that
\[
\Psi' (\sigma) = -\sigma^2 \left[ \phi \left( \frac{x_+ (\sigma) - \theta_b}{\sigma} \right) \left( 1 - \frac{(x_+ (\sigma) - \theta_b)^2}{\sigma^2} \right) + \frac{1}{\sigma} \frac{(x_+ (\sigma) - \theta_b)^2}{\sigma} \left( \frac{(x_+ (\sigma) - \theta_b)^2}{\sigma} - 1 \right)^2 \right] < 0.
\]
Before proving Lemma 10, we first make the following observation.
Lemma A.2. Let $\sigma^*$ be defined as in a.9. Along the $R = 0$ locus,

(i) $x_-(\sigma) = x_+(\sigma) > \theta_b + \sigma$ if, and only if, $\sigma \in (\bar{\sigma}, \sigma^*)$;

(ii) $x_-(\sigma) = x_+(\sigma) < \theta_b + \sigma$ if, and only if, $\sigma > \sigma^*$;

(iii) $x_-(\sigma) = x_+(\sigma) = \theta_b + \sigma$ if, and only if, $\sigma = \sigma^*$.

Proof. At any point $(\sigma, \beta)$ along the $R = 0$, locus characterized by equation (a.8), we have that $x_-(\sigma) = x_+(\sigma) = \frac{\sigma^2 \theta_g - \sigma^2 \theta_b}{\sigma^2 - \sigma^2}$ (look at (a.6)). From this, we obtain that, along the locus, $x_-(\sigma) = x_+(\sigma) > \theta_b + \sigma$ if, and only if, $-\sigma^2 + \sigma(\theta_g - \theta_b) + \sigma^2 > 0$. One of the two roots is negative. The other root is $\sigma^*$, as defined in (a.9). Then, $x_-(\sigma) = x_+(\sigma) > \theta_b + \sigma$ if, and only if, $\sigma \in (\bar{\sigma}, \sigma^*)$. Similarly, along the locus we have $x_-(\sigma) = x_+(\sigma) < \theta_b + \sigma$ if, and only if, $\sigma > \sigma^*$. Finally, at $(\sigma, \beta) = (\sigma^*, \beta_t), x_-(\sigma^*) = x_+(\sigma^*) = \theta_b + \sigma^*$.

Proof of Lemma 10. The proof of this Lemma consists in showing that, if (11) fails, for any $\sigma$ such that the roots $x_-(\sigma)$ and $x_+(\sigma)$ of a bounded retention equilibrium are well-defined and distinct, we have that either $\theta_b + \sigma < x_-(\sigma) < x_+(\sigma)$ or $\theta_b + \sigma > x_+(\sigma) > x_-(\sigma)$. But this is inconsistent with a bounded retention equilibrium, because by Lemma 3 (iii) type $b$ responds to the retention interval by choosing $\sigma_b$ that satisfies $x_-(\sigma) < \theta_b + \sigma_b < x_-(\sigma)$, so a fixed-point is impossible.

But the Lemma asserts that any class of nontrivial equilibria is impossible, so we rule out monotone equilibria first.

Recall Lemma A.1: the failure of 11 is equivalent to $\beta < \beta_t$. Furthermore, $\beta_t < \beta_h$, so $\beta < \beta_h$. Referring once again to Lemma A.1, this means we are in the realm of condition (12), so $x^*(\sigma) > \theta_b$, indicating that a monotone equilibrium is not possible.

Now let’s turn our attention to bounded retention. If $\beta \leq \beta_t$, Figure A.1 indicates that we have at most two values of $\sigma$ such that $R = 0$. Let us denote these values by $\sigma^*_t$ and $\sigma^*_h$, with $\sigma^*_t \leq \sigma^* \leq \sigma^*_h$. Then, the only way we can have a bounded retention equilibrium is if either $\sigma < \sigma^*_t$ or $\sigma > \sigma^*_h$.

We treat the cases $\beta < \beta_t$ and $\beta = \beta_t$ separately.

In the first case, $x_-(\sigma^*_t) > \theta_b + \sigma^*_t$ as asserted by Lemma A.2(i). This means that $x_-(\sigma)$ is increasing for $\sigma < \sigma^*_t$ and close enough to $\sigma^*_t$, as we can see from (a.12). The key step is to show that $x_-(\sigma) > \theta_b + \sigma$ for all $\sigma \in (\bar{\sigma}, \sigma^*_t)$. If for some $\sigma$ in this interval we have that $x_-(\sigma) \leq \theta_b + \sigma$, then it must that $x_-(\sigma)$ crosses $\theta_b + \sigma$ from below at some point, because we know that $x_-(\sigma^*_t) > \theta_b + \sigma^*_t$ and, since $R > 0$, $x_-(\sigma)$ is well-defined and therefore continuous over the entire interval $(\bar{\sigma}, \sigma^*_t)$. Crossing $\theta_b + \sigma$ from below requires $\theta_b + \sigma = 1$, but an inspection of (a.12) reveals that $x_+(\sigma) = 0$ at the intersection point. So this cannot happen and therefore $\theta_b + \sigma < x_-(\sigma) < x_+(\sigma)$ for all $\sigma \in (\bar{\sigma}, \sigma^*_t)$.

The equalities are not considered because being on the locus means $x_-(\sigma) = x_+(\sigma)$: It’s a trivial equilibrium.
Now consider $\sigma > \sigma_h^\beta$. At $\sigma^\beta_h$ we have $x_+(\sigma^\beta_h) < \theta_b + \sigma^\beta_h$ by Lemma A.2(ii). Looking at (a.12), this implies that $x_+(\sigma)$ is increasing for $\sigma$ close enough to $\sigma^\beta_h$. What we want to show now is that $x_+(\sigma) < \theta_b + \sigma$ for all $\sigma > \sigma^\beta_h$. If for some $\sigma$ in this interval we have that $x_+(\sigma) \geq \theta_b + \sigma$, then it must that $x_+(\sigma)$ crosses $\theta_b + \sigma$ from below at some point, because $x_+(\sigma^\beta_h) < \theta_b + \sigma^\beta_h$ and, since $R > 0$, $x_+(\sigma)$ is well-defined and continuous at any $\sigma > \sigma^\beta_h$. Crossing $\theta_b + \sigma$ from below requires $x_+(\sigma) \geq 1$, but again referring to (a.12) we can see that $x_+(\sigma) = 0$ at such a point. So $\theta_b + \sigma > x_+(\sigma) > x_-(\sigma)$ for all $\sigma > \sigma^\beta_h$.

Finally, consider the case $\beta = \beta_1$. Here, we have that $\sigma^\beta_1 = \sigma_h^\beta = \sigma^*$, where $\sigma^*$ is defined in (a.9). Looking at (a.12) we can see that $x_+(\sigma^*) = x_+(\sigma_1^*) = 0$ and, as established by Lemma A.2(iii), $x_-(\sigma^*) = x_+(\sigma^*) = \theta_b + \sigma^*$. Then, if we consider $\sigma < \sigma^*$, it is clear that for $\sigma$ close enough to $\sigma^*$ we have $x_-(\sigma) > \theta_b + \sigma$, and as we showed before this leads to the conclusion that $\theta_b + \sigma < x_-(\sigma) < x_+(\sigma)$ for all $\sigma \in (\sigma, \sigma^\beta_1)$). Similarly, for $\sigma > \sigma^*$ and close enough to $\sigma^*$, we have that $x_+(\sigma) < \theta_b + \sigma$, which leads to $\theta_b + \sigma > x_+(\sigma) > x_-(\sigma)$ for all $\sigma > \sigma^\beta_h$.

Proof of the Assertion in Footnote 11. This assertion states that the non-existence of bounded replacement is robust to allowing for a finite upper bound to the choice of noise. In what follows, then, assume that there exists an upper bound on noise, so $\sigma_k \in [\underline{\sigma}, \bar{\sigma}]$ is costless, whereas the cost of going below $\sigma$ or above $\bar{\sigma}$ is prohibitively high. Let us first state

Lemma A.3. Under bounded replacement, $x_+ < \frac{x_++x_-}{2} < \theta_b$.

Proof. When $\sigma_b \neq \sigma_g$ and $x_-$ and $x_+$ are both finite and given by (4), we have that $(x_+ + x_-)/2 = (\sigma^2_b\theta_g - \sigma^2_g\theta_b)/(\sigma^2_b - \sigma^2_g)$. So if $\sigma_b < \sigma_g$ then $x_+ < (x_+ + x_-)/2 < \theta_b$. ■

Now suppose that a bounded replacement equilibrium exists. Any type inside the replacement interval chooses $\bar{\sigma}$, trying to escape from the danger zone. That immediately tells us that $\theta_b \notin [x_+, x_-]$, otherwise $\sigma_b$ cannot be lower than $\sigma_g$. Then, since $x_+ < \theta_b$ by Lemma A.3, we need $x_- < \theta_b < \sigma_g$, and therefore both types must be in the retention zone, $X$. Any type in $X$ will choose $\sigma$ or $\bar{\sigma}$, depending on which one of the two leads to a smaller probability mass in the replacement set.\footnote{To see this, notice that for any type $k$, in this case the probability of retention converges to 1 as $\sigma_k \to 0$ and as $\sigma_k \to \infty$. The probability of retention achieves its minimum when the first-order derivative is equal to zero.} It follows that, to maintain $\sigma_b > \sigma_g$, the bad type must choose $\sigma$ and the good type $\bar{\sigma}$, so that the first-order derivatives must satisfy

$\phi\left(\frac{x_+ - \theta_g}{\bar{\sigma}}\right)\left(\frac{x_+ - \theta_g}{\bar{\sigma}^2}\right) + \phi\left(\frac{x_- - \theta_g}{\bar{\sigma}}\right)\left(\frac{x_- - \theta_g}{\bar{\sigma}^2}\right) \geq 0$

(a.13)

for the good type, and

$\phi\left(\frac{x_+ - \theta_b}{\bar{\sigma}}\right)\left(\frac{x_+ - \theta_b}{\bar{\sigma}^2}\right) + \phi\left(\frac{x_- - \theta_b}{\bar{\sigma}}\right)\left(\frac{x_- - \theta_b}{\bar{\sigma}^2}\right) \leq 0$

(a.14)
for the bad type. Combining (a.14) with the principal’s indifference condition (6), we have:

\[-\beta \frac{1}{\sigma} \phi \left( \frac{x_+ - \theta_g}{\sigma} \right) \left( \frac{x_+ - \theta_b}{\sigma} \right) + \beta \frac{1}{\sigma} \phi \left( \frac{x_- - \theta_g}{\sigma} \right) \left( \frac{x_- - \theta_b}{\sigma} \right) \leq 0.\]

But this, together with type-g’s first-order condition (a.13), yields:

\[\left[ \phi \left( \frac{x_+ - \theta_g}{\sigma} \right) - \phi \left( \frac{x_- - \theta_g}{\sigma} \right) \right] (\theta_g - \theta_b) \geq 0.\]

Because \(\theta_g > \theta_b\), this implies a contradiction since \(\phi \left( \frac{x_+ - \theta_g}{\sigma} \right) < \phi \left( \frac{x_- - \theta_g}{\sigma} \right)\) by Lemma 11. ■

2. OMITTED PROOFS FOR DYNAMICS WITH TERM LIMITS, SECTION 6.2

The following Lemma will be used to prove Proposition 4.

**Lemma A.4.** Assume \(\beta \in (\beta_1, \beta_h)\), then

(i) \(\frac{\partial x^*_+}{\partial \beta} < 0\) and \(\frac{\partial x^*_+}{\partial \beta} > 0\)

(ii) \(\lim_{\beta \to \beta_1^+} x^*_+ = \lim_{\beta \to \beta_1^+} x^*_+ = \theta_b + \sigma^*\) and \(\lim_{\beta \to \beta_1^+} \sigma^*_b = \sigma^*\), where \(\sigma^*\) is defined in (a.9).

**Proof.** (i) When \(\beta \in (\beta_1, \beta_h)\), both (11) and (12) hold by Lemma A.1, and therefore Proposition 2 tells us that there exists a unique equilibrium, which is a bounded retention equilibrium where \(\sigma_b > \sigma_g = \sigma\) and the principal retains in a bounded interval \(X = [x_-, x_+]\), with \(x_- < x_+\). The equilibrium values \((\sigma^*_b, x^*_-, x^*_+)\) are determined by

\[\beta \frac{1}{\sigma} \phi \left( \frac{x_+ - \theta_g}{\sigma} \right) = \frac{1}{\sigma_b} \phi \left( \frac{x_- - \theta_b}{\sigma_b} \right),\]

for \(x = x^*_-, x^*_+\),

and

\[\phi \left( \frac{x_- - \theta_b}{\sigma_b} \right) (x_- - \theta_b) = \phi \left( \frac{x_+ - \theta_b}{\sigma_b} \right) (x_+ - \theta_b).\]

Let us differentiate these equations with respect to \(\beta\). In the case of the first equation, we obtain

\[\frac{\sigma'_b}{\sigma_b} \left( \frac{x_+ - \theta_b}{\sigma_b} \right)^2 - 1 = \frac{R (\sigma_b)}{\sigma_b \bar{\sigma}} x' \frac{1}{\beta} \]

\[\frac{\sigma'_b}{\sigma_b} \left( 1 - \left( \frac{x_+ - \theta_b}{\sigma_b} \right)^2 \right) = \frac{R (\sigma_b)}{\sigma_b \bar{\sigma}} x' \frac{1}{\beta}.\]

In the case of the second equilibrium equation we obtain the same expression as in (a.11), where \(\Psi (\sigma)\) is now \(\sigma_b\), and the derivatives are those with respect to \(\beta\). By combining it with (a.15),
and after some heavy algebra, we obtain:

(a.16) \[
\begin{align*}
x'\_ &= -\frac{y\_ (y\_ + y\_)}{\beta} \left( \frac{\sigma^2}{\theta g - \theta b} \right) \left( \frac{\theta g - \theta b}{\sigma} \right) \left( y\_ + y\_ \right) \left( y\_ + y\_ \right) \left( y\_ - 1 \right) \\
x'\_ &= \frac{y\_ (y\_ + y\_)}{\beta} \left( \frac{\sigma^2}{\theta g - \theta b} \right) \left( \frac{\theta g - \theta b}{\sigma} \right) \left( y\_ - y\_ \right) \left( y\_ + y\_ \right) \left( y\_ - 1 \right)
\end{align*}
\]

where the notation \( x'_i \) means \( \frac{\partial x_i}{\partial \beta} \), and \( y_i := \frac{x_i - \theta b}{\sigma} \), for \( i = -, + \).

Since \( \sigma_b > \sigma \) it must be that \( x_- > \theta_b \) (see Lemma 3(ii)), so \( y_- > 0 \). Also, from Lemma 3(iii), \( y_- > 1 > y_- \). Then, from the above expressions we can see that \( x'_- < 0 \) and \( x'_+ > 0 \), which means that the interval shrinks as \( \beta \) decreases.

(ii) By Lemma 3(iii), equilibrium \( \sigma_b \) satisfies \( x_- (\sigma_b) < \theta b + \sigma_b < x_+ (\sigma_b) \). In the limit as \( \beta \to \beta^+_i \), condition (11) holds with equality, and \( x_- (\sigma^*) = x_+ (\sigma^*) = \theta b + \sigma^* \) where \( \sigma^* = \sigma \left( \alpha + \sqrt{\alpha^2 + 1} \right) \) by Lemma A.2 (iii). Then, \( \sigma^*_b \to \sigma^* \) as \( \beta \to \beta^+_i \), which completes the proof.

Proof of Proposition 4. For some (provisionally given) value of \( \beta \), obtain the baseline static model and then use Proposition 2 to generate retention probabilities \( \Pi_g \) and \( \Pi_b \). The circle is closed by the additional condition that \((\beta, \Pi_g, \Pi_b)\) must solve (19), reproduced here for convenience:

(a.17) \[
\beta = \frac{q}{1 - q} \frac{1 - p}{p} = \frac{1 + \delta \Pi_b}{1 + \delta \Pi_g}
\]

As argued in the main text, it has to be the case that \( \Pi_g \geq \Pi_b \), because the principal will choose a retention zone that retains the high type at least as often as the low type. This says that, in the dynamic model, \( \beta \leq 1 \), and therefore condition (12) trivially holds. Then, following Proposition 2 and Lemma A.1, we can separate the analysis into two cases: either (11) fails and \( \beta \leq \beta^+_i < 1 \), or (11) holds and \( \beta \in (\beta^+_1, 1] \).

In the former case, Proposition 2 tells us that in the static model, only trivial equilibria exist (see Lemma 10). Then, \( \Pi_b = \Pi_g \). But equilibrium condition (a.17) then says that \( \beta = 1 \), a contradiction.

That is, if an equilibrium exists in this dynamic version of the costless model, it must be the case that \( \beta \in (\beta^+_1, 1] \subset (\beta^+_1, \beta_b) \), so it must be a bounded retention equilibrium. We now prove its existence.

For any given \( \beta \in (\beta^+_1, 1] \), by Proposition 2, there is a unique equilibrium in the static model, and it involves bounded retention thresholds \( \{ x_- (\beta), x_+ (\beta) \} \). Given \( \{ \sigma_b (\beta), \sigma_g (\beta) \} \) in that
equilibrium (with \(\sigma_b(\beta) > \sigma_g(\beta) = \sigma\) as already established), define, for \(k = b, g\):

(a.18) \[
\Pi_k(\beta) = \int_x \pi_k(x)dx = \frac{1}{\sigma_k(\beta)} \int_{x^{+}(\beta)}^{x^{-}(\beta)} \phi \left( \frac{x - \theta_k}{\sigma_k(\beta)} \right) dx.
\]

Now, in line with (a.17), define a mapping \(\beta' = \psi(\beta)\) by

(a.19) \[
\beta' = \frac{1 + \delta \Pi_b(\beta)}{1 + \delta \Pi_g(\beta)}
\]

Because the equilibrium is unique for every \(\beta \in (\beta_l, 1]\), it is easy to see that \(\psi\) is a continuous map. Next, when \(\beta = 1\), we know from the non-triviality of the corresponding static equilibrium that \(\Pi_b(1) < \Pi_g(1)\), so that \(\beta' = \psi(1) < 1\). Finally, as \(\beta \to \beta_l^+\), the boundaries of the static equilibrium retention thresholds \(x^*_-\) and \(x^*_+\) converge to each other (see Lemma A.4(ii)), so that

\[
\lim_{\beta \to \beta_l^+} \Pi_g = \lim_{\beta \to \beta_l^+} \Pi_b = 0, \text{ and therefore}
\]

\[\beta' = \psi(\beta) = \frac{1 + \delta \Pi_b(\beta)}{1 + \delta \Pi_g(\beta)} \to 1\]

as \(\beta \to \beta_l^+\). This verifies a second end-point condition \(\lim_{\beta \to \beta_l^+} \psi(\beta) > \beta_l\). By the intermediate value theorem, there is at least one value of \(\beta\) with \(\psi(\beta) = \beta\), and this — along with the corresponding values of \(\sigma_b\) and \(\sigma_g\) — is easily seen to be an equilibrium of the dynamic game.

To complete the proof, we establish uniqueness of equilibrium. Begin by differentiating the expression in (a.18) with respect to \(\beta\), taking care to use an envelope argument for type \(b\) (his first-order condition) and the fact that \(\sigma_g(\beta) = \sigma\) for type \(g\). We obtain:

(a.20) \[
\frac{\partial \Pi_k(\beta)}{\partial \beta} = \frac{1}{\sigma_k(\beta)} \left[ \phi \left( \frac{x^{+}(\beta) - \theta_k}{\sigma_k(\beta)} \right) x^{+}_+(\beta) - \phi \left( \frac{x^{-}(\beta) - \theta_k}{\sigma_k(\beta)} \right) x^{-}_-(\beta) \right].
\]

Next, observe that

(a.21) \[
\frac{\partial}{\partial \beta} \frac{1 + \delta \Pi_b(\beta)}{1 + \delta \Pi_g(\beta)} = \frac{\frac{\partial \Pi_b(\beta)}{\partial \beta} (1 + \delta \Pi_g(\beta)) - (1 + \delta \Pi_b(\beta)) \frac{\partial \Pi_g(\beta)}{\partial \beta}}{(1 + \delta \Pi_g(\beta))^2}.
\]

Substitute (a.20) in (a.21) and use the fact that \(x^{\pm}(\beta)\) and \(x^{\pm}_+(\beta)\) solve (6) with equality to obtain (after some manipulation)

\[
\frac{\partial}{\partial \beta} \frac{1 + \delta \Pi_b(\beta)}{1 + \delta \Pi_g(\beta)} = \frac{1}{\sigma_g(\beta)} \phi \left( \frac{x^{+}(\beta) - \theta_g}{\sigma_g(\beta)} \right) x^{+}_+(\beta) - \frac{1}{\sigma_b(\beta)} \phi \left( \frac{x^{-}(\beta) - \theta_b}{\sigma_b(\beta)} \right) x^{-}_-(\beta)
\]

\[
\left( \beta - \frac{1 + \delta \Pi_b(\beta)}{1 + \delta \Pi_g(\beta)} \right).
\]

Because \(x^{+}_+(\beta) > 0\) and \(x^{-}_-(\beta) < 0\) (Lemma A.4(i)), we must conclude that

(a.22) \[
\text{Sign} \left\{ \frac{\partial}{\partial \beta} \frac{1 + \delta \Pi_b(\beta)}{1 + \delta \Pi_g(\beta)} \right\} = \text{Sign} \left\{ \beta - \frac{1 + \delta \Pi_b(\beta)}{1 + \delta \Pi_g(\beta)} \right\}.
\]

This last observation, along with the end-point condition \(\lim_{\beta \to \beta_l} \phi(\beta) > \beta_l\), eliminates two solutions to the equation

\[
\beta = \frac{1 + \delta \Pi_b(\beta)}{1 + \delta \Pi_g(\beta)},
\]

for that would require the sign equality (a.22) to be violated for some \(\beta\). ■
3. omitted details for mean-shifting effort, section 6.3

In section 6.3 of the main text we claim that $\theta_g > \theta_b$ even when these values are endogenously chosen.

We begin by eliminating the possibility that $\theta_b > \theta_g$. From the definition in (a.6) it is clear that a bounded retention regime is still associated with $\sigma_b > \sigma_g$ and it is of the form $X = [x_-, x_+]$, and a bounded replacement regime is associated with $\sigma_b < \sigma_g$, and the principal replaces inside $X^c = [x_+, x_-]$. Then, under any one of these two regimes, the first-order condition with respect to $\theta_k$ is

\begin{equation}
\left( a.23 \right)
\frac{1}{\sigma_k} \phi \left( \frac{x_- - \theta_k}{\sigma_k} \right) - \frac{1}{\sigma_k} \phi \left( \frac{x_+ - \theta_k}{\sigma_k} \right) \leq d' \left( \theta_k - \theta_b \right),
\end{equation}

with equality holding if $\theta_b > \theta_g$.

Under bounded retention, we have

$$
x_- < \frac{x_+ + x_-}{2} = \frac{\sigma_b^2 \theta_g - \sigma_g^2 \theta_b}{\sigma_b^2 - \sigma_g^2} < \theta_g < \theta_b,
$$

so that

$$
\frac{x_- - \theta_k}{\sigma_k} < \frac{x_+ - \theta_k}{\sigma_k} < \frac{\theta_g - x_-}{\sigma_k}.
$$

Because $\phi(\cdot)$ is single-peaked and symmetric around 0,

$$
\phi \left( \frac{x_- - \theta_k}{\sigma_k} \right) > \phi \left( \frac{x_+ - \theta_k}{\sigma_k} \right),
$$

But then (a.23) cannot hold with equality for any $k$, so $\theta_b > \theta_g$ is impossible if $\theta_g > \theta_b$.

Similarly, under bounded replacement, we have $\sigma_b < \sigma_g$, so that

$$
\theta_g < \theta_b < \frac{x_+ + x_-}{2} = \frac{\sigma_b^2 \theta_g - \sigma_g^2 \theta_b}{\sigma_b^2 - \sigma_g^2} < x_-.
$$

Then, once again,

$$
\frac{\theta_k - x_-}{\sigma_k} < \frac{x_+ - \theta_k}{\sigma_k} < \frac{x_- - \theta_k}{\sigma_k},
$$

and the same contradiction follows. Finally, with monotone retention, $\sigma_g = \sigma_b = \sigma$, and the retention rule is: retain iff

$$
x \leq x^*(\sigma) := \frac{\theta_g + \theta_b}{2} + \frac{\sigma^2}{\theta_b - \theta_g} \ln(\beta).
$$

The first-order derivative with respect to $\theta_k$ is then

$$
-\frac{1}{\sigma_k} \phi \left( \frac{x^* - \theta_k}{\sigma_k} \right) - d' \left( \theta_k - \theta_b \right),
$$

which is always negative, so given that $\theta_g > \theta_b$, $\theta_b > \theta_g$ can never hold.

Moreover, there cannot be an equilibrium where $\theta_g = \theta_b = \theta$. For if there were, the induced “second-stage game” with choice of noise must have exactly the same equilibrium payoffs, as well as the same marginal payoffs with respect to the common value $\theta$, not counting the effort
cost \(d\). But since \(\theta_g \neq \theta_b\), and \(d'\) is injective, it is clear that at least one of the agents is not satisfying the optimality conditions in the “first stage”, when \(\theta\) is chosen. Therefore \(\theta_g \neq \theta_b\). ■

### 4. The Non-Normal Case, Section 6.4

For any \(\sigma_b\) and \(\sigma_g\), define

\[
h(x) := \frac{f \left( \frac{x-\theta_b}{\sigma_b} \right)}{f \left( \frac{x-\theta_g}{\sigma_g} \right)},
\]

and let \(k := \beta \sigma_b / \sigma_g\). Following (3) in the main text, the retention zone is then given by

\[
X(k) := \{x|h(x) \leq k\}.
\]

**Lemma A.5.** (i) If \(\sigma_b = \sigma_g\), then \(h(x)\) is strictly decreasing in \(x\) with \(\lim_{x \to -\infty} h(x) = \infty\) and \(\lim_{x \to \infty} h(x) = 0\).

(ii) If \(\sigma_b > \sigma_g\), then \(\lim_{x \to \infty} h(x) = \lim_{x \to -\infty} h(x) = \infty\).

(iii) If \(\sigma_b < \sigma_g\), then \(\lim_{x \to \infty} h(x) = \lim_{x \to -\infty} h(x) = 0\).

**Remark.** The symmetric statements of parts (ii) and (iii), despite the fact that \(\theta_b < \theta_g\), reflect our observation in the main text that “spreads dominate means.”

**Proof.** (i) Define \(z(x) \equiv (x - \theta_b) / \sigma\) and \(a \equiv (\theta_g - \theta_b) / \sigma\), where \(\sigma = \sigma_b = \sigma_g\). Then

\[
h(x) = \frac{f(z(x))}{f(z(x) - a)}.
\]

Because \(z(x)\) is affine and increasing in \(x\), the result follows directly from strong MLRP.

(ii) There is \(\epsilon > 0\) such that for \(x\) sufficiently large, \((x - \theta_b) / \sigma_b \leq (x - [\theta_g + \epsilon]) / \sigma_g\). Because \(f'(z) \leq 0\) for all \(z > 0\) (see main text), it follows that for all \(x\) large enough,

\[
\frac{f \left( \frac{x-\theta_b}{\sigma_b} \right)}{f \left( \frac{x-\theta_g}{\sigma_g} \right)} \geq \frac{f \left( \frac{x-\theta_g+\epsilon}{\sigma_g} \right)}{f \left( \frac{x-\theta_g}{\sigma_g} \right)},
\]

and now, using strong MLRP, the right hand side of this inequality goes to infinity as \(x \to \infty\).

The case \(x \to -\infty\) follows parallel lines: switch \((\theta_b, \sigma_b)\) and \((\theta_g, \sigma_g)\) in the argument above, notice that \(f\) is increasing for \(z < 0\), and use part (i) again.

Noticing that the relative magnitudes of \(\theta_b\) and \(\theta_g\) played no role in part (ii), the same argument with appropriately switched symbols works for part (iii).

We can now complete the

**Proof of Proposition 5, Part (i).** Suppose the assertion is false. Then, by Lemma A.5 and the definition of the retention zone in (a.25), we must have \(\sigma_g > \sigma_b\), and retention for all \(x\) large in
absolute value. But then, given such a zone, the probability of retention of any type converges to 1 as $\sigma_k \to \infty$. Therefore, for any candidate pair $(\sigma_y, \sigma_b)$, any type finds a profitable deviation. ■

For the remainder of Proposition 5 as well as Proposition 6, we need additional steps.

**Lemma A.6.** For $z \geq 0$, $f(z)$ is increasing for $z \in [0, z^\ast)$, decreasing for $z > z^\ast$, and maximized at $z^\ast > 0$, the unique solution to $f'(z)/f(z) = -1/z$.

**Proof.** The derivative of $f(z)$ with respect to $z$ is $f(z) = \left[ \frac{f'(z)}{f(z)} + \frac{1}{z} \right]$. $f'(z)/f(z)$ is non-increasing by MLRP, and $1/z$ is strictly decreasing as well. We also have that $\frac{f'(z)}{f(z)} + \frac{1}{z} \to \infty$ as $z \to 0$ and that $\frac{f'(z)}{f(z)} + \frac{1}{z}$ is negative for $z$ sufficiently large. Then $z^\ast$, the unique maximizer of $f(z)$ for $z \geq 0$, satisfies $\frac{f'(z)}{f(z)} + \frac{1}{z} = 0$. ■

**Lemma A.7.** (i) If $X = [x^\ast, \infty)$ and $\theta_k > x^\ast$, the agent of type $k$ chooses $\sigma_k = \overline{\sigma}$; if $\theta_k < x^\ast$, the problem has no solution, in particular, the agent always wants to inject additional noise; if $\theta_k = x^\ast$, the agent is indifferent across all choices of $\sigma$.

(ii) Assume a retention zone of the form $[x_-, x_+]$ with $x_- < x_+$. If $x_- \leq \theta_k$ and $x_+ > \theta_k$ then $\sigma_k = \overline{\sigma}$.

(iii) Assume a retention zone of the form $[x_-, x_+]$ with $x_- < x_+$. If $x_- > \theta_k$, then for each $k$ define

$$d_k(\sigma_k) := f \left( \frac{x_- - \theta_k}{\sigma_k} \right) (x_- - \theta_k) - f \left( \frac{x_+ - \theta_k}{\sigma_k} \right) (x_+ - \theta_k) \text{ for all } \sigma_k > 0.$$

Then type $k$’s payoff derivative with respect to her choice of noise $\sigma_k$ is precisely given by $\sigma_k^2 d_k(\sigma_k)$. The function $d_k$ is continuous, initially positive then negative, with a unique root to $d_k(\sigma_k) = 0$, $\sigma_k^\ast$, satisfying

$$\sigma_k^\ast \in \left( \frac{x_- - \theta_b}{z^\ast}, \frac{x_+ - \theta_b}{z^\ast} \right),$$

where $z^\ast > 0$ is defined in Lemma A.6, and agent $k$ sets $\sigma_k = \max\{\overline{\sigma}, \sigma_k^\ast\}$.

**Proof.** (i) In the case of monotone retention, the first-order derivative with respect to $\sigma_k$ is

$$f \left( \frac{x^\ast - \theta_k}{\sigma_k} \right) \frac{x^\ast - \theta_k}{\sigma_k^2}.$$

It is always negative if $x^\ast < \theta_k$, so $\sigma_k = \overline{\sigma}$; always positive if $x^\ast > \theta_k$, so the agent always wants to increase the noise and the problem has no solution; and always equal to 0 if $x^\ast = \theta_k$, so the agent is indifferent across all choices of $\sigma$.

(ii) A type-$k$ agent wishes to maximize the probability of being in the retention zone $[x_-, x_+]$, so he chooses $\sigma_k \geq \overline{\sigma}$, to maximize

$$F \left( \frac{x_+ - \theta_k}{\sigma_k} \right) - F \left( \frac{x_- - \theta_k}{\sigma_k} \right),$$
where $F$ is the cdf of $f$. The first-order derivative of the objective function with respect to $\sigma_k$ is

$$
\frac{d_k(\sigma_k)}{\sigma_k^2} = \frac{1}{\sigma_k^2} \left[ f\left(\frac{x_+ - \theta_k}{\sigma_k}\right) (x_- - \theta_k) - f\left(\frac{x_- - \theta_k}{\sigma_k}\right) (x_+ - \theta_k) \right],
$$

where $d_k$ is defined in (a.26). If $x_- \leq \theta_k$ and $x_+ > \theta_k$, the sign is negative, and the agent optimally chooses $\sigma_k = \sigma$.

(iii). If $x_- > \theta_k$, rewrite the above expression as

$$
\frac{d_k(\sigma_k)}{\sigma_k^2} = \frac{1}{\sigma_k^2} f\left(\frac{x_+ - \theta_k}{\sigma_k}\right) (x_- - \theta_k) \left[ f\left(\frac{x_- - \theta_k}{\sigma_k}\right) - f\left(\frac{x_+ - \theta_k}{\sigma_k}\right) \right],
$$

so the sign is determined by the sign of the term inside the square brackets. By the strong MLRP, $f\left(\frac{x_- - \theta_k}{\sigma_k}\right) / f\left(\frac{x_+ - \theta_k}{\sigma_k}\right)$ is strictly decreasing in $\sigma_k$, with limit $\infty$ as $\sigma_k \to 0$ and limit $0$ as $\sigma_k \to \infty$, so there exists a unique $\sigma_k > 0$ that satisfies $d_k(\sigma_k) = 0$. Agent $k$ sets $\sigma_k = \max\{\sigma, \sigma_k^*\}$.

Since $\sigma_k^*$ satisfies $d_k(\sigma_k) = 0$, $f\left(\frac{x_- - \theta_k}{\sigma_k}\right) / f\left(\frac{x_+ - \theta_k}{\sigma_k}\right)$ is decreasing at $z \in [0, z^*)$ and decreasing at $z > z^*$, it must be that $\frac{x_- - \theta_k}{\sigma_k} < z^* < \frac{x_+ - \theta_k}{\sigma_k}$, so $\sigma_k^*$ satisfies (a.27).

**Proof of Proposition 5, Part (ii).** By Lemmas A.5 and A.7(i), monotone equilibria are only possible if $\sigma_g = \sigma_b = \sigma$. Using the definition of the retention zone in (a.25) and the strong MLRP, the principal retains in such an equilibrium if and only if $x \geq x^*(\sigma)$, where $x^*(\sigma)$ is uniquely defined by

(a.29) \[ \beta f\left(\frac{x^*(\sigma) - \theta_g}{\sigma}\right) = f\left(\frac{x^*(\sigma) - \theta_b}{\sigma}\right). \]

By Lemma A.7(i), we have $\theta_b \geq x^*(\sigma)$, and using the strong MLRP along with (a.29), we see that

(a.30) \[ \beta f\left(-\frac{\theta_g - \theta_b}{\sigma}\right) \geq f(0). \]

It follows that $\sigma \geq \sigma(\beta)$, where recall the definition of $\sigma(\beta)$ from (24) in the main text.

Conversely, if $\sigma \geq \sigma(\beta)$, then allow both types to choose $\sigma_b = \sigma_g = \sigma$; then the principal will select the monotone retention threshold $x^*(\sigma)$, where this threshold solves (a.29). Because $\sigma \geq \sigma(\beta)$, (a.30) holds, and it follows that $x^*(\sigma) \leq \theta_b$. Applying Lemma A.7(i) yet again, we must conclude that $\sigma_b = \sigma_g = \sigma$ is an optimal response by each of the types to the retention zone $[x^*(\sigma), \infty)$, and the proof is complete.

Now we move on to the existence issues raised in Proposition 6. It will be useful to define $s(x) := f'(x) / f(x)$. Using MLRP, it is easy to check that $s$ is a decreasing function. Moreover, by single-peakedness around 0, $s(x)$ is positive for $x < 0$, negative for $x > 0$, and zero at $x = 0$.  

\[ \text{In any monotone equilibrium, } \theta_g > \theta_b \geq x^*(\sigma), \text{ and so by Lemma A.7(i), } \sigma_g = \sigma. \text{ By Lemma A.5, } \sigma_b = \sigma_g. \]
Lemma A.8. For any $\sigma_b > \sigma_g$:

(i) $h(x)$ is decreasing for $x \leq \theta_g$ and increasing for $x \geq \frac{\sigma_b \theta_g - \sigma_g \theta_b}{\sigma_b - \sigma_g} > \theta_g$. In particular, $X(k)$ is an interval for all $k \geq 1$, and because $k = \beta \sigma_b / \sigma_g$, this is a fortiori true for all $\beta \geq 1$.

(ii) Under the additional assumption that $\frac{\partial \ln f(x)}{\partial x}$ is convex for all $x > 0$. $h(x)$ decreases and then increases on $\left( \theta_g, \frac{\sigma_b \theta_g - \sigma_g \theta_b}{\sigma_b - \sigma_g} \right)$, with its minimum achieved at the unique solution to

\[ \frac{1}{\sigma_b} s \left( \frac{x - \theta_b}{\sigma_b} \right) = \frac{1}{\sigma_g} s \left( \frac{x - \theta_g}{\sigma_g} \right), \]

so that $X(k)$ is an interval for all $k$ higher than the minimum value of $h$.

(iii) Combining cases (i) and (ii), a nonempty retention zone is an interval $[x_-, x_+]$, where $x_-, x_+$ are the two real roots to

\[ \beta \frac{1}{\sigma_g} f \left( \frac{x - \theta_g}{\sigma_g} \right) = \frac{1}{\sigma_b} f \left( \frac{x - \theta_b}{\sigma_b} \right), \]

and the upper root always exceeds $\theta_g$.

Proof. For notational convenience, define $z_k(x) := (x - \theta_k)/\sigma_k$ for $k = b, g$. Then differentiate $h$ in (a.24) to see that

\[ \text{Sign } h'(x) = \text{Sign } \left[ \frac{1}{\sigma_b} s(z_b(x)) - \frac{1}{\sigma_g} s(z_g(x)) \right]. \]

Figure A.2 can be used to follow the proof.

Part (i). Break the region $x \leq \theta_g$ into two parts. If $x \leq \theta_b < \theta_g$, then $0 \geq z_b(x) > z_g(x)$, so $0 \leq s(z_b(x)) < s(z_g(x))$. Therefore, $\frac{1}{\sigma_b} s(z_b(x)) < \frac{1}{\sigma_g} s(z_g(x))$, and so by (a.33), $h'(x) < 0$. 
If $x \in (\theta_b, \theta_g)$, then $z_b(x) > 0$ but $z_g(x) \leq 0$, so (a.33) implies right away that $h'(x) < 0$.

At the other extreme, if $x \geq \frac{\sigma_b \theta_g - \sigma_g \theta_b}{\sigma_b - \sigma_g} > \theta_b$, it is easy to verify that $0 < z_b(x) \leq z_g(x)$. It follows that $0 > s(z_b(x)) \geq s(z_g(x))$, so that $\frac{1}{\sigma_b} s(z_b(x)) > \frac{1}{\sigma_g} s(z_g(x))$ and therefore (a.33) implies that $h'(x) > 0$.

By Lemma A.5(ii), $\lim_{|x| \to \infty} h(x) = \infty$. Also, $h(\theta_g) < 1$, and $h\left(\frac{\sigma_b \theta_g - \sigma_g \theta_b}{\sigma_b - \sigma_g}\right) = 1$. Finally, if $x \in \left(\theta_g, \frac{\sigma_b \theta_g - \sigma_g \theta_b}{\sigma_b - \sigma_g}\right)$, $z_b(x) > z_g(x) > 0$, so by the single-peakedness of $f$ around 0, $f(z_b(x)) < f(z_g(x))$ and therefore $h(x) < 1$. In particular, $X(k)$ must be an interval for all $k \geq 1$.

(ii) Suppose that the assertion is false. Then there exist $y, w \in \left(\theta_g, \frac{\sigma_b \theta_g - \sigma_g \theta_b}{\sigma_b - \sigma_g}\right)$, with $y > w$ such that $h(y) = h(w), h'(y) \leq 0$ and $h'(w) \geq 0$. These inequalities together imply

$$
\frac{\sigma_g}{\sigma_b} s\left(\frac{y - \theta_b}{\sigma_b}\right) \leq s\left(\frac{y - \theta_g}{\sigma_g}\right) < s\left(\frac{w - \theta_g}{\sigma_g}\right) \leq \frac{\sigma_g}{\sigma_b} s\left(\frac{w - \theta_b}{\sigma_b}\right).
$$

Since $y, w \in \left(\theta_g, \frac{\sigma_b \theta_g - \sigma_g \theta_b}{\sigma_b - \sigma_g}\right)$, we have $\frac{y - \theta_b}{\sigma_b} > \frac{y - \theta_g}{\sigma_g}$ and $\frac{w - \theta_b}{\sigma_b} > \frac{w - \theta_g}{\sigma_g}$. Then, $s(x)$ convex for all $x > 0$ implies

$$
\frac{y - \theta_b}{\sigma_b} - s\left(\frac{w - \theta_b}{\sigma_b}\right) \geq \frac{y - \theta_g}{\sigma_g} - s\left(\frac{w - \theta_g}{\sigma_g}\right),
$$

or, equivalently,

$$
\frac{\sigma_b}{\sigma_g} \left(\frac{w - \theta_b}{\sigma_b} - s\left(\frac{y - \theta_b}{\sigma_b}\right)\right) \leq \frac{w - \theta_g}{\sigma_g} - s\left(\frac{y - \theta_g}{\sigma_g}\right).
$$

Since $\frac{w - \theta_b}{\sigma_b} - s\left(\frac{y - \theta_b}{\sigma_b}\right) > 0$ and $\sigma_b > \sigma_g$ we also have

$$
\frac{\sigma_g}{\sigma_b} \left(\frac{w - \theta_b}{\sigma_b} - s\left(\frac{y - \theta_b}{\sigma_b}\right)\right) < \frac{w - \theta_g}{\sigma_g} - s\left(\frac{y - \theta_g}{\sigma_g}\right),
$$

which contradicts (a.34).

(iii) The assertion that the retention zone is an interval follows from the arguments in parts (i) and (ii). The equation (a.32) is equivalent to $X(k) = k$ and therefore must define the edges of the retention zone. Because $h$ is decreasing all the way up to $x = \theta_g$, the upper root $x_+$ that defines the retention zone must exceed $\theta_g$. See Figure A.2 for an illustration.

Lemma A.9. Under the conditions of Lemma A.8, consider any situation in which $\sigma_b > \sigma_g \geq \sigma$, in which the principal retains if and only if $x \in [x_-, x_+]$ with $x_+ > x_-$, where these roots solve (a.32), and in which type $b$ is playing a best response to the principal’s choice of retention zone. Then, the derivative of the payoff of type $g$ evaluated at $\sigma_g$ is strictly negative.

Proof. Because $\sigma_b > \sigma_g \geq \sigma$, $\sigma_b$ is an interior solution to type $b$’s optimization problem. Therefore, the first order condition for type $b$’s optimization holds with equality, and

$$
f\left(\frac{x_+ - \theta_b}{\sigma_b}\right) (x_+ - \theta_b) = f\left(\frac{x_- - \theta_b}{\sigma_b}\right) (x_- - \theta_b),
$$

where $x_-$ and $x_+$ are defined as above.
which also shows in passing that \( x_- > \theta_b \). (For \( x_+ > \theta_g > \theta_b \) by Lemma A.8(iii), so that every term in (a.35) must be strictly positive.)

Now, let’s study the derivative of type-\(g\)’s retention probability evaluated at \( \sigma_g \). This is:

\[
\frac{1}{\sigma_g^2} f \left( \frac{x_- - \theta_g}{\sigma_g} \right) (x_- - \theta_g) - \frac{1}{\sigma_g^2} f \left( \frac{x_+ - \theta_g}{\sigma_g} \right) (x_+ - \theta_g) = \frac{1}{\beta \sigma_b \sigma_g} \left[ f \left( \frac{x_- - \theta_b}{\sigma_b} \right) (x_- - \theta_g) - f \left( \frac{x_+ - \theta_b}{\sigma_b} \right) (x_+ - \theta_g) \right] = \frac{1}{\beta \sigma_g} f \left( \frac{x_+ - \theta_b}{\sigma_g} \right) \left[ \frac{x_+ - \theta_b}{{\sigma_g}} \right] (x_- - \theta_g) - (x_+ - \theta_g) \right],
\]

where the first equality invokes the formula (a.32) for the roots, and the second equality uses the first order condition (a.35) for the low type.

Because \( x_- > \theta_b, x_+ > x_- \) and \( \theta_g > \theta_b \), the sign of this derivative is negative. It follows that \( \sigma_g \) must be at the corner solution \( \sigma \), and the proof is complete.

Lemma A.9 is suggestive of the fact that in any bounded retention equilibrium, \( \sigma_g = \sigma \). So in our hunt for such equilibria, we will provisionally fix \( \sigma_g \) at \( \sigma \), and to save on notation we denote \( \sigma_g \) by simply \( \sigma \).

Lemma A.13 below will guarantee that the principal will employ nontrivial bounded retention intervals for any \( \sigma > \sigma \), under some conditions. For this, we first prove some technical results (Lemmas A.10–A.12 below). Let \( x^{**}(\sigma) \) be the unique minimizer of \( h(x) \), defined by (a.31) in Lemma A.8, when type \( b \) employs \( \sigma_b = \sigma > \sigma \) and type \( g \) plays \( \sigma_g = \sigma \). That is,

\[
(a.36) \quad \frac{1}{\sigma} \left( \frac{x^{**}(\sigma) - \theta_b}{\sigma} \right) = \frac{1}{\sigma} \left( \frac{x^{**}(\sigma) - \theta_g}{\sigma} \right).
\]

**Lemma A.10.** \( \frac{x^{**}(\sigma) - \theta_b}{\sigma} \) is strictly decreasing in \( \sigma \) and there exists a unique \( \sigma^{**} \) that solves

\[
(a.37) \quad x^{**}(\sigma) = \theta_b + z^{**} \sigma.
\]

**Proof.** Differentiate (a.36) with respect to \( \sigma \) to obtain

\[
(a.38) \quad \frac{d}{d\sigma} x^{**}(\sigma) = \frac{1}{\sigma^2} s'(\frac{x^{**}(\sigma) - \theta_b}{\sigma}) \frac{x^{**}(\sigma) - \theta_b}{\sigma} + \frac{1}{\sigma^2} s'(\frac{x^{**}(\sigma) - \theta_g}{\sigma}) \frac{x^{**}(\sigma) - \theta_g}{\sigma}.
\]

By Lemma A.8(ii), \( x^{**}(\sigma) > \theta_g \), and \( s(x) \) is decreasing and negative for \( x > 0 \), so the numerator is negative. By Lemma A.8(ii), \( h(x) \) is decreasing for \( x < x^{**}(\sigma) \) and increasing for \( x > x^{**}(\sigma) \). That means that

\[
\frac{1}{\sigma} s \left( \frac{x}{\sigma} - \theta_g \right) > \frac{1}{\sigma} s \left( \frac{x}{\sigma} - \theta_b \right) \quad \text{for all } x < x^{**}(\sigma),
\]

\[
\frac{1}{\sigma} s \left( \frac{x}{\sigma} - \theta_g \right) < \frac{1}{\sigma} s \left( \frac{x}{\sigma} - \theta_b \right) \quad \text{for all } x > x^{**}(\sigma),
\]

so \( \sigma^{**} \) must be in the interval \( (x^{**}(\sigma), \infty) \).
and
\[
\frac{1}{\sigma} s\left(\frac{x - \theta_g}{\sigma}\right) < \frac{1}{\sigma} s\left(\frac{x - \theta_b}{\sigma}\right) \quad \text{for all } x > x^\ast(\sigma).
\]

Therefore, at \( x = x^\ast(\sigma) \) we have
\[
\frac{1}{\sigma^2} s'(\sigma) \left(\frac{x^\ast(\sigma) - \theta_b}{\sigma}\right) > \frac{1}{\sigma^2} s'(\sigma) \left(\frac{x^\ast(\sigma) - \theta_g}{\sigma}\right)
\]
so the denominator in (a.38) is positive, and \( x^\ast(\sigma) < 0 \): \( x^\ast(\sigma) \) is decreasing in \( \sigma \). Then, so is \( \frac{x^\ast(\sigma) - \theta_b}{\sigma} \).

Finally, notice that, as \( \sigma \to \sigma \), \( x^\ast(\sigma) \) cannot converge to a finite value, because \( x^\ast(\sigma) \) solves (a.36) and \( s(x) \) is strictly decreasing. Since \( x^\ast(\sigma) > \theta_g \) it must be that \( x^\ast(\sigma) \to \infty \) as \( \sigma \to \sigma \). So \( \frac{x^\ast(\sigma) - \theta_b}{\sigma} \to \infty \) as \( \sigma \to \sigma \). And \( \frac{x^\ast(\sigma) - \theta_b}{\sigma} \to 0 \) as \( \sigma \to \infty \). Then, since \( z^\ast > 0 \), there exists a unique \( \sigma \) that satisfies (a.37).

To proceed further, define
\[
(a.39) \quad \beta_l := \frac{1}{\sigma^2} f\left(\frac{x^\ast(\sigma^\ast) - \theta_b}{\sigma^\ast}\right).
\]

**Lemma A.11.** \( \beta_l < 1 \).

**Proof.** By Lemma A.8, the minimizer of \( h(x) \), \( x^\ast(\sigma) \), is in the interval \( (\theta_g, \frac{\sigma \theta_g - \theta \theta_b}{\sigma - \theta}) \). That means that \( \frac{x^\ast(\sigma) - \theta_b}{\sigma} > \frac{x^\ast(\sigma) - \theta_b}{\sigma} > 0 \) and therefore \( f\left(\frac{x^\ast(\sigma) - \theta_b}{\sigma}\right) < f\left(\frac{x^\ast(\sigma) - \theta_b}{\sigma}\right) \). Then, for \( \sigma > \sigma \) we also have \( \frac{1}{\sigma} f\left(\frac{x^\ast(\sigma) - \theta_b}{\sigma}\right) < \frac{1}{\sigma} f\left(\frac{x^\ast(\sigma) - \theta_b}{\sigma}\right) \). \( \beta_l < 1 \) results from taking \( \sigma = \sigma^\ast \), defined as the solution to (a.37).

**Lemma A.12.** Let \( \beta(\sigma) \) be defined as
\[
(a.40) \quad \beta(\sigma) = \frac{1}{\sigma} f\left(\frac{x^\ast(\sigma) - \theta_b}{\sigma}\right).
\]

(i) If \( \beta = \beta(\sigma) \), \( X = \{x^\ast(\sigma)\} \);

If \( \beta > \beta(\sigma) \), \( X = [x_-(\sigma), x_+(\sigma)] \) with \( x_+(\sigma) > x_-(\sigma) \), which are the two roots to (a.32);

If \( \beta < \beta(\sigma) \), \( X = \emptyset \);

(ii) \( \beta(\sigma) \) is increasing at all \( \sigma \in (\sigma, \sigma^\ast) \); it is decreasing at all \( \sigma > \sigma^\ast \); it attains a maximum at \( \sigma = \sigma^\ast \), and its maximum value is \( \beta_l \), defined in (a.39).

**Proof.** (i) By Lemma A.8(ii), \( x^\ast(\sigma) \) is the unique minimizer of \( h(x) \). Recall the retention zone is \( X = \{x : h(x) \leq k\} \).

If \( \beta = \beta(\sigma) \) or, equivalently, if \( h(x^\ast(\sigma)) = k \), \( X = \{x^\ast(\sigma)\} \).
If \( \beta > \beta(\sigma) \), \( h(x^*(\sigma)) < k \). By Lemma A.8(ii), \( X = [x_-(\sigma), x_+\sigma)] \) with \( x_+\sigma) > x_-(\sigma) \), which are the two roots to (a.32).

If \( \beta < \beta(\sigma) \), \( h(x^*(\sigma)) > k \) and therefore \( h(x) > k \) for all \( x \), so \( X = \emptyset \).

(ii) Take (a.40) and differentiate with respect to \( \sigma \). After some algebra, and using that \( \beta(\sigma) \) satisfies (a.40) and \( x^*(\sigma) \) satisfies (a.36), we obtain

\[
\beta'(\sigma) = -\frac{\beta(\sigma)}{\sigma} \cdot \left[ \frac{1}{f(z)} \cdot \frac{\partial f(z)}{\partial z} \right]_{z=x^*(\sigma)-\theta_b}.
\]

By Lemma A.8(ii), \( x^*(\sigma) > \theta_g > \theta_b \). By Lemma A.6, for \( z \geq 0 \), \( \partial f(z)/\partial z \) is first positive, then negative, and zero at \( z^* \). By Lemma A.10, \( x^*(\sigma)-\theta_b \) is decreasing in \( \sigma \) and there exists a unique \( \sigma^* \), defined in (a.37), such that \( \frac{x^*(\sigma)-\theta_b}{\sigma} = z^* \). Then, for \( \sigma \in (\sigma^*, \sigma^*) \), \( \beta'(\sigma) > 0 \), and for \( \sigma > \sigma^* \), \( \beta'(\sigma) < 0 \). Finally, this means that \( \beta(\sigma) \) attains a maximum at \( \sigma^* \). This maximum value of \( \beta \) is \( \beta_1 \), defined in (a.39).

We now establish a sufficient condition for the existence of bounded retention intervals for any possible pair \((\sigma_b, \sigma_g)\) with \( \sigma_b > \sigma_g = \sigma \).

**Lemma A.13.** For any pair \((\sigma_b, \sigma_g)\) with \( \sigma_b > \sigma \) and \( \sigma_g = \sigma \), the principal employs a nontrivial bounded retention interval if \( \beta > \beta_1 \), or, equivalently, if \( \sigma < \sigma^*(\beta) \) where \( \sigma^*(\beta) \) is defined as the value of \( \sigma \) that solves (a.39) when we replace \( \beta_1 \) with \( \beta < 1 \) and \( \sigma^*(\beta) = \infty \) for \( \beta \geq 1 \).

**Proof.** By Lemma A.12(ii), if \( \beta > \beta_1 \), \( \beta > \beta(\sigma) \) for all \( \sigma > \sigma \). Then, by Lemma A.12(i), the principal retains if and only if \( x \in [x_-(\sigma), x_+\sigma)] \), with \( x_-(\sigma) < x_+\sigma) \), for all \( \sigma > \sigma \).

Now, we see that \( \beta > \beta_1 \) is equivalent to \( \sigma < \sigma^*(\beta) \).

First, recall that, for any \( \sigma > 0 \), \( \sigma^* \) and \( x^*(\sigma) \), defined in (a.37) and (a.36), respectively, are well-defined. Then, so is \( \beta_1(\sigma) \), defined in (a.39). Let us now show that \( \beta_1(\sigma) \) is strictly increasing in \( \sigma \). Let us define \( \bar{\sigma}(\sigma) \) as \( \bar{\sigma}(\sigma) := x^*(\sigma^*(\sigma), \sigma) \). So, \( \bar{\sigma}(\sigma) \) is our function \( x^*(\sigma, \sigma) \), defined in (a.36), evaluated at \( \sigma^*(\sigma) \), defined in (a.37). Since

\[
\frac{\bar{\sigma}(\sigma) - \theta_b}{\sigma^*(\sigma)} = z^*;
\]

(a.36), when evaluated at \( \sigma^*(\sigma) \), becomes

\[
\frac{\bar{\sigma}(\sigma) - \theta_b}{\sigma} \cdot \left( \frac{\bar{\sigma}(\sigma) - \theta_g}{\sigma} \right) = -1;
\]

(a.42), and (a.39), also evaluated at \( \sigma^*(\sigma) \), becomes

\[
\beta_1(\sigma) = \frac{z^* f(z^*)}{\frac{\bar{\sigma}(\sigma) - \theta_b}{\sigma}}.
\]

(a.43)

Differentiate (a.43) with respect to \( \sigma \) to get

\[
\beta_1'(\sigma) = \beta_1(\sigma) \cdot \left[ \frac{1}{\sigma} \left( 1 + \frac{1}{\sigma} \left( \frac{\bar{\sigma}(\sigma) - \theta_g}{\sigma} \right) \left( \bar{\sigma}(\sigma) - \theta_b \right) \right) - \left( \frac{1}{\bar{\sigma}(\sigma) - \theta_b} + \frac{1}{\sigma} \left( \frac{\bar{\sigma}(\sigma) - \theta_g}{\sigma} \right) \right) \cdot \bar{\sigma}'(\sigma) \right].
\]
Now using (a.42) we can substitute \( \frac{1}{z} s \left( \frac{\bar{x}(\sigma) - \theta_b}{\sigma} \right) \) by \(-1/(\bar{x}(\sigma) - \theta_b)\), and we obtain

\[
\beta_1' (\sigma) = \beta_1 (\sigma) \cdot \frac{1}{\bar{x}(\sigma) - \theta_b} \left( 1 - \frac{\bar{x}(\sigma) - \theta_b}{\bar{x}(\sigma) - \theta_b} \right)
\]

\[
= \beta_1 (\sigma) \cdot \frac{\theta_g - \theta_b}{\bar{x}(\sigma) - \theta_b} > 0.
\]

The final question is what’s the limit of \( \beta_1 (\sigma) \) as \( \sigma \to 0 \). If \( \sigma \to 0 \), the value that maximizes the likelihood of the good type, \( x^{**} (\sigma) \) converges to \( \theta_g \), for any \( \sigma \). That means that \( \frac{1}{z} f \left( \frac{\bar{x}(\sigma) - \theta_b}{\sigma} \right) \to \infty \) as \( \sigma \to 0 \), and therefore

\[
\beta_1 (\sigma) = \frac{z^* f(x^*)}{\bar{x}(\sigma) - \theta_b} \to 0.
\]

Then, we can conclude that \( \beta > \beta_1 \) is equivalent to \( \sigma < \hat{\sigma} (\beta) \) where \( \hat{\sigma} (\beta) \) is defined as the value of \( \sigma \) that solves (a.39) when we replace \( \beta_1 \) with \( \beta \).

Now we determine a second upper bound on \( \sigma \) (recall that a parallel bound — \( \sigma < \sigma(\beta) \) — was used to negate the existence of a monotone retention equilibrium):

(a.44) \( \sigma < \hat{\sigma} (\beta) \) if \( \beta \in (0, 1) \).

By Lemma A.11, \( \beta_1 < 1 \), and therefore if \( \beta \geq 1 \), (a.44) is trivially satisfied.

For the arguments to follow, it will be useful to indicate clearly the way in which the two upper bounds on \( \sigma \) relate to each other. To do so, let \( \beta_h \) be the value of \( \beta > 1 \) such that \( \sigma = \sigma(\beta) \).

Now look at Figure A.3. Notice that (a) \( \sigma \geq \hat{\sigma} (\beta) \) implies \( \sigma < \sigma(\beta) \); (b) \( \sigma \geq \sigma(\beta) \) implies \( \sigma < \hat{\sigma} (\beta) \); whereas (c) \( \sigma < \sigma(\beta) \) and \( \sigma < \hat{\sigma} (\beta) \) can occur simultaneously.
The condition \( \sigma < \hat{\sigma}(\beta) \) is a sufficient condition under which the principal, when conjecturing that the agent will play \( \sigma_a = \sigma \) and \( \sigma_b > \sigma \), will employ a nontrivial, bounded retention interval, for any such \( \sigma \). The next step is to show that there exists a fixed point between the noise \( \sigma_b \) conjectured by the principal and the one optimally chosen by type-b. For this, we need to analyze the way in which the retention interval \([x_-(\sigma), x_+(\sigma)]\) behaves, in particular as \( \sigma \to \sigma \) and \( \sigma \to \infty \). This is what we do next.

Lemma A.14. Let \( x_-(\sigma) \) and \( x_+(\sigma) \) be the roots to

\[
\beta \frac{1}{\sigma} f\left(\frac{x - \theta_b}{\sigma}\right) = \frac{1}{\sigma} f\left(\frac{x - \theta_b}{\sigma}\right),
\]

for \( \sigma > \sigma \). Then,

(i) \( \lim_{\sigma \to \sigma} x_-(\sigma) = x^*(\sigma) \) and \( \lim_{\sigma \to \sigma} x_+(\sigma) = \infty \)

(ii) \( \lim_{\sigma \to \infty} x_-(\sigma) < \theta_b \).

(iii) The derivatives are

\[
x'_i(\sigma) = -\frac{1}{\sigma^2} \beta \frac{1}{\sigma} f' \left( \frac{x_i(\sigma) - \theta_b}{\sigma} \right) - \frac{1}{\sigma} f' \left( \frac{x_i(\sigma) - \theta_b}{\sigma} \right).
\]

for \( i = -, + \).

(iv) If \( \sigma \geq \sigma(\beta) \), then \( x_-(\sigma) < \theta_b \) for all \( \sigma > \sigma \).

Proof. (i) \( x_-(\sigma) \) must satisfy \( h'(x_-) < 0 \), whereas \( x_+(\sigma) \) must satisfy \( h'(x_+) > 0 \). Recalling (a.33), this means that:

\[
\frac{1}{s} \left( \frac{x_-(\sigma) - \theta_g}{\sigma} \right) > \frac{1}{s} \left( \frac{x_-(\sigma) - \theta_b}{\sigma} \right),
\]

and

\[
\frac{1}{s} \left( \frac{x_+(\sigma) - \theta_g}{\sigma} \right) < \frac{1}{s} \left( \frac{x_+(\sigma) - \theta_b}{\sigma} \right).
\]

Similarly, the monotone threshold \( x^*(\sigma) \) satisfies

\[
\beta f \left( \frac{x^*(\sigma) - \theta_g}{\sigma} \right) = f \left( \frac{x^*(\sigma) - \theta_b}{\sigma} \right)
\]

and

\[
\frac{1}{s} \left( \frac{x^*(\sigma) - \theta_g}{\sigma} \right) > \frac{1}{s} \left( \frac{x^*(\sigma) - \theta_b}{\sigma} \right).
\]

In the limit as \( \sigma \to \sigma \), the condition (a.45) is the same as (a.49), and the condition (a.47) is the same as (a.50) (equality of (a.47) at \( \sigma = \sigma \) cannot hold because \( s(x) \) is decreasing). Since, by the MLRP, there exists a unique \( x^*(\sigma) \), we have that \( \lim_{\sigma \to \sigma} x_-(\sigma) = x^*(\sigma) \).
Notice that, as \( \sigma \to \sigma^* \), (a.48) becomes
\[
s \left( \frac{x_+ (\sigma) - \theta_b}{\sigma} \right) \geq s \left( \frac{x_+ (\sigma) - \theta_g}{\sigma} \right),
\]
But since \( s (x) \) is decreasing and \( \theta_g > \theta_b \), it must be that \( \lim_{\sigma \to \sigma^*} x_+ (\sigma) = \infty \) or \( \lim_{\sigma \to \sigma^*} x_+ (\sigma) = -\infty \). But it cannot be that \( \lim_{\sigma \to \sigma^*} x_+ (\sigma) = -\infty \) because \( x_+ (\sigma) \geq x_- (\sigma) \) for all \( \sigma > \sigma^* \), so it must be that \( \lim_{\sigma \to \sigma^*} x_+ (\sigma) = \infty \).

(ii) Notice that, for \( \sigma \) large enough, at \( x = \theta_b \) we have
\[
\beta \frac{1}{\sigma} f \left( \frac{\theta_b - \theta_g}{\sigma} \right) > \frac{1}{\sigma} f (0).
\]
That means that, for \( \sigma \) large enough, the principal retains the agent at \( x = \theta_b \), so \( \lim_{\sigma \to \infty} x_- (\sigma) < \theta_b \).

(iii) Differentiate (a.53) with respect to \( \sigma \) to obtain (a.46).

(iv) By part (i) of this Lemma, \( x_- (\sigma) \to x^* (\sigma) \) as \( \sigma \to \sigma^* \). If \( \sigma \geq \sigma^* (\beta) \), a monotone equilibrium exists by Proposition 5(ii), and therefore \( x^* (\sigma) \leq \theta_b \), by Lemma A.7(i). \( \sigma \geq \sigma^* (\beta) \) also implies that \( \beta > 1 > \beta_l \) and therefore \( x_- (\sigma) \) is well-defined for any \( \sigma > \sigma^* \) by Lemma A.12(i), and it is clearly continuous in \( \sigma \).

We note again that \( x_- (\sigma) \) must satisfy \( h' (x_-) < 0 \), which implies (a.47). Then, the denominator in (a.46) for \( x_- (\sigma) \) is positive, and we get that \( x_- (\sigma) \) is also well-defined and continuous in \( \sigma > \sigma^* \), and also
\[
\text{(a.51)} \quad \text{Sign} (x_- (\sigma)) = \text{Sign} \left( - \left[ \frac{\partial f (z)}{\partial z} \right] \bigg|_{z = s_*(\sigma) - \theta_b} \right).
\]
If \( x^* (\sigma) < \theta_b \), continuity of \( x_- (\sigma) \) implies that \( x_- (\sigma) < \theta_b \) for \( \sigma > \sigma^* \) close enough to \( \sigma^* \).

If \( x^* (\sigma) = \theta_b \), (a.51) and Lemma A.6 say that \( x'_- (\sigma) < 0 \) for \( \sigma > \sigma^* \) close enough to \( \sigma^* \), so once again \( x_- (\sigma) < \theta_b \) for \( \sigma > \sigma^* \) close enough to \( \sigma^* \).

Then, for the assertion to be false, it is required that \( x'_- (\sigma) \geq 0 \) for some \( \sigma > \sigma^* \) at which \( x'_- (\sigma) = \theta_b \), but this contradicts (a.51) and Lemma A.6.

Guided by Lemma A.7, let us now consider the following mapping, defined for all \( \sigma > \sigma^* \):
\[
\Psi (\sigma) = \max \{ \sigma, \sigma^* (\sigma) \},
\]
where \( \sigma^* (\sigma) \) is the unrestricted maximizer of type-\( b \)'s retention probability when the principal retains if and only if \( x \in [x_- (\sigma), x_+ (\sigma)] \), so \( \sigma^* (\sigma) \) solves (see Lemma A.7(iii))
\[
\text{(a.52)} \quad f \left( \frac{x_- (\sigma) - \theta_b}{\sigma^* (\sigma)} \right) (x_- (\sigma) - \theta_b) = f \left( \frac{x_+ (\sigma) - \theta_b}{\sigma^* (\sigma)} \right) (x_+ (\sigma) - \theta_b),
\]
and \( x_- (\sigma) \) and \( x_+ (\sigma) \) are the roots to
\[
\text{(a.53)} \quad \beta \frac{1}{\sigma} f \left( \frac{x - \theta_g}{\sigma} \right) = \frac{1}{\sigma} f \left( \frac{x - \theta_b}{\sigma} \right).
\]
The following Lemma determines an important feature of $\Psi(\sigma)$.

**Lemma A.15.** If a fixed point $\sigma = \Psi(\sigma)$ exists, $\Psi'(\sigma) < 0$.

**Proof.** At any such fixed point we have $\underline{\sigma} < \sigma = \Psi(\sigma)$, so $\Psi(\sigma)$ must solve (a.52). Then, differentiate (a.52) with respect to $\sigma$, replacing $\sigma^*(\sigma)$ by $\Psi(\sigma)$, to obtain

$$
\Psi'(\sigma) = \left( f' \left( \frac{\sigma - \theta_b}{\Psi(\sigma)} \right) \frac{\sigma - \theta_b}{\Psi(\sigma)} + f \left( \frac{\sigma - \theta_b}{\Psi(\sigma)} \right) \right) \left( \sigma - \theta_b \right) - \left( f' \left( \frac{\sigma + \theta_b}{\Psi(\sigma)} \right) \frac{\sigma + \theta_b}{\Psi(\sigma)} + f \left( \frac{\sigma + \theta_b}{\Psi(\sigma)} \right) \right) \left( \sigma + \theta_b \right).
$$

Evaluating at $\Psi(\sigma) = \sigma$ and plugging the expressions for $x'_{-}(\sigma)$ and $x'_{+}(\sigma)$ in (a.46) yields (a.54)

$$
\Psi'(\sigma) = \frac{1}{\sigma^2} \left( \frac{f' \left( \frac{\sigma - \theta_b}{\Psi(\sigma)} \right) \frac{\sigma - \theta_b}{\Psi(\sigma)} + f \left( \frac{\sigma - \theta_b}{\Psi(\sigma)} \right)}{f' \left( \frac{\sigma + \theta_b}{\Psi(\sigma)} \right) \frac{\sigma + \theta_b}{\Psi(\sigma)} + f \left( \frac{\sigma + \theta_b}{\Psi(\sigma)} \right)} \right)^2 - \frac{\beta}{\sigma^2} f' \left( \frac{\sigma - \theta_b}{\Psi(\sigma)} \right) \left( \frac{\sigma - \theta_b}{\Psi(\sigma)} \right)^2 - \frac{\beta}{\sigma^2} f' \left( \frac{\sigma + \theta_b}{\Psi(\sigma)} \right) \left( \frac{\sigma + \theta_b}{\Psi(\sigma)} \right)^2.
$$

Since $h' \left( x_{-}(\sigma) \right) < 0$ and $h' \left( x_{+}(\sigma) \right) > 0$,

$$
\beta \frac{1}{\sigma^2} f' \left( \frac{x_{+}(\sigma) - \theta_b}{\sigma} \right) < \frac{1}{\sigma^2} f' \left( \frac{x_{+}(\sigma) - \theta_b}{\sigma} \right), \quad \text{and}
$$

$$
\beta \frac{1}{\sigma^2} f' \left( \frac{x_{-}(\sigma) - \theta_b}{\sigma} \right) > \frac{1}{\sigma^2} f' \left( \frac{x_{-}(\sigma) - \theta_b}{\sigma} \right),
$$

so the numerator in (a.54) is positive. The second-order condition of $\sigma$ is

$$
f' \left( \frac{x_{+} - \theta_b}{\sigma} \right) \left( \frac{x_{+} - \theta_b}{\sigma^2} \right)^2 - f' \left( \frac{x_{-} - \theta_b}{\sigma} \right) \left( \frac{x_{-} - \theta_b}{\sigma^2} \right)^2 < 0,
$$

which is the denominator in (a.54). So $\Psi'(\sigma) < 0$ at any fixed point. \hfill \blacksquare

**Lemma A.16.** If both the conditions $\sigma < \sigma(\beta)$ and $\sigma < \hat{\sigma}(\beta)$ hold, there is a unique nontrivial equilibrium. It has bounded retention.

**Proof.** Under the condition $\sigma < \sigma(\beta)$, both bounded replacement equilibria and monotone equilibria are ruled out by Proposition 5. Importantly, the nonexistence of monotone equilibria is equivalent to $x^*(\sigma) > \theta_b$.

By Lemma A.14(i), $[x_{-}(\sigma), x_{+}(\sigma)] \to [x^*(\sigma), \infty)$ as $\sigma \to \sigma$. Since $x^*(\sigma) \geq \theta_b$, by Lemma A.7(i) we have that $\Psi(\sigma) \to \infty$ as $\sigma \to \sigma$.

By Lemma A.14(ii), for $\sigma$ large enough, $x_{-}(\sigma) < \theta_b$, and by Lemma A.8, $x_{+}(\sigma) > \theta_b$, so $\theta_b \in [x_{-}(\sigma), x_{+}(\sigma)]$ for $\sigma$ large enough. Then, $\Psi(\sigma) = \sigma$ for $\sigma$ large enough, by Lemma A.7(ii).

By Lemma A.13, (a.44) guarantees that the principal always employs a bounded retention interval for any pair $(\sigma_g, \sigma_b)$ with $\sigma_b = \sigma > \sigma = \sigma_g$. Then, it is clear that $x_{-}(\sigma)$ and $x_{+}(\sigma)$ are continuous in $\sigma$, and therefore so is $\Psi(\sigma)$. Then, the above end-point verifications and continuity guarantee that $\Psi$ has at least one fixed point. Lemma A.15 guarantees that such a fixed point is unique. At this fixed point, both the principal and the bad type are playing best responses.
It remains to show that the good type is also playing a best response at $\sigma$. Notice that $x(\sigma) > \theta_g$, so that Lemma A.7(iii) applies, and the derivative of the payoff of the good type with respect to $\sigma_g$ is given by $\sigma_g^2 d_g(\sigma_g)$, where $d_g$ is defined in equation (a.26). By Lemma A.9, this derivative is negative at $\sigma_g = \sigma$, and by part (iii) of Lemma A.7, it must continue to be negative for all $\sigma_g > \sigma$. Therefore the best response of the good type is indeed to play $\sigma$, as claimed.

Lemma A.17. If $\sigma \geq \sigma(\beta)$ (in which case $\sigma < \hat{\sigma}(\beta)$ automatically holds), then a bounded retention equilibrium cannot exist.

Proof. Every bounded retention equilibrium involves $\sigma_g = \sigma$. This follows from Lemma A.9 and the decreasing payoff derivative as noted in part (iii) of Lemma A.7. It follows that every bounded retention equilibrium can be expressed as a fixed point of the mapping $\Psi(\sigma)$, where at the fixed point, $\sigma > \sigma$. However, if $\sigma \geq \sigma(\beta)$, $x_-(\sigma) < \theta_b$ for all $\sigma > \sigma$, by Lemma A.14(iv). Moreover, $x_+(\sigma) > \theta_b$ for all $\sigma > \sigma$, by Lemma A.8(iii), so $\theta_b \in [x_-(\sigma), x_+(\sigma)]$ for all $\sigma > \sigma$. Then, by Lemma A.7(ii), $\Psi(\sigma) = \sigma$ for all $\sigma > \sigma$, and a fixed point of $\Psi$ cannot exist.

Lemma A.18. If $\sigma \geq \hat{\sigma}(\beta)$, then $\sigma < \sigma(\beta)$ and a nontrivial equilibrium does not exist.

Proof. By Proposition 5, a bounded replacement equilibrium does not exist. Moreover, $\sigma \geq \hat{\sigma}(\beta)$ implies $\beta \leq 1$. But then $\sigma < \sigma(\beta) = \infty$, and again by Proposition 5, a monotone retention equilibrium cannot exist. It only remains to show that a bounded retention equilibrium cannot exist either.

Consider $\beta < \beta_l$. By Lemma A.12 a nontrivial bounded retention equilibrium is possible only if $\sigma < \sigma_i^\beta$ or if $\sigma > \sigma_h^\beta$, where $\beta\left(\sigma_i^\beta\right) = \beta\left(\sigma_h^\beta\right) = \beta$, and $\sigma_i^\beta < \sigma** < \sigma_h^\beta$. What we now have to show is that, for any $\sigma$ such that a nontrivial retention regime is employed by the principal, type-$b$’s best response will never coincide with such $\sigma$.

Suppose $\sigma_i^\beta$ exists, and consider $\sigma < \sigma_i^\beta$. By Lemma A.12(i), $x_-(\sigma_i^\beta) = x**(\sigma_i^\beta)$, and by Lemma A.12(ii) $\sigma_i^\beta < \sigma**$ implies $\beta'(\sigma_i^\beta) > 0$ or, equivalently (see (a.41))

$$\frac{\partial f(z)}{\partial z} \bigg|_{z = \frac{x_-(\sigma)-\theta_b}{\sigma}} < 0.$$ 

By Lemma A.6, this means that $\frac{x_-(\sigma_i^\beta)-\theta_b}{\sigma_i^\beta} > z^*$, but type-$b$’s optimal $\sigma$ satisfies $\frac{x_-(\sigma)-\theta_b}{\sigma} < z^*$ by Lemma A.7(iii), so $\sigma_i^\beta$ cannot be a fixed point. The next step is to show that $\frac{x_-(\sigma)-\theta_b}{\sigma} > z^*$ for all $\sigma < \sigma_i^\beta$. So suppose not: there exists $\sigma$ such that $\frac{x_-(\sigma)-\theta_b}{\sigma} \leq z^*$. That means that $x_-(\sigma)$ crosses $\sigma^* + \theta_b$ from below at some point, which requires $x'_-(\sigma) > z^*> 0$ at such intersection point. But $x_-(\sigma) = \sigma z^* + \theta_b$ implies $\frac{x_-(\sigma)-\theta_b}{\sigma} = z^*$, and therefore by Lemma A.6 and inspection of (a.46), $x'_-(\sigma) = 0$, a contradiction.
Now suppose \( \sigma^\beta_h \) exists, and consider \( \sigma > \sigma^\beta_h \). By Lemma A.12(i), \( x_+\left(\sigma^\beta_h\right) = x^{**}\left(\sigma^\beta_h\right) \), and by Lemma A.12(ii), \( \sigma^\beta_h > \sigma^{**} \) implies \( \beta'\left(\sigma^\beta_h\right) < 0 \) or, equivalently (see (a.41) again),
\[
\frac{\partial f(z)}{\partial z} \bigg|_{z = \frac{x_+\left(\sigma\right) - \theta_h}{\sigma}} > 0.
\]
By Lemma A.6, this means \( \frac{x_+\left(\sigma^\beta_h\right) - \theta_h}{\sigma^\beta_h} < z^* \), but type-\( b \)'s optimal \( \sigma \) satisfies \( \frac{x_+\left(\sigma\right) - \theta_h}{\sigma} > z^* \) by Lemma A.7(iii), so \( \sigma^\beta_h \) cannot be a fixed point. The next step is to show that \( \frac{x_+\left(\sigma\right) - \theta_h}{\sigma} > z^* \) for all \( \sigma > \sigma^\beta_h \). So suppose not: there exists \( \sigma \) such that \( \frac{x_+\left(\sigma\right) - \theta_h}{\sigma} \geq z^* \). That means that \( x_+\left(\sigma\right) \) crosses \( \sigma z^* + \theta_h \) from below at some point, which requires \( x'_+\left(\sigma\right) > 0 \) at such intersection point. This implies that \( \frac{x_+\left(\sigma\right) - \theta_h}{\sigma} = z^* \), but by Lemma A.6 and inspection of (a.46) this implies \( x'_+\left(\sigma\right) = 0 \), a contradiction.

Finally, consider the case \( \beta = \beta_1 \). In this case we have \( \sigma^\beta_1 = \sigma^{**} = \sigma^\beta_h \), so at \( \sigma = \sigma^{**} \), \( x_+\left(\sigma\right) = x_+\left(\sigma\right) = \sigma z^* + \theta_h \) and \( x'_-\left(\sigma\right) = x'_+\left(\sigma\right) = 0 \). Then, if we consider \( \sigma < \sigma^{**} \), it is clear that for \( \sigma \) close enough to \( \sigma^{**} \) we have \( x_-\left(\sigma\right) > \theta_1 + z^* \sigma \), and as we showed before this leads to the conclusion that \( \theta_1 + z^* \sigma < x_-\left(\sigma\right) \) for all \( \sigma \in \left(\sigma_1, \sigma^\beta_h\right) \). Similarly, for \( \sigma > \sigma^{**} \) and close enough to \( \sigma^{**} \), we have that \( x_+\left(\sigma\right) < \theta_1 + z^* \sigma \), which leads to \( \theta_1 + z^* \sigma > x_+\left(\sigma\right) \) for all \( \sigma > \sigma^\beta_h \).

**Proof of Proposition 6.** We combine the previous Lemmas. Parts (i) and (ii): Consider first \( \sigma \geq \hat{\sigma}(\beta) \): Lemma A.18 says a nontrivial equilibrium does not exist. Now let \( \sigma < \hat{\sigma}(\beta) \). Bounded replacement equilibria are ruled out by Proposition 5(i). If \( \sigma \geq \sigma(\beta) \), by Lemma A.17 a bounded retention equilibrium cannot exist, but by Proposition 5(ii) a unique monotone retention equilibrium does exist. If \( \sigma < \sigma(\beta) \), by Proposition 5(ii) and Lemma A.16 there is a unique nontrivial equilibrium, and it has bounded retention. The equilibrium strategies are given by Lemmas A.8, A.9, and A.7(iii). Part (iii): when \( \beta \leq 1 \), \( \sigma < \hat{\sigma}(\beta) \), and therefore if a nontrivial equilibrium exists, it must involve bounded retention by Proposition 5(ii).

5. PROOF OF PROPOSITION 7, SECTION 6.5

Agents 1 and 2 simultaneously signal their types in a one-shot game:
\[
x_i = \theta_{k(i)} + \sigma_{k(i)}\varepsilon_i,
\]
where \( i = 1, 2 \), \( k\left(i\right) \) denotes \( i \)'s type, and \( \varepsilon_1 \) and \( \varepsilon_2 \) are i.i.d. standard normal. When the principal observes a pair \((x_1, x_1)\), her posterior probabilities are:
\[
\Pr\left(k\left(1\right) = g\mid (x_1, x_2)\right) = \frac{\text{Density} \left( (x_1, x_2) \mid k\left(1\right) = g \right)}{\text{Density} \left( (x_1, x_2) \right)} = \frac{\text{Density} \left( (x_1, x_2) \mid k\left(1\right) = g, k\left(2\right) = b \right)}{\text{Density} \left( (x_1, x_2) \right)}.
\]
So, after observing \((x_1, x_2)\), our principal will prefer agent 1 over agent 2 if (and modulo indifference, only if)
\[
\frac{1}{\sigma_g} \phi\left(\frac{x_1 - \theta_g}{\sigma_g}\right) \frac{1}{\sigma_b} \phi\left(\frac{x_2 - \theta_b}{\sigma_b}\right) \geq \frac{1}{\sigma_g} \phi\left(\frac{x_2 - \theta_g}{\sigma_g}\right) \frac{1}{\sigma_b} \phi\left(\frac{x_1 - \theta_b}{\sigma_b}\right),
\]
where \( \phi \) is the standard normal density.
which, after some manipulation, yields:

\[(a.55) \quad (\sigma_b^2 - \sigma_g^2) x_1^2 - 2 (\sigma_b^2 \theta_g - \sigma_g^2 \theta_b) x_1 \leq (\sigma_b^2 - \sigma_g^2) x_2^2 - 2 (\sigma_b^2 \theta_g - \sigma_g^2 \theta_b) x_2.\]

To prove Proposition 7, we begin by ruling out the possibility that \(\sigma_b = \sigma_g = \sigma\) in equilibrium. If that were the case, then \((a.55)\) reduces to

\[x_1 \geq x_2;\]

that is, the principal retains the agent with the higher signal. In this case, it is easy to compute the retention probability for agent \(j\) for both realizations of types:

\[
\int_{-\infty}^{\infty} \frac{1}{\sigma_g} \phi \left( \frac{x_j - \theta_g}{\sigma_g} \right) \left( 1 - \Phi \left( \frac{x_j - \theta_b}{\sigma_b} \right) \right) dx_j = 1 - \Phi \left( \frac{\theta_g - \theta_b}{\sqrt{\sigma_g^2 + \sigma_b^2}} \right) \text{ if } k(i) = b,
\]

\[
\int_{-\infty}^{\infty} \frac{1}{\sigma_b} \phi \left( \frac{x_j - \theta_b}{\sigma_b} \right) \left( 1 - \Phi \left( \frac{x_j - \theta_g}{\sigma_g} \right) \right) dx_j = \Phi \left( \frac{\theta_g - \theta_b}{\sqrt{\sigma_g^2 + \sigma_b^2}} \right) \text{ if } k(i) = g,
\]

where we have used the property that \(\int_{-\infty}^{\infty} \phi(w) \Phi \left( \frac{w-a}{b} \right) dw = \Phi \left( \frac{-a}{\sqrt{1+b^2}} \right)\). But it is clear from these expressions that \(b\) will want to increase \(\sigma_b\), whereas \(g\) will seek to lower \(\sigma_g\) — there will always be an agent who would deviate, and therefore there is no equilibrium in which both types choose the same noise. Also, as we will soon see (but it is already quite clear) there can be no monotonic equilibrium either, since the only way the principal will keep the agent with the higher signal is when both agents communicate with the same level of noise.

Next, we eliminate the possibility that \(\sigma_b < \sigma_g\). In this case, let \(\hat{x}\) be the value of \(x\) that minimizes the likelihood ratio \[\left[ \frac{1}{\sigma_g} \phi \left( \frac{x - \theta_g}{\sigma_g} \right) \right] / \left[ \frac{1}{\sigma_b} \phi \left( \frac{x - \theta_b}{\sigma_b} \right) \right].\] It is easy enough to verify that

\[\hat{x} = \frac{\sigma_b^2 \theta_g - \sigma_g^2 \theta_b}{\sigma_b^2 - \sigma_g^2} < \theta_b,
\]

and that \((a.55)\) becomes

\[|x_1 - \hat{x}| \geq |x_2 - \hat{x}|;\]

that is, the principal retains the agent whose signal is further away from \(\hat{x}\). With these in hand, player \(i\)'s retention probability, when his type is \(\theta_i\), is given by:

\[
\Pi_i = \int_{-\infty}^{\hat{x}} \frac{1}{\sigma_j} \phi \left( \frac{x_j - \theta_j}{\sigma_j} \right) \left( 1 - \Phi \left( \frac{2\hat{x} - x_j - \theta_i}{\sigma_i} \right) + \Phi \left( \frac{x_j - \theta_i}{\sigma_i} \right) \right) dx_j
\]

\[+ \int_{\hat{x}}^{\infty} \frac{1}{\sigma_j} \phi \left( \frac{x_j - \theta_j}{\sigma_j} \right) \left( 1 - \Phi \left( \frac{x_j - \theta_i}{\sigma_i} \right) + \Phi \left( \frac{2\hat{x} - x_j - \theta_i}{\sigma_i} \right) \right) dx_j.
\]
We want to evaluate the derivative of \( \Pi_i \) with respect to \( \sigma_i \) at \( \hat{x} = \frac{\sigma_i^2 \theta_j - \sigma_j^2 \theta_i}{\sigma_i^2 - \sigma_j^2} \), which is given by:

\[
\sigma_i \frac{\partial \Pi_i}{\partial \sigma_i} = \int_{-\infty}^{\hat{x}} \frac{1}{\sigma_i} \phi \left( \frac{x_j - \theta_j}{\sigma_j} \right) \left( \frac{1}{\sigma_i} \phi \left( \frac{2\hat{x} - x_j - \theta_i}{\sigma_i} \right) (2\hat{x} - x_j - \theta_i) - \frac{1}{\sigma_i} \phi \left( \frac{x_j - \theta_i}{\sigma_i} \right) (x_j - \theta_i) \right) dx_j \\
+ \int_{\hat{x}}^{\infty} \frac{1}{\sigma_j} \phi \left( \frac{x_j - \theta_j}{\sigma_j} \right) \left( \frac{1}{\sigma_i} \phi \left( \frac{2\hat{x} - x_j - \theta_i}{\sigma_i} \right) (2\hat{x} - x_j - \theta_i) - \frac{1}{\sigma_i} \phi \left( \frac{x_j - \theta_i}{\sigma_i} \right) (x_j - \theta_i) \right) dx_j \\
= \int_{-\infty}^{\hat{x}} \frac{1}{\sigma_j} \phi \left( \frac{x_j - \theta_j}{\sigma_j} \right) \left( \frac{1}{\sigma_i} \phi \left( \frac{2\hat{x} - x_j - \theta_i}{\sigma_i} \right) (2\hat{x} - x_j - \theta_i) - \frac{1}{\sigma_i} \phi \left( \frac{x_j - \theta_i}{\sigma_i} \right) (x_j - \theta_i) \right) dx_j \\
- \int_{\hat{x}}^{\infty} \frac{1}{\sigma_j} \phi \left( \frac{x_j - \theta_j}{\sigma_j} \right) \left( \frac{1}{\sigma_i} \phi \left( \frac{x_j - \theta_i}{\sigma_i} \right) (x_j - \theta_i) dx_j \\
+ \int_{\hat{x}}^{\infty} \frac{1}{\sigma_j} \phi \left( \frac{x_j - \theta_j}{\sigma_j} \right) \left( \frac{1}{\sigma_i} \phi \left( \frac{x_j - \theta_i}{\sigma_i} \right) (x_j - \theta_i) dx_j \\
- \int_{\hat{x}}^{\infty} \frac{1}{\sigma_j} \phi \left( \frac{x_j - \theta_j}{\sigma_j} \right) \left( \frac{1}{\sigma_i} \phi \left( \frac{2\hat{x} - x_j - \theta_i}{\sigma_i} \right) (2\hat{x} - x_j - \theta_i) dx_j 

The above equation has four terms on the right-hand side, which we will need to manipulate separately. The following expressions involving the normal density will be used repeatedly:

**Lemma A.19.** The normal density \( \phi \) satisfies:

\[
(a.56) \quad \int_a^b \frac{w}{\sigma} \phi \left( \frac{w - \mu}{\sigma} \right) dw = \mu \left[ \Phi \left( \frac{b - \mu}{\sigma} \right) - \Phi \left( \frac{a - \mu}{\sigma} \right) \right] - \sigma \left[ \phi \left( \frac{b - \mu}{\sigma} \right) - \phi \left( \frac{a - \mu}{\sigma} \right) \right]
\]

and

\[
(a.57) \quad \frac{1}{\sigma} \phi \left( \frac{x - \mu}{\sigma} \right) \frac{1}{\omega} \phi \left( \frac{x - \tau}{\omega} \right) = \frac{1}{\sqrt{\sigma^2 + \omega^2}} \phi \left( \frac{x - \mu + \sigma^2 \tau}{\sigma^2 + \omega^2} \right) \frac{1}{\sqrt{\sigma^2 + \omega^2}} \phi \left( \frac{\mu - \tau}{\sigma^2 + \omega^2} \right).
\]

**Proof.** Equation (a.57) follows from standard algebra. As for (a.56), suppose that \( w \sim N \left( \mu, \sigma^2 \right) \). Then, integrating by parts, it is easy to see that:

\[
(a.58) \quad \mathbb{E} \left( w \mid w \in [a, b] \right) = \frac{\int_a^b \frac{w}{\sigma} \phi \left( \frac{w - \mu}{\sigma} \right) dw}{\Phi \left( \frac{b - \mu}{\sigma} \right) - \Phi \left( \frac{a - \mu}{\sigma} \right)} = \frac{b \Phi \left( \frac{b - \mu}{\sigma} \right) - a \Phi \left( \frac{a - \mu}{\sigma} \right) - \int_a^b \phi \left( \frac{w - \mu}{\sigma} \right) dw}{\Phi \left( \frac{b - \mu}{\sigma} \right) - \Phi \left( \frac{a - \mu}{\sigma} \right)}.
\]

Observe that

\[
\Phi \left( \frac{w - \mu}{\sigma} \right) = \frac{\partial}{\partial w} \left( (w - \mu) \Phi \left( \frac{w - \mu}{\sigma} \right) + \sigma \phi \left( \frac{w - \mu}{\sigma} \right) \right).
\]
Using this information in (a.58), we must conclude that

\[
\mathbb{E}(w \in [a, b]) = \frac{b \Phi \left( \frac{b - \mu}{\sigma} \right) - a \Phi \left( \frac{a - \mu}{\sigma} \right) - \int_a^b \frac{\partial}{\partial x} \left( (w - \mu) \Phi \left( \frac{w - \mu}{\sigma} \right) + \sigma \phi \left( \frac{w - \mu}{\sigma} \right) \right) dw}{\Phi \left( \frac{b - \mu}{\sigma} \right) - \Phi \left( \frac{a - \mu}{\sigma} \right)}
\]

\[
= \frac{b \Phi \left( \frac{b - \mu}{\sigma} \right) - a \Phi \left( \frac{a - \mu}{\sigma} \right) - \left( (b - \mu) \Phi \left( \frac{b - \mu}{\sigma} \right) + \sigma \phi \left( \frac{b - \mu}{\sigma} \right) \right) - \left( (a - \mu) \Phi \left( \frac{a - \mu}{\sigma} \right) + \sigma \phi \left( \frac{a - \mu}{\sigma} \right) \right)}{\Phi \left( \frac{b - \mu}{\sigma} \right) - \Phi \left( \frac{a - \mu}{\sigma} \right)}
\]

\[
= \frac{\mu \Phi \left( \frac{b - \mu}{\sigma} \right) - \mu \Phi \left( \frac{a - \mu}{\sigma} \right) - \sigma \phi \left( \frac{b - \mu}{\sigma} \right) + \sigma \phi \left( \frac{a - \mu}{\sigma} \right)}{\Phi \left( \frac{b - \mu}{\sigma} \right) - \Phi \left( \frac{a - \mu}{\sigma} \right)} = \mu - \frac{\phi \left( \frac{b - \mu}{\sigma} \right) - \phi \left( \frac{a - \mu}{\sigma} \right)}{\Phi \left( \frac{b - \mu}{\sigma} \right) - \Phi \left( \frac{a - \mu}{\sigma} \right)}.
\]

With this expression in hand, we see that

\[
\int_a^b \frac{1}{\sigma} \phi \left( \frac{w - \mu}{\sigma} \right) dw = \mathbb{E}[w | w \in [a, b]] \cdot \left( \Phi \left( \frac{b - \mu}{\sigma} \right) - \Phi \left( \frac{a - \mu}{\sigma} \right) \right)
\]

\[
= \mu \left( \Phi \left( \frac{b - \mu}{\sigma} \right) - \Phi \left( \frac{a - \mu}{\sigma} \right) \right) - \sigma \left( \phi \left( \frac{b - \mu}{\sigma} \right) - \phi \left( \frac{a - \mu}{\sigma} \right) \right)
\]

\[
= \frac{\mu}{\Phi \left( \frac{b - \mu}{\sigma} \right) - \Phi \left( \frac{a - \mu}{\sigma} \right)} - \frac{\sigma}{\Phi \left( \frac{b - \mu}{\sigma} \right) - \Phi \left( \frac{a - \mu}{\sigma} \right)} \cdot \frac{\phi \left( \frac{b - \mu}{\sigma} \right) - \phi \left( \frac{a - \mu}{\sigma} \right)}{\Phi \left( \frac{b - \mu}{\sigma} \right) - \Phi \left( \frac{a - \mu}{\sigma} \right)}
\]

We will use (a.56) and (a.57) repeatedly below to analyze the four terms in the expression for \(\sigma_t \frac{\partial}{\partial t} \sigma_t \). The first of these terms is given by:

\[
\int_{-\infty}^{\hat{x}} \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2}} \phi \left( \frac{x_j - \theta_j}{\sigma_j} \right) \frac{1}{\sigma_i} \phi \left( \frac{2\hat{x} - x_j - \theta_i}{\sigma_i} \right) \left( 2\hat{x} - x_j - \theta_i \right) dx_j
\]

\[
= (2\hat{x} - \theta_i) \int_{-\infty}^{\hat{x}} \frac{1}{\sigma_j} \phi \left( \frac{x_j - \theta_j}{\sigma_j} \right) \frac{1}{\sigma_i} \phi \left( \frac{2\hat{x} - x_j - \theta_i}{\sigma_i} \right) dx_j
\]

\[
= (2\hat{x} - \theta_i) \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2}} \phi \left( \frac{\theta_j + \theta_i - 2\hat{x}}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right) \int_{-\infty}^{\hat{x}} \frac{1}{\sigma_j} \phi \left( \frac{x_j - \frac{\sigma_1^2 \theta_j + \sigma_2^2 (2\hat{x} - \theta_i)}{\sigma_1^2 + \sigma_2^2}}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right) dx_j
\]

\[
= (2\hat{x} - \theta_i) \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2}} \phi \left( \frac{\theta_j + \theta_i - 2\hat{x}}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right) \int_{-\infty}^{\hat{x}} \frac{1}{\sigma_j} \phi \left( \frac{x_j - \frac{\sigma_1^2 \theta_j + \sigma_2^2 (2\hat{x} - \theta_i)}{\sigma_1^2 + \sigma_2^2}}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right) \Phi \left( \frac{\hat{x} - \frac{\sigma_1^2 \theta_j + \sigma_2^2 (2\hat{x} - \theta_i)}{\sigma_1^2 + \sigma_2^2}}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right) dx_j
\]

\[
= (2\hat{x} - \theta_i) \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2}} \phi \left( \frac{\theta_j + \theta_i - 2\hat{x}}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right) \times \left[ \frac{\sigma_1^2 \theta_j + \sigma_2^2 (2\hat{x} - \theta_i)}{\sigma_1^2 + \sigma_2^2} \Phi \left( \frac{\hat{x} - \frac{\sigma_1^2 \theta_j + \sigma_2^2 (2\hat{x} - \theta_i)}{\sigma_1^2 + \sigma_2^2}}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right) - \frac{\sigma_j \sigma_i}{\sqrt{\sigma_1^2 + \sigma_2^2}} \Phi \left( \frac{\hat{x} - \frac{\sigma_1^2 \theta_j + \sigma_2^2 (2\hat{x} - \theta_i)}{\sigma_1^2 + \sigma_2^2}}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right) \right]
\]
In similar vein, and again using (a.57), the second term is given by:

\[
\begin{align*}
- \int_{-\infty}^{\infty} & \frac{1}{\sigma_j} \phi \left( \frac{x_j - \theta_j}{\sigma_j} \right) \frac{1}{\sigma_i} \phi \left( \frac{x_j - \theta_i}{\sigma_i} \right) (x_j - \theta_i) \, dx_j \\
& = - \int_{-\infty}^{\infty} \frac{1}{\sigma_j} \phi \left( \frac{x_j - \theta_j}{\sigma_j} \right) \frac{1}{\sigma_i} \phi \left( \frac{x_j - \theta_i}{\sigma_i} \right) x_j \, dx_j + \theta_i \int_{-\infty}^{\infty} \frac{1}{\sigma_j} \phi \left( \frac{x_j - \theta_j}{\sigma_j} \right) \frac{1}{\sigma_i} \phi \left( \frac{x_j - \theta_i}{\sigma_i} \right) \, dx_j \\
& = - \frac{1}{\sqrt{\sigma_i^2 + \sigma_j^2}} \phi \left( \frac{\theta_j - \theta_i}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{\sigma_i^2 + \sigma_j^2}} \phi \left( \frac{x_j - \frac{\sigma_i^2 \theta_j + \sigma_i^2 \theta_i}{\sigma_i^2 + \sigma_j^2}}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) x_j \, dx_j \\
+ & \theta_i \frac{1}{\sqrt{\sigma_i^2 + \sigma_j^2}} \phi \left( \frac{\theta_j - \theta_i}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{\sigma_i^2 + \sigma_j^2}} \phi \left( \frac{x_j - \frac{\sigma_i^2 \theta_j + \sigma_i^2 \theta_i}{\sigma_i^2 + \sigma_j^2}}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \, dx_j \\
& = - \frac{1}{\sqrt{\sigma_i^2 + \sigma_j^2}} \phi \left( \frac{\theta_j - \theta_i}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \int_{-\infty}^{\infty} \phi \left( \frac{x_j - \frac{\sigma_i^2 \theta_j + \sigma_i^2 \theta_i}{\sigma_i^2 + \sigma_j^2}}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) dx_j \\
+ & \theta_i \frac{1}{\sqrt{\sigma_i^2 + \sigma_j^2}} \phi \left( \frac{\theta_j - \theta_i}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \Phi \left( \frac{\hat{x} - \frac{\sigma_i^2 \theta_j + \sigma_i^2 \theta_i}{\sigma_i^2 + \sigma_j^2}}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) - \frac{\sigma_i \sigma_j}{\sqrt{\sigma_i^2 + \sigma_j^2}} \phi \left( \frac{\hat{x} - \frac{\sigma_i^2 \theta_j + \sigma_i^2 \theta_i}{\sigma_i^2 + \sigma_j^2}}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \\
& + \theta_i \frac{1}{\sqrt{\sigma_i^2 + \sigma_j^2}} \phi \left( \frac{\theta_j - \theta_i}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \Phi \left( \frac{\hat{x} - \frac{\sigma_i^2 \theta_j + \sigma_i^2 \theta_i}{\sigma_i^2 + \sigma_j^2}}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right).
\end{align*}
\]

The third term translates to:

\[
\begin{align*}
\int_{-\infty}^{\infty} & \frac{1}{\sigma_j} \phi \left( \frac{x_j - \theta_j}{\sigma_j} \right) \frac{1}{\sigma_i} \phi \left( \frac{x_j - \theta_i}{\sigma_i} \right) (x_j - \theta_i) \, dx_j \\
& = \int_{-\infty}^{\infty} \frac{1}{\sigma_j} \phi \left( \frac{x_j - \theta_j}{\sigma_j} \right) \frac{1}{\sigma_i} \phi \left( \frac{x_j - \theta_i}{\sigma_i} \right) x_j \, dx_j - \theta_i \int_{-\infty}^{\infty} \frac{1}{\sigma_j} \phi \left( \frac{x_j - \theta_j}{\sigma_j} \right) \frac{1}{\sigma_i} \phi \left( \frac{x_j - \theta_i}{\sigma_i} \right) \, dx_j \\
& = \frac{1}{\sqrt{\sigma_i^2 + \sigma_j^2}} \phi \left( \frac{\theta_j - \theta_i}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{\sigma_i^2 + \sigma_j^2}} \phi \left( \frac{x_j - \frac{\sigma_i^2 \theta_j + \sigma_i^2 \theta_i}{\sigma_i^2 + \sigma_j^2}}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) x_j \, dx_j \\
- & \theta_i \frac{1}{\sqrt{\sigma_i^2 + \sigma_j^2}} \phi \left( \frac{\theta_j - \theta_i}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{\sigma_i^2 + \sigma_j^2}} \phi \left( \frac{x_j - \frac{\sigma_i^2 \theta_j + \sigma_i^2 \theta_i}{\sigma_i^2 + \sigma_j^2}}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \, dx_j \\
& = \frac{1}{\sqrt{\sigma_i^2 + \sigma_j^2}} \phi \left( \frac{\theta_j - \theta_i}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \left[ \frac{\sigma_i^2 \theta_j + \sigma_i^2 \theta_i}{\sigma_i^2 + \sigma_j^2} \right] - \Phi \left( \frac{\hat{x} - \frac{\sigma_i^2 \theta_j + \sigma_i^2 \theta_i}{\sigma_i^2 + \sigma_j^2}}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) + \frac{\sigma_i \sigma_j}{\sqrt{\sigma_i^2 + \sigma_j^2}} \phi \left( \frac{\hat{x} - \frac{\sigma_i^2 \theta_j + \sigma_i^2 \theta_i}{\sigma_i^2 + \sigma_j^2}}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \\
- & \theta_i \frac{1}{\sqrt{\sigma_i^2 + \sigma_j^2}} \phi \left( \frac{\theta_j - \theta_i}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \left[ 1 - \Phi \left( \frac{\hat{x} - \frac{\sigma_i^2 \theta_j + \sigma_i^2 \theta_i}{\sigma_i^2 + \sigma_j^2}}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \right].
\end{align*}
\]
and finally, the fourth term is given by

\[- \int_{\mathbb{R}} \frac{1}{\sigma_j} \phi \left( \frac{x_j - \theta_j}{\sigma_j} \right) \frac{1}{\sigma_i} \phi \left( \frac{2 \hat{x}_j - x_j - \theta_j}{\sigma_i} \right) (2 \hat{x}_j - x_j - \theta_j) \, dx_j \]

\[= - \frac{(2 \hat{x}_j - \theta_j)}{\sqrt{\sigma_i^2 + \sigma_j^2}} \int_{\mathbb{R}} \frac{1}{\sigma_j} \phi \left( \frac{x_j - \theta_j - 2 \hat{x}_j}{\sigma_j} \right) \phi \left( \frac{x_j - \theta_j + \sigma_i^2 (2 \hat{x}_j - \theta_j)}{\sigma_i \sigma_j} \sigma_j^2 + \sigma_i^2 \right) \, dx_j \]

\[\quad + \frac{1}{\sqrt{\sigma_i^2 + \sigma_j^2}} \phi \left( \frac{\theta_j + \theta_i - 2 \hat{x}_j}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \int_{\mathbb{R}} x_j d\phi \left( \frac{x_j - \theta_j + \sigma_i^2 (2 \hat{x}_j - \theta_j)}{\sigma_i \sigma_j} \sigma_j^2 + \sigma_i^2 \right) \]

Summing all these terms up, we can conclude that:

\[\sigma_i \frac{\partial \Pi_i}{\partial \sigma_i} = \frac{\sigma_i^2}{\sqrt{\sigma_i^2 + \sigma_j^2}} \phi \left( \frac{\theta_j + \theta_i - 2 \hat{x}_j}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \left( 2 \Phi \left( \frac{\hat{x}_j - \sigma_i^2 \theta_j + \sigma_i^2 (2 \hat{x}_j - \theta_j)}{\sigma_i \sigma_j} \sqrt{\sigma_i^2 + \sigma_j^2} \right) - 1 \right) \left( \frac{2 \hat{x}_j - \theta_i - \theta_j}{\sigma_i^2 + \sigma_j^2} \right) \]

\[\quad + \frac{\sigma_i^2}{\sqrt{\sigma_i^2 + \sigma_j^2}} \phi \left( \frac{\theta_j - \theta_i}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \left( 2 \Phi \left( \frac{\hat{x}_j - \sigma_i^2 \theta_j + \sigma_i^2 \theta_j}{\sigma_i \sigma_j} \sqrt{\sigma_i^2 + \sigma_j^2} \right) - 1 \right) \left( \frac{\theta_i - \theta_j}{\sigma_i^2 + \sigma_j^2} \right) \]

\[\quad + 2 \frac{\sigma_i \sigma_j}{\sigma_i^2 + \sigma_j^2} \phi \left( \frac{\theta_j + \theta_i - 2 \hat{x}_j}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \phi \left( \frac{\hat{x}_j - \sigma_i^2 \theta_j + \sigma_i^2 (2 \hat{x}_j - \theta_j)}{\sigma_i \sigma_j} \sqrt{\sigma_i^2 + \sigma_j^2} \right) \]

\[\quad + 2 \frac{\sigma_i \sigma_j}{\sigma_i^2 + \sigma_j^2} \phi \left( \frac{\theta_j - \theta_i}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \phi \left( \frac{\hat{x}_j - \sigma_i^2 \theta_j + \sigma_i^2 \theta_j}{\sigma_i \sigma_j} \sqrt{\sigma_i^2 + \sigma_j^2} \right) \cdot \]

Now notice that

\[\phi \left( \frac{\theta_j + \theta_i - 2 \hat{x}_j}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \phi \left( \frac{\hat{x}_j - \sigma_i^2 \theta_j + \sigma_i^2 (2 \hat{x}_j - \theta_j)}{\sigma_i \sigma_j} \sqrt{\sigma_i^2 + \sigma_j^2} \right) = \phi \left( \frac{\theta_j - \theta_i}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \phi \left( \frac{\hat{x}_j - \sigma_i^2 \theta_j + \sigma_i^2 \theta_j}{\sigma_i \sigma_j} \sqrt{\sigma_i^2 + \sigma_j^2} \right), \]
so that

\[
\frac{\partial \Pi_i}{\partial \sigma_i} = \frac{\sigma_i^2}{\sqrt{\sigma_i^2 + \sigma_j^2}} \phi \left( \frac{\theta_j - \theta_i}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \left( 2 \Phi \left( \frac{\hat{x} - \frac{\sigma_i^2 \theta_i + \sigma_j^2 \theta_j}{\sigma_i^2 + \sigma_j^2}}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) - 1 \right) \frac{\sqrt{\sigma_i^2 + \sigma_j^2}}{\sigma_i^2 + \sigma_j^2}
\]

\[
+ \frac{\sigma_i^2}{\sqrt{\sigma_i^2 + \sigma_j^2}} \phi \left( \frac{\theta_j - \theta_i}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \left( 2 \Phi \left( \frac{\hat{x} - \frac{\sigma_i^2 \theta_i + \sigma_j^2 \theta_j}{\sigma_i^2 + \sigma_j^2}}{\sigma_i^2 + \sigma_j^2} \right) - 1 \right) \frac{\theta_i - \theta_j}{\sigma_i^2 + \sigma_j^2}
\]

\[
+ 4 \frac{\sigma_j \sigma_i}{\sigma_i^2 + \sigma_j^2} \phi \left( \frac{\theta_j - \theta_i}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \phi \left( \frac{\hat{x} - \frac{\sigma_i^2 \theta_i + \sigma_j^2 \theta_j}{\sigma_i^2 + \sigma_j^2}}{\sigma_i^2 + \sigma_j^2} \right)
\]

Evaluated at \( \hat{x} = \frac{\sigma_i^2 \theta_i - \sigma_j^2 \theta_j}{\sigma_i^2 - \sigma_j^2} \), we obtain

\[
\frac{\partial \Pi_i}{\partial \sigma_i} = \frac{\sigma_i^2}{\sqrt{\sigma_i^2 + \sigma_j^2}} \phi \left( \frac{\theta_j - \theta_i}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \left( 2 \Phi \left( \frac{\hat{x} - \frac{\sigma_i^2 \theta_i + \sigma_j^2 \theta_j}{\sigma_i^2 + \sigma_j^2}}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) - 1 \right) \frac{\theta_i - \theta_j}{\sigma_i^2 + \sigma_j^2}
\]

\[
+ 4 \frac{\sigma_j \sigma_i}{\sigma_i^2 + \sigma_j^2} \phi \left( \frac{\theta_j - \theta_i}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \phi \left( \frac{\hat{x} - \frac{\sigma_i^2 \theta_i + \sigma_j^2 \theta_j}{\sigma_i^2 + \sigma_j^2}}{\sigma_i^2 + \sigma_j^2} \right)
\]

Because \( \hat{x} < \theta_b < \frac{\sigma_i^2 \theta_i + \sigma_j^2 \theta_j}{\sigma_i^2 + \sigma_j^2} \), we have that \( \Phi \left( \frac{\hat{x} - \frac{\sigma_i^2 \theta_i + \sigma_j^2 \theta_j}{\sigma_i^2 + \sigma_j^2}}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) < \frac{1}{2} \). Therefore \( \partial \Pi_b / \partial \sigma_b > 0 \), whereas the sign of \( \partial \Pi_g / \partial \sigma_g \) is ambiguous. The fact that \( \partial \Pi_b / \partial \sigma_b > 0 \) indicated that the bad type wants to deviate, and the equilibrium is falls apart.

The result also holds in the case of costly noise. If we are in the model of Section 6.6, notice that

\[
\frac{\partial \Pi_b}{\partial \sigma_b} - \frac{\partial \Pi_g}{\partial \sigma_g} = \frac{\sigma_b^2 + \sigma_g^2}{\sqrt{\sigma_b^2 + \sigma_g^2}} \phi \left( \frac{\theta_g - \theta_b}{\sqrt{\sigma_b^2 + \sigma_g^2}} \right) \left[ 2 \Phi \left( \frac{\hat{x} - \frac{\sigma_b^2 \theta_b + \sigma_g^2 \theta_g}{\sigma_b^2 + \sigma_g^2}}{\sqrt{\sigma_b^2 + \sigma_g^2}} \right) - 1 \right] \frac{\theta_g - \theta_b}{\sigma_b^2 + \sigma_g^2} > 0,
\]

so that, because \( \partial \Pi_k / \partial \sigma_k = c'(\sigma_k) \) for \( k = g, b \), we have

\[
c'(\sigma_b) \sigma_b > c'(\sigma_g) \sigma_g,
\]

and moreover, \( \sigma_b > \sigma \) (because \( \partial \Pi_b / \partial \sigma_b > 0 \)). Therefore the above inequality implies that \( \sigma_b > \sigma_g \), a contradiction.

We are left with only one possibility, \( \sigma_b > \sigma_g \), where the principal retains 1 if, and only if,

\[
|x_1 - \hat{x}| \leq |x_2 - \hat{x}|,
\]
so the principal retains the agent whose signal is now closer to \( \hat{x} := \frac{\theta_i - \theta_g}{\sigma_i^2 - \sigma_g^2} > \theta_g \), which now maximizes the likelihood ratio \( \left[ \frac{1}{\sigma_g} \phi \left( \frac{x - \theta_g}{\sigma_g} \right) \right] \left[ \frac{1}{\sigma_i} \phi \left( \frac{x - \theta_i}{\sigma_i} \right) \right] \). The objective function of \( i \) is:

\[
\Pi_i (\sigma_i; \sigma_j; \hat{x}) = \int_{-\infty}^{\hat{x}} \frac{1}{\sigma_j} \phi \left( \frac{x_j - \theta_j}{\sigma_j} \right) \left( \Phi \left( \frac{2\hat{x} - x_j - \theta_i}{\sigma_i} \right) - \Phi \left( \frac{x_j - \theta_i}{\sigma_i} \right) \right) dx_j + \int_{\hat{x}}^{\infty} \frac{1}{\sigma_j} \phi \left( \frac{x_j - \theta_j}{\sigma_j} \right) \left( \Phi \left( \frac{x_j - \theta_i}{\sigma_i} \right) - \Phi \left( \frac{2\hat{x} - x_j - \theta_i}{\sigma_i} \right) \right) dx_j.
\]

Momentarily ignoring the fact that \( \hat{x} \) has a different value than before (because we have different values of \( \sigma \)), clearly, this probability and \( i \)'s previous probability of retention in the case where \( \sigma_g > \sigma_i \) add up to 1: before, \( i \) was elected if, for a given \( x_j \), his own signal fell outside a given interval, whereas now \( i \) is elected if the signal falls in the complementary set. Therefore, the derivative with respect to \( \sigma_i \) evaluated at \( \hat{x} = \frac{\theta_i - \theta_g}{\sigma_i^2 - \sigma_g^2} \) satisfies

\[
\frac{\partial \Pi_i (\sigma_i; \sigma_j; \hat{x})}{\partial \sigma_i} = -\frac{\sigma_i^2}{\sqrt{\sigma_i^2 + \sigma_j^2}} \phi \left( \frac{\theta_j - \theta_i}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \left( 2\Phi \left( \frac{\hat{x} - \frac{\theta_i + \theta_j}{\sigma_i^2 + \sigma_j^2}}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) - 1 \right) \left( \frac{\theta_i - \theta_j}{\sigma_i^2 + \sigma_j^2} \right) - 4 \frac{\sigma_j \sigma_i}{\sigma_i^2 + \sigma_j^2} \phi \left( \frac{\theta_j - \theta_i}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) \phi \left( \hat{x} - \frac{\theta_i + \theta_j}{\sigma_i^2 + \sigma_j^2} \right) \left( \frac{\theta_i + \theta_j}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right).
\]

Now \( \hat{x} > \theta_g > \frac{\theta_i + \theta_j}{\sigma_i^2 + \sigma_j^2} \), so \( \Phi \left( \frac{\hat{x} - \frac{\theta_i + \theta_j}{\sigma_i^2 + \sigma_j^2}}{\sqrt{\sigma_i^2 + \sigma_j^2}} \right) > \frac{1}{2} \). Then,

\[
\frac{\partial \Pi_g (\sigma_g; \sigma_b; \hat{x})}{\partial \sigma_g} < 0,
\]

whereas the sign of \( \frac{\partial \Pi_b (\sigma_b; \sigma_g; \hat{x})}{\partial \sigma_b} \) is ambiguous. There is no clear contradiction here.

Once again, we can see that the result holds in the case of costly noise. Notice that

\[
\frac{\partial \Pi_b (\sigma_b; \sigma_g; \hat{x})}{\partial \sigma_b} - \frac{\partial \Pi_g (\sigma_g; \sigma_b; \hat{x})}{\partial \sigma_g} \sigma_g = \frac{\sigma_b^2}{\sqrt{\sigma_g^2 + \sigma_b^2}} \phi \left( \frac{\theta_g - \theta_b}{\sqrt{\sigma_g^2 + \sigma_b^2}} \right) \left( 2\Phi \left( \frac{\hat{x} - \frac{\theta_b + \theta_g}{\sigma_g^2 + \sigma_b^2}}{\sqrt{\sigma_g^2 + \sigma_b^2}} \right) - 1 \right) \left( \frac{\theta_g - \theta_b}{\sigma_g^2 + \sigma_b^2} \right) > 0.
\]

Then

\[
c' (\sigma_b) \sigma_b > c' (\sigma_g) \sigma_g.
\]

So, once again, in principle there is no contradiction here. Moreover, if \( c' (\sigma) \sigma \) is always increasing, this inequality is consistent with \( \sigma_b > \sigma_g \).
6. PROOF OF PROPOSITION 11 IN SECTION 6.7

Consider a situation in which each type \( \theta \) chooses some noise \( \sigma(\theta) \). Then signal emitted by type \( \theta \) has density

\[
\pi_\theta(x) = \frac{1}{\sigma(\theta)} \phi \left( \frac{x - \theta}{\sigma(\theta)} \right).
\]

Let \( U(x) \) be the expected payoff to the principal when the signal \( x \) is received. This is just the expected value of \( u(\theta) \) weighted by the posterior distribution of \( \theta \) using Bayes’ Rule and the strategies, as described above. So

\[
U(x) = \frac{1}{\int \pi_\theta(x) q(\theta) d\theta} \int_{-\infty}^{\infty} u(\theta) \frac{1}{\sigma(\theta)} \phi \left( \frac{x - \theta}{\sigma(\theta)} \right) q(\theta) d\theta.
\]  

(a.59)

**Lemma A.20.** Suppose that \( \sigma(\theta) \) is continuous in \( \theta \) and has a unique maximum at \( \theta^* \). Then \( U(x) \) converges to \( u(\theta^*) \) as \( |x| \to \infty \).

**Proof.** Pick any sequence \( x_n \) such that \( x_n \to \infty \) (the argument for \( x_n \to -\infty \) will be identical). Define a corresponding sequence of density functions on \( \mathbb{R} \), \( h_n \), by

\[
h_n(\theta) = \frac{1}{\int \pi_t(x_n) q(t) dt} \frac{q(\theta)}{\sigma(\theta)} \phi \left( \frac{x - \theta}{\sigma(\theta)} \right),
\]

and let \( H_n(\theta) = \int_{-\infty}^{\theta} h_n(s) ds \) be the corresponding sequence of cdfs. We claim that this sequence of probability measures converges weakly to the degenerate probability measure placing probability 1 on \( \theta^* \).

To prove the claim, first pick any \( \theta < \theta^* \). Let \( \sigma_1 \) be the maximum value of \( \sigma(s) \) for \( s \leq \theta \). Because \( \sigma(\theta) \) is uniquely maximized at \( \theta^* \) and \( \theta^* > \theta \), there exists an interval of length \( \epsilon \) around \( \theta^* \) such that \( \min \sigma(s) \) for \( s \) in that interval --- call it \( \sigma_2 \) --- strictly exceeds \( \sigma_1 \). Denote by \( Q(\theta) \) the prior mass of types up to \( \theta \), and by \( \Delta_Q \) the prior mass in the \( \epsilon \)-interval around \( \theta^* \). With these values fixed, observe that for \( n \) large enough so that \( x_n > \theta \),

\[
H_n(\theta) = \frac{\int_{-\infty}^{\theta} \frac{q(s)}{\sigma(s)} \exp \left\{ -\frac{1}{2} \left[ \frac{x_n - s}{\sigma(s)} \right]^2 \right\} ds}{\int_{-\infty}^{\infty} \frac{q(t)}{\sigma(t)} \exp \left\{ -\frac{1}{2} \left[ \frac{x_n - t}{\sigma(t)} \right]^2 \right\} dt} \\
\leq \frac{\frac{Q(\theta)}{\sigma_1} \exp \left\{ -\frac{1}{2} \left[ \frac{x_n - \theta}{\sigma_1} \right]^2 \right\}}{\frac{\Delta_Q}{\sigma_2} \exp \left\{ -\frac{1}{2} \left[ \frac{x_n - (\theta^* - \epsilon)}{\sigma_2} \right]^2 \right\}} \\
\to 0
\]
as \( n \to \infty \), where the very last conclusion uses \( \sigma_1 < \sigma_2 \).

By symmetrically applying the same logic to the “other side” of \( \theta^* \), we must also conclude that \( 1 - H_n(\theta) \to 0 \) for each \( \theta > \theta^* \). It follows that \( H_n(\theta) \to 1 \) for each \( \theta > \theta^* \). That completes the proof of convergence to the degenerate cdf placing all weight on \( \theta^* \).

By a standard characterization of weak convergence, and using the fact that \( u(\theta) \) is a bounded, continuous function, it follows that

\[
U(x_n) = \int_{-\infty}^{\infty} u(\theta) h_n(\theta) d\theta \to u(\theta^*).
\]

**Lemma A.21.** Assume Condition U. Consider any monotone retention threshold \( x^* \). Then any optimal choice function by an agent of type \( \theta \) only depends on the difference \( t = x^* - \theta \) and on that agent’s payoffs; in particular, it does not depend on the type distribution \( q(\theta) \). Call this function \( s(t) \). It is continuous. If the retention zone is \( [x^*, \infty) \), then \( s(t) \) attains a unique maximum at some \( t_1 > 0 \). If the retention zone is \( (-\infty, x^*] \), then \( s(t) \) attains a unique maximum at some \( t_2 < 0 \).

**Proof.** An agent of type \( \theta \) chooses \( \sigma \) to maximize

\[
1 - \Phi\left( \frac{x^* - \theta}{\sigma} \right) - c(\sigma)
\]

if the retention zone is \( [x^*, \infty) \), and

\[
\Phi\left( \frac{x^* - \theta}{\sigma} \right) - c(\sigma)
\]

if the retention zone is \( (-\infty, x^*] \). Just these expressions make it clear that the solution \( \sigma \) can only depend on \( t = x^* - \theta \). By Condition U, the solution is unique and therefore easily seen to be continuous. The first order condition with retention zone \( [x^*, \infty) \) is given by

\[
(\text{a.60}) \quad \phi\left( \frac{x^* - \theta}{\sigma} \right) \frac{x^* - \theta}{\sigma^2} - c'(\sigma) = 0.
\]

By Condition U, (a.60) is necessary and sufficient for a maximum. When \( x^* > \theta \), the corresponding value of \( \sigma \) exceeds \( \sigma^* \), and using the fact that \( \sigma c'(\sigma) \) is increasing when \( \sigma \geq \sigma^* \), we see that the maximum possible value of \( \sigma \) satisfying (a.60) is achieved when

\[
\sigma c'(\sigma) = \phi\left( \frac{x^* - \theta}{\sigma} \right) \frac{x^* - \theta}{\sigma} = \phi(z^*) z^* = \phi(z^*) z^* = \phi(z^*) z^*,
\]

where \( z^* \) is the value that maximizes \( \phi(z) z \). That is, define \( \sigma_1 \) by the first and last terms in the equality above and then set \( x^* - \theta = t_1 = \sigma_1 z^* \) to define \( t_1 \). When the retention zone is \( (-\infty, x^*] \), the first order condition is given by

\[
(\text{a.61}) \quad -\phi\left( \frac{x^* - \theta}{\sigma} \right) \frac{x^* - \theta}{\sigma^2} - c'(\sigma) = 0.
\]

\footnote{The values \( \sigma_* \) and \( \sigma^* \) are the lowest and highest values that noise could optimally have, as per the discussion in the main text.}
Now the corresponding value of $\sigma$ exceeds $\sigma$ when $x^* < \theta$. By a parallel argument to the one just made, the maximum possible value of $\sigma$ satisfying (a.60) is achieved when

$$\sigma c'(\sigma) = -\phi \left( \frac{x^* - \theta}{\sigma} \right) \frac{x^* - \theta}{\sigma} = -\phi(z_\ast)z_\ast,$$

where $z_\ast$ is the value that minimizes $\phi(z)z$ ($z_\ast$ will be negative). Define $\sigma_2$ by the first and last terms in the equality above and then set $x^* - \theta = t_2 = \sigma_2 z_\ast$ to define $t^*$.

Lemma A.22. Let $t^*$ stand for $t_1$ or $t_2$ as defined in Lemma A.21. Then $u(x^* - t^*) = V$.

Proof. We consider the retention zone $[x^*, \infty)$ where $t^* = t_1$; the other case is dealt with in identical fashion. By Lemmas A.20 and A.21, $U(x)$ converges to $u(x^* - t_1)$ as $|x| \to \infty$. Suppose that $u(x^* - t_1) > V$. Then for $x$ negative and large in absolute value — in particular for some $x < x^*$ — we would have $U(x) > V$, so that the principal must retain for such values. That contradicts monotone retention. Similarly, if $u(x^* - t_1) < V$, then for $x$ large — in particular for some $x > x^*$ — we would have $U(x) < V$, so that the principal must replace for such values. Once again, that contradicts monotone retention. We are therefore left with just one possibility: $u(x^* - t_1) = V$.

Lemma A.23. $U(x^*) = V$.

Proof. By monotone retention, $U(x^* - \epsilon) \leq V \leq U(x^* + \epsilon)$ (or $U(x^* - \epsilon) \geq V \geq U(x^* + \epsilon)$). $U$ is obviously continuous, so the result follows.

Lemma A.23 combined with (a.59) tells us that

$$\frac{1}{\int \pi_\theta(x^*) q(\theta) d\theta} \int_{-\infty}^{\infty} u(\theta) \frac{1}{s(x^* - \theta)} \phi \left( \frac{x^* - \theta}{s(x^* - \theta)} \right) q(\theta) d\theta = V,$$

where $s(t)$ is the optimal noise choice function as defined in Lemma A.21. Using the formula for $\pi_\theta(x)$ and transposing terms, we have

$$\int_{-\infty}^{\infty} u(\theta) - V \frac{1}{s(x^* - \theta)} \phi \left( \frac{x^* - \theta}{s(x^* - \theta)} \right) q(\theta) d\theta = 0.$$

Lemma A.22 pins down $x^*$ uniquely:

$$x^* = u^{-1}(V) + t^*,$$

so that combining these two inequalities, we conclude that

(a.62) \[ \int_{-\infty}^{\infty} h(\theta) q(\theta) d\theta = 0, \]

where

$$h(\theta) = \frac{u(\theta) - V}{s(u^{-1}(V) + t^* - \theta)} \phi \left( \frac{u^{-1}(V) + t^* - \theta}{s(u^{-1}(V) + t^* - \theta)} \right)$$

is a function that depends on model parameters but is entirely independent of the particular density $\{ q(\theta) \}$; see Lemma A.21. Let $Q$ be the set of all densities on $\mathbb{R}$ equipped with the topology induced by the sup norm, and let $Q^0$ be the subset of densities in $Q$ that satisfy (a.62). It is obvious that $Q - Q^0$ is open and dense in $Q$. ■
7. Examples

7.1. (Non-Generic) Example of a Monotone Equilibrium in the Costly Noise Model. In the main text we have argued that, in the costly noise model of Section 6.6, with a monotone regime \( X = [x^*, \infty) \) with \( x^* \in [\theta_b, \theta_g] \) the bad type will optimally respond by choosing \( \sigma_b > \sigma \), whereas the good type will play \( \sigma_g < \sigma \), so this cannot be an equilibrium. Consider then the case \( x^* \notin [\theta_b, \theta_g] \). Type-\( k \) agent’s objective function is:

\[
1 - \Phi \left( \frac{x^* - \theta_k}{\sigma_k} \right) - c(\sigma_k)
\]

and the corresponding first order condition is

\[
\phi \left( \frac{x^* - \theta_k}{\sigma_k} \right) \frac{x^* - \theta_k}{\sigma_k^2} - c'(\sigma_k) = 0
\]

If \( \sigma_g = \sigma_b = \sigma \), the two first-order condition together imply that

\[
\phi \left( \frac{x^* - \theta_g}{\sigma} \right) (x^* - \theta_g) = \phi \left( \frac{x^* - \theta_b}{\sigma} \right) (x^* - \theta_b).
\]

Furthermore, from the principal’s indifference condition, we have that

\[
\beta \phi \left( \frac{x^* - \theta_g}{\sigma} \right) = \phi \left( \frac{x^* - \theta_b}{\sigma} \right),
\]

which determines the value of \( x^* \):

\[
x^* = \frac{\theta_g + \theta_b}{2} - \frac{\sigma^2}{\theta_g - \theta_b} \ln(\beta).
\]

If we use this indifference condition in equation (a.63) we obtain the following:

\[
(x^* - \theta_g) = \beta (x^* - \theta_b),
\]

or

\[
x^* = \frac{\theta_g - \beta \theta_b}{1 - \beta}.
\]

Now, combining this expression for \( x^* \) with (a.64) the equilibrium value of \( \sigma \) is fully determined:

\[
\sigma = \Delta \sqrt{\frac{1}{2 \ln(\beta)}} \frac{1 + \beta}{\beta - 1},
\]

where \( \Delta := \theta_g - \theta_b \).

Assume \( \beta < 1 \), so \( x^* > \theta_k \) \( \forall k \). The good type’s first-order condition is

\[
c'(\sigma) = \phi \left( \frac{\beta \Delta}{1 - \beta} \right) \frac{\Delta}{\sigma^2 (1 - \beta)} \beta
\]

\[
= \phi \left( \sqrt{2 \ln(\beta)} \frac{\beta - 1}{1 + \beta} \frac{\beta}{1 - \beta} \right) \frac{2 \ln(1/\beta)}{\Delta} \frac{\beta}{1 + \beta} > 0
\]
Let \( c(\sigma) = \frac{1}{2\sigma}(\sigma - \sigma)^2 \forall \sigma \geq \sigma. \) Then the condition reads

\[
\frac{\sigma - \sigma}{\sigma} = \phi \left( \sqrt{2 \ln (\beta)} \frac{\beta - 1}{1 + \beta} \frac{\beta}{1 - \beta} \right) \frac{2}{\Delta} \ln \left( \frac{1}{\beta} \right) \frac{\beta}{1 + \beta},
\]

or

\[
\sigma = \frac{\Delta \sqrt{\frac{1}{2 \ln (\beta)} \frac{1 + \beta}{\beta - 1}}}{1 + \phi \left( \sqrt{2 \ln (\beta)} \frac{\beta - 1}{1 + \beta} \frac{\beta}{1 - \beta} \right) \frac{2}{\Delta} \ln \left( \frac{1}{\beta} \right) \frac{\beta}{1 + \beta}}.
\]

### 7.2. Bounded Replacement Equilibria in the Costly Noise Model

The following Proposition describes necessary conditions for bounded replacement equilibria in the costly noise model of Section 6.6.

**Proposition A.1.** Suppose a bounded replacement equilibrium exists. Then, either

(i) \( \sigma > \sigma_g > \sigma_b \) and \( \beta > 1 \) and large enough so that \( x_+ < x_- < \theta_b < \theta_g; \) or

(ii) \( \sigma_g > \sigma_b > \sigma \) and \( \beta < 1 \) and small enough so that \( x_+ < \theta_b < \theta_g < x_- \).

The proof of Proposition A.1 follows from the following set of lemmas.

**Lemma A.24.** Suppose the principal retains and replaces according to some non-trivial, non-monotone rule that satisfies \( x_- \in [\theta_b, \theta_g] \). Then, the agents’ best responses satisfy \( \sigma_b > \sigma_g \).

**Proof.** As we did in the proof of Proposition 10, it is enough to show that, under the premise that \( x_- \in [\theta_b, \theta_g] \), type-b’s marginal benefit of noise strictly exceeds that for the good type at every noise level. That is, \( B_b(\sigma) > B_g(\sigma) \) for all \( \sigma \). Then, by a simple single-crossing argument, we must have \( \sigma_b > \sigma_g \).

In the case in which the principal conjectures that \( \sigma_g > \sigma_b \), so she replaces inside \([x_+, x_-]\) with \( x_+ < x_- \), we obtain \( B_b(\sigma) > B_g(\sigma) \) for all \( \sigma \) by following the exact same steps as in (50). For this we need inequality (46) of Lemma 11, and \( x_+ < \theta_b \), which is established by Lemma A.3.
In the case in which the principal conjectures that \( \sigma_g < \sigma_b \), she retains inside \([x_-, x_+]\) with \( x_- < x_+ \). Then

\[
MgB_g(\sigma) = \phi \left( \frac{x_+ - \theta_g}{\sigma} \right) \frac{x_+ - \theta_g}{\sigma^2} - \phi \left( \frac{x_- - \theta_g}{\sigma} \right) \frac{x_+ - \theta_g}{\sigma^2} \\
\leq \phi \left( \frac{x_+ - \theta_g}{\sigma} \right) \frac{x_+ - \theta_g}{\sigma^2} - \phi \left( \frac{x_- - \theta_g}{\sigma} \right) \frac{x_- - \theta_g}{\sigma^2} \\
= \phi \left( \frac{x_+ - \theta_g}{\sigma} \right) \frac{x_+ - x_+}{\sigma^2} \\
< \phi \left( \frac{x_+ - \theta_g}{\sigma} \right) \frac{x_- - x_+}{\sigma^2} \\
= \phi \left( \frac{x_+ - \theta_g}{\sigma} \right) \frac{x_+ - \theta_g}{\sigma^2} - \phi \left( \frac{x_+ - \theta_g}{\sigma} \right) \frac{x_- - \theta_g}{\sigma^2} \\
\leq \phi \left( \frac{x_- - \theta_g}{\sigma} \right) \frac{x_- - \theta_g}{\sigma^2} - \phi \left( \frac{x_- - \theta_g}{\sigma} \right) \frac{x_- - \theta_g}{\sigma^2} \\
= MgB_b(\sigma),
\]

where the first weak inequality follows from \( x_- \leq \theta_g \) and inequality (33) of Lemma 2, the strict inequality follows from \( \phi \) single-peaked around zero and \( 0 < x_+ - \theta_g < x_+ - \theta_b \) (see Lemma 1), and the last weak inequality follows from \( x_- \geq \theta_b \) and (again) inequality (33) of Lemma 2.

Lemma A.25. When \( \beta = 1 \), every equilibrium involves bounded retention. More generally:

(i) It cannot be that \( \sigma_b \leq \sigma \leq \sigma_g \).

(ii) If \( \sigma_g < \sigma \) and \( \beta \leq 1 \), then \( \sigma_b > \sigma_g \) and there can only be bounded retention.

(iii) If \( \sigma_g > \sigma \) and \( \beta \geq 1 \), then \( \sigma_b > \sigma_g \) and there can only be bounded retention.

Proof. First notice that, in any equilibrium with bounded retention or replacement, so \( \sigma_b \neq \sigma_g \), we have

(a.65) \[
\frac{1}{\sigma_b} c'(\sigma_b) > \beta \frac{1}{\sigma_g} c'(\sigma_g) \quad \text{and} \quad \sigma_b c'(\sigma_b) > \beta \sigma_g c'(\sigma_g).
\]

To see the first inequality, combine the inequalities in (7) and (8), while invoking the two first-order conditions in (27), to conclude that

\[
\frac{1}{\sigma_b} c'(\sigma_b) = \phi \left( \frac{x_+ - \theta_g}{\sigma_b} \right) \frac{x_+ - \theta_g}{\sigma_b^3} - \phi \left( \frac{x_- - \theta_g}{\sigma_b} \right) \frac{x_+ - \theta_b}{\sigma_b^3} \\
> \beta \left[ \phi \left( \frac{x_+ - \theta_g}{\sigma_g} \right) \frac{x_+ - \theta_g}{\sigma_g^3} - \phi \left( \frac{x_- - \theta_g}{\sigma_g} \right) \frac{x_+ - \theta_g}{\sigma_g^3} \right] = \beta \frac{1}{\sigma_g} c'(\sigma_g).
\]

To prove the second inequality, use (6) in (27) for the good type to obtain

(a.66) \[
\phi \left( \frac{x_+ - \theta_g}{\sigma_b} \right) \frac{x_+ - \theta_g}{\sigma_b} - \phi \left( \frac{x_- - \theta_g}{\sigma_b} \right) \frac{x_+ - \theta_g}{\sigma_b} = \beta \sigma_g c'(\sigma_g),
\]
and compare this to the first-order condition for the bad type, which is given by:

\[
(a.67) \quad \phi \left( \frac{x_- - \theta_b}{\sigma_b} \right) \left( \frac{x_- - \theta_b}{\sigma_b} \right) - \phi \left( \frac{x_+ - \theta_b}{\sigma_b} \right) \left( \frac{x_+ - \theta_b}{\sigma_b} \right) = \sigma_b c'(\sigma_b)
\]

Invoking (33) of Lemma 2 and (46) of Lemma 11, we see that the expression

\[
\left[ \phi \left( \frac{x_- - \theta_b}{\sigma_b} \right) - \phi \left( \frac{x_+ - \theta_b}{\sigma_b} \right) \right] \left[ \frac{\theta_g - \theta_b}{\sigma_b} \right]
\]

is strictly positive. But adding this term to the left-hand side of (a.66) yields the left-hand side of (a.67). We must therefore conclude that \(\sigma_b c'(\sigma_b) > \beta \sigma_g c'(\sigma_g)\).

The first assertion in Lemma A.25 is a simple consequence of (i)–(iii), to which we now turn. If (i) is false, then \(c'(\sigma_b) \leq 0\) and \(c'(\sigma_g) \geq 0\), which contradicts (a.65). For (ii), if \(\sigma_g < \sigma\) then \(c'(\sigma_g) < 0\). Then (a.65) implies

\[
c'(\sigma_b) > \beta \min \left\{ \frac{\sigma_b}{\sigma_g}, \frac{\sigma_g}{\sigma_b} \right\} < \beta,
\]

so that \(c'(\sigma_b)/c'(\sigma_g) < 1\) when \(\beta \leq 1\). Rearranging (and keeping in mind that \(c'(\sigma_g) < 0\)), we have \(c'(\sigma_b) > c'(\sigma_g)\), or \(\sigma_b > \sigma_g\).

To prove (iii), assume \(\sigma_g > \sigma\). Then \(c'(\sigma_g) > 0\), and (a.65) implies that

\[
c'(\sigma_b) > \beta \max \left\{ \frac{\sigma_b}{\sigma_g}, \frac{\sigma_g}{\sigma_b} \right\} > \beta.
\]

If \(\beta \geq 1\), this inequality implies \(\sigma_b > \sigma_g\).

---

**Proof of Proposition A.1.** If a bounded replacement equilibrium exists, \(\sigma_g > \sigma_b\) by Proposition 1. By Lemma A.3, we have \(x_+ < \theta_b < \theta_g\) in any such equilibrium. Lemma A.25 says that if a bounded replacement equilibrium exists, it must be that either \(\beta < 1\) and \(\sigma_g > \sigma_b > \sigma\) (parts (i) and (ii)), or \(\beta > 1\) and \(\sigma_b < \sigma_g < \sigma\) (parts (i) and (iii)).

Lemma A.24 tell us that we must also have either \(x_- > \theta_g\) or \(x_- < \theta_b\). The case \(x_- < \theta_b\) implies \(\beta > 1\): The principal is feeling so pessimistic about future agents that she is strictly willing to retain the agent even after observing \(x = \theta_b\). That is, (3) holds with strict inequality at \(x = \theta_b\). But that requires \(-(\theta_g - \theta_b)^2 + 2A\sigma_g^2 > 0\), so \(A\) must be positive, which means \(\beta \sigma_b/\sigma_g > 1\), so \(\beta > 1\). A contradiction. The case of \(x_- > \theta_g\) is inconsistent with \(\beta > 1\). If this were not true, we have \(x_- > \theta_g\) together with \(\sigma_b < \sigma_g < \sigma\) by Lemma A.25. But now look at (27): both agents are inside the replacement zone, and therefore the marginal benefit from \(\sigma_g\) is positive. Since \(\sigma\) involves zero marginal cost, both agents must be optimally choosing noise above the ambient level, implying a contradiction.

We will now construct two examples of bounded replacement, one for \(\beta < 1\) and another for \(\beta > 1\). For the case \(\beta < 1\), Proposition A.1 says that both choices of noise must be above the ambient noise. Let us then take \(\theta_b = 1\), \(\theta_g = 2\) and \(x_- = 2.3\). The idea is that both types are inside the replacement zone, as Proposition A.1 determines, but the bad type is deep inside it. Then, the good type will pay a bigger cost of escaping the zone, whereas there is not much the bad type can do. Provided we construct the “right” marginal cost function, this yields \(\sigma_g > \sigma_b\).
Let us impose $g = 0$ and $b = 0$. We have now pinned down the value of $x_+$:

$$\frac{x_+ + x_-}{2} = \frac{\sigma_g^2 \theta_g - \sigma_b^2 \theta_b}{\sigma_g^2 - \sigma_b^2},$$

so that

$$x_+ = 2 \frac{\sigma_g^2 \theta_g - \sigma_b^2 \theta_b}{\sigma_g^2 - \sigma_b^2} - x_- \approx -1.38$$

The value of $\beta < 1$ is now also determined:

$$\beta = \frac{1}{\sigma_g} \phi \left( \frac{x_+ - \theta_g}{\sigma_g} \right) = \frac{1}{\sigma_b} \phi \left( \frac{x_- - \theta_b}{\sigma_b} \right) \approx 2.92 \cdot 10^{-6}.$$

Finally, the two first-order conditions need to be satisfied. Both types’ marginal benefits are always positive, so we just have to care about marginal cost for values of $\sigma$ above $\bar{\sigma}$. We choose $c'(\sigma) = A \ln(\sigma) + B$.

The cost function that yields this expression for the marginal cost is

$$c(\sigma) = A (\sigma \ln(\sigma) - \bar{\sigma} \ln(\bar{\sigma})) + (B - A) (\sigma - \bar{\sigma}).$$

We have two free parameters, for the two first-order conditions:

$$\phi \left( \frac{x_+ - \theta_g}{\sigma_g} \right) \frac{x_+ - \theta_g}{\sigma_g} - \phi \left( \frac{x_- - \theta_g}{\sigma_g} \right) x_- - \theta_g \frac{x_+ - \theta_g}{\sigma_g} = A \ln(\sigma_g) + B,$$

$$\phi \left( \frac{x_- - \theta_b}{\sigma_b} \right) \frac{x_- - \theta_b}{\sigma_b} - \phi \left( \frac{x_+ - \theta_b}{\sigma_b} \right) x_+ - \theta_b \frac{x_- - \theta_b}{\sigma_b} = A \ln(\sigma_b) + B.$$

Therefore,

$$A = \frac{\phi \left( \frac{x_+ - \theta_g}{\sigma_g} \right) \frac{x_+ - \theta_g}{\sigma_g} - \phi \left( \frac{x_- - \theta_g}{\sigma_g} \right) x_- - \theta_g \frac{x_+ - \theta_g}{\sigma_g} - \phi \left( \frac{x_- - \theta_b}{\sigma_b} \right) \frac{x_- - \theta_b}{\sigma_b} - \phi \left( \frac{x_+ - \theta_b}{\sigma_b} \right) x_+ - \theta_b \frac{x_- - \theta_b}{\sigma_b}}{\ln(\sigma_g) - \ln(\sigma_b)} \approx 1,$$

$$B = \phi \left( \frac{x_- - \theta_b}{\sigma_b} \right) \frac{x_- - \theta_b}{\sigma_b} - \phi \left( \frac{x_+ - \theta_b}{\sigma_b} \right) x_+ - \theta_b \frac{x_- - \theta_b}{\sigma_b} - A \ln(\sigma_b) \approx 1.39.$$

The resulting value of the ambient noise ($c'(\sigma) = 0$) is $\bar{\sigma} \approx 1/4$.

Figure A.4 depicts the equilibrium.

Now we find an example of a bounded replacement equilibrium for the case $\beta > 1$. By Proposition A.1 it must be the case that $x_+ < x_- < \theta_b < \theta_g$. Both agents are now in the retention zone so they want to stay there: $\sigma_g, \sigma_b < \sigma$. The bad type is closer to the replacement zone, though, so he will make a bigger effort than the good type to stay safe: $\sigma_b < \sigma_g < \sigma$. 


Let us then choose $\theta_b = 3$, $\theta_g = 5$ and $x_- = 2.5$. For the choices of noise, let’s take $\sigma_b = 0.3$ and $\sigma_g = 0.6$. All this again pins down the value of $x_+$:

$$x_+ = 2 \frac{\sigma_b^2 \theta_g - \sigma_g^2 \theta_b}{\sigma_b^2 - \sigma_g^2} - x_- \approx 2.17.$$ 

For $\beta$ we have:

$$\beta = \frac{1}{\sigma_b} \phi \left( \frac{x_+ - \theta_b}{\sigma_b} \right) \approx 2936. \quad \beta$$

is now immensely big.

For the cost function, once again take: $c^\prime (\sigma) = A \ln (\sigma) + B$, and solve to implement the first-order conditions:

$$A = \left( \phi \left( \frac{x_- - \theta_g}{\sigma_g} \right) \frac{x_- - \theta_g}{\sigma_g^2} - \phi \left( \frac{x_+ - \theta_g}{\sigma_g} \right) \frac{x_+ - \theta_g}{\sigma_g^2} \right) - \left( \phi \left( \frac{x_- - \theta_b}{\sigma_b} \right) \frac{x_- - \theta_b}{\sigma_b^2} - \phi \left( \frac{x_+ - \theta_b}{\sigma_b} \right) \frac{x_+ - \theta_b}{\sigma_b^2} \right) \ln (\sigma_g) - \ln (\sigma_b) \approx 0.68$$

$$B = \phi \left( \frac{x_+ - \theta_b}{\sigma_b} \right) \frac{x_+ - \theta_b}{\sigma_b^2} - \phi \left( \frac{x_- - \theta_b}{\sigma_b} \right) \frac{x_- - \theta_b}{\sigma_b^2} - A \ln (\sigma_b) \approx 0.35.$$ 

Ambien noise is now $\sigma \approx 0.6004$. Figure A.5 depicts the equilibrium.

8. MISCELLANEOUS DETAILS

8.1. The Behavior of $\sigma (\theta)$ in the Costly Noise Model. We will analyze the behavior of $\sigma (\theta)$ in the costly noise model of Section 6.6, when the agents face a retention rule $X = [x_-, x_+]$, where $x_+$ can be equal to infinity (monotone regime). See Figure 3. Even though the analysis will not be complete in the case of bounded retention, it will shed some light on the way $\sigma$...
changes with $\theta$. The first-order condition of an agent of type $\theta$ is
\[ \phi \left( \frac{x_- - \theta}{\sigma} \right) \frac{x_- - \theta}{\sigma^2} - \phi \left( \frac{x_+ - \theta}{\sigma} \right) \frac{x_+ - \theta}{\sigma^2} = c' (\sigma). \]

By differentiating the first-order condition at $\sigma (\theta)$, we can find an expression for $\sigma' (\theta)$:
\[ \frac{\partial \sigma (\theta)}{\partial \theta} = - \frac{1}{\sigma (\theta)^2} \frac{\partial h (\theta)}{\partial \sigma} \left[ \phi \left( \frac{x_- - \theta}{\sigma (\theta)} \right) \frac{x_- - \theta}{\sigma^2 (\theta)} - \phi \left( \frac{x_+ - \theta}{\sigma (\theta)} \right) \frac{x_+ - \theta}{\sigma^2 (\theta)} - c' (\sigma) \right] \big|_{\sigma = \sigma (\theta)}, \]
where
\[ h (\theta) := \phi \left( \frac{x_- - \theta}{\sigma (\theta)} \right) \left( \frac{x_- - \theta}{\sigma (\theta)} \right)^2 - 1 - \phi \left( \frac{x_+ - \theta}{\sigma (\theta)} \right) \left( \frac{x_+ - \theta}{\sigma (\theta)} \right)^2 - 1. \]

The denominator is the second-order derivative, which is negative at the optimum. Therefore:
\[ \text{Sign} \left\{ \frac{\partial \sigma (\theta)}{\partial \theta} \right\} = \text{Sign} \{ h (\theta) \}. \]

Let us study the case of monotone retention first (see Panel A of Figure 3). In this case the term involving $x_+$ in $h (\theta)$ disappears, and therefore
\[ \text{Sign} \left\{ \frac{\partial \sigma (\theta)}{\partial \theta} \right\} = \text{Sign} \{ |x_- - \theta| - \sigma (\theta) \}. \]

Then, all we have to do is to compare $\sigma (\theta)$ and $|x_- - \theta|$. Remember that at $\theta = x_-$, $\sigma (\theta) = \sigma > 0$, so $\sigma (\theta)$ is decreasing as we enter the retention zone, and it will be so until
\[ \sigma (\theta) = \theta - x_- , \]
at which stage the derivative is 0. From this point onwards $\sigma (\theta)$ is always increasing\(^{6}\). However, $\sigma (\theta)$ cannot grow unboundedly, since this would mean $c' (\sigma) \to \infty$, but $\phi (z) z \to 0$ as $|z| \to \infty$, so $\sigma (\theta)$ approaches $\sigma$ from below as $\theta \to \infty$.

\(^6\) $\sigma (\theta)$ cannot cross the $\theta - x_-$ function again since this would require $\sigma' (\theta) \geq 1$ at the intersection point, but crossing $\theta - x_-$ means $\sigma' (\theta) = 0$. 

---

**Figure A.5.** A Bounded Replacement Equilibrium for $\beta$ Large.
If we move away from \( x_- \) but in the opposite direction; that is, as we decrease \( \theta \), \( \sigma (\theta) \) cannot always stay above \( x_- - \theta \) because this would mean \( \sigma (\theta) \to \infty \) as \( \theta \to -\infty \), and we have just argued that this is inconsistent with optimality. This means there is an intersection point at which \( \sigma (\theta) = x_- - \theta \) and \( \sigma (\theta) \) reaches its maximum. Then, \( \sigma (\theta) \downarrow \sigma \) as \( \theta \to -\infty \).

Now turn to the case of bounded retention, depicted in Panel B of Figure 3. The symmetry of \( \sigma (\theta) \) around the midpoint \( \frac{x_- + x_+}{2} \) of the retention interval is evident from the first-order condition. If \( x_+ < \infty \), \( \sigma (\theta) = \sigma (x_- + x_+ - \theta) \) for all \( \theta \leq \frac{x_- + x_+}{2} \). Let us therefore study the behavior of \( \sigma (\theta) \) for \( \theta \geq \frac{x_- + x_+}{2} \).

First, consider type \( \theta = \frac{x_- + x_+}{2} \). By the symmetry and the existence of \( \sigma' (\theta) \), at this point \( \sigma' (\theta) = 0 \). Effectively:

\[
h \left( \frac{x_- + x_+}{2} \right) = 0.
\]

For the type at the edge, \( \theta = x_+ \), the first-order condition is

(a.68)

\[-\phi \left( \frac{x_+ - x_-}{\sigma} \right) \frac{x_+ - x_-}{\sigma^2} = \sigma' (\sigma), \]

so \( \sigma (\theta) < \sigma \). But his chosen \( \sigma \) is bigger than the one of the midpoint type: the marginal benefit at \( \theta = \frac{x_- + x_+}{2} \) is equal to

\[-\phi \left( \frac{x_+ - x_-}{2\sigma} \right) \frac{x_+ - x_-}{\sigma^2}, \]

and it is smaller than the left-hand side of (a.68). Also, \( \sigma' (\theta) \) is positive at \( \theta = x_+ \):

\[
h (x_+) = \left( \phi \left( \frac{x_+ - x_-}{\sigma (x_+)} \right) \left( \frac{x_+ - x_-}{\sigma (x_+)} \right)^2 + \phi (0) - \phi \left( \frac{x_+ - x_-}{\sigma (x_+)} \right) \right) > 0.
\]

What about types above \( x_+ \)? There exists a type \( \theta > x_+ \) such that \( \sigma (\theta) = \sigma \). For such a type, the value of \( \sigma \) that maximizes his probability of retention is equal to \( \sigma \), that is,

\[
\phi \left( \frac{x_- - \theta}{\sigma} \right) \frac{x_- - \theta}{\sigma} - \phi \left( \frac{x_+ - \theta}{\sigma} \right) \frac{x_+ - \theta}{\sigma} = 0.
\]

Observe that for type \( \theta \) the solution is represented in Panel B of Figure 4 by those \( z_1 \) and \( z_2 \) such that \( \phi (z_1) z_1 = \phi (z_2) z_2 \) and \( z_2 - z_1 = \frac{x_+ - x_-}{\sigma} \).

Now consider any type \( \theta \) above \( \theta \) who considers playing \( \sigma \). His first-order derivative would be

\[
\phi \left( \frac{\theta - x_+}{\sigma} \right) \frac{\theta - x_+}{\sigma} - \phi \left( \frac{\theta - x_-}{\sigma} \right) \frac{\theta - x_-}{\sigma}.
\]

But Panel B of Figure 4 reveals that the sign of this expression will be positive, since by increasing \( \theta \) we are considering bigger values of both \( z_1 \) and \( z_2 \). This means that \( \sigma (\theta) > \sigma \forall \theta > \theta \). Similarly, for any type \( \theta \in (x_+, \theta) \), the first-order derivative at \( \sigma \) will be negative, so \( \sigma (\theta) < \sigma \) for such \( \theta \).
Let us focus on types \( \theta > \theta \). For such types, since \( \sigma (\theta) > \sigma \), \( \sigma c ' (\sigma) \) is increasing in \( \sigma \). This means that \( \sigma \) cannot grow unboundedly with \( \theta \), since that would mean \( \sigma c ' (\sigma) \to \infty \), whereas \( \phi \left( \frac{\theta-x_+}{\sigma} \right) \frac{\theta-x_+}{\sigma} - \phi \left( \frac{\theta-x_-}{\sigma} \right) \frac{\theta-x_-}{\sigma} \) is a bounded function (each term is between 0 and 1).

Depart from \( \theta = \theta \). Notice that type \( \theta \) satisfies that \( \sigma \in [\theta - x_+, \theta - x_-] \) (take a look at Panel B of Figure 4 again to convince yourself: \( z_1 < 1 < z_2 \)). This says that \( h (\theta) > 0 \) so \( \sigma ' (\theta) > 0 \). Notice then that the distance \( \frac{\theta-x_+}{\sigma(\theta)} - \frac{\theta-x_-}{\sigma(\theta)} = \frac{x_+-x_-}{\sigma(\theta)} \) is decreasing in \( \theta \) for values close to \( \theta \). Furthermore, it has to be the case that \( \frac{\theta-x_+}{\sigma(\theta)} \) is increasing in \( \theta \), since otherwise \( \frac{\theta-x_-}{\sigma(\theta)} \) would be decreasing (remember their distance decreases), and therefore \( \phi \left( \frac{\theta-x_+}{\sigma(\theta)} \right) \frac{\theta-x_+}{\sigma(\theta)} - \phi \left( \frac{\theta-x_-}{\sigma(\theta)} \right) \frac{\theta-x_-}{\sigma(\theta)} \) would decrease (remember that \( \frac{\theta-x_+}{\sigma(\theta)} < 1 < \frac{\theta-x_-}{\sigma(\theta)} \)) at the same time that \( \sigma (\theta) c' (\sigma (\theta)) \) increases.

So as long as \( \frac{\theta-x_+}{\sigma(\theta)} < 1 \), \( \frac{\theta-x_+}{\sigma(\theta)} \) is increasing and so it \( \sigma (\theta) \) (see function \( h (\theta) \)). Furthermore, since \( \sigma (\theta) \) is bounded, this means that \( \frac{\theta-x_+}{\sigma(\theta)} \) eventually goes above 1. Function \( h (\theta) \) indicates that as soon as this happens \( \frac{\theta-x_+}{\sigma(\theta)} = 1 \), \( \sigma (\theta) \) is still increasing with \( \theta \). Since we know \( \sigma (\theta) \) is bounded and it converges to \( \sigma \) as \( \theta \to \infty \), there exists a point \( \theta \) at which \( \sigma ' (\theta) = h (\theta) = 0 \).

Now take a look at Figure A.6, which plots functions \( \phi (z) z \) (the orange curve, related to the first-order derivative) and \( \phi (z) (z^2 - 1) \) (the blue curve, related to function \( h (\theta) \)). \( h (\theta) = 0 \) means that there are two points on the \( x \) axis that reach the same height on the blue curve. The smaller point corresponds to \( \frac{\theta-x_+}{\sigma(\theta)} \), and the larger point to \( \frac{\theta-x_-}{\sigma(\theta)} \). But then, both \( \frac{\theta-x_+}{\sigma(\theta)} \) and \( \frac{\theta-x_-}{\sigma(\theta)} \) are increasing in \( \theta \) at such a point (because \( \sigma ' (\theta) \) but \( \theta \) increases), and therefore \( h (\theta) < 0 \) forever after: both \( \frac{\theta-x_+}{\sigma(\theta)} \) and \( \frac{\theta-x_-}{\sigma(\theta)} \) will always be increasing forever after because \( \sigma (\theta) \) is decreasing and the numerators increase with \( \theta \).
8.2. **Sufficient Conditions for Uniqueness.** Condition U in Section 6.6 states that for every monotone or bounded retention zone, and for every agent type, the optimal choice of noise is unique. Here we show a condition on the cost function \( c(\sigma) \) that guarantees the desired uniqueness.

Consider the case \( \sigma_b \geq \sigma_g \), so \( X = [x_-, x_+] \) with \( x_+ < \infty \) iff \( \sigma_b > \sigma_g \). Recall the necessary first-order condition:

\[
-\phi \left( \frac{x_+ - \theta}{\sigma} \right) \frac{x_+ - \theta}{\sigma^2} + \phi \left( \frac{x_- - \theta}{\sigma} \right) \frac{x_- - \theta}{\sigma^2} = c'(\sigma).
\]

We want to impose conditions such that the objective function is always strictly concave, this generating an unique optimal choice for each parameter. For this, we will ask \( c''(\sigma) \) to be always bigger than the second derivative of the marginal benefit, which is the derivative of the left-hand side with respect to \( \sigma \):

\[
\frac{1}{\sigma^2} \left[ \phi \left( \frac{x_+ - \theta}{\sigma} \right) \frac{x_+ - \theta}{\sigma} \left( 2 - \left( \frac{x_+ - \theta}{\sigma} \right)^2 \right) - \phi \left( \frac{x_- - \theta}{\sigma} \right) \frac{x_- - \theta}{\sigma} \left( 2 - \left( \frac{x_- - \theta}{\sigma} \right)^2 \right) \right].
\]

This expression is related to the function \( \phi(z) z (2 - z^2) \), where the value of \( z \) could be anywhere in the real line: \( x_+ - \theta \) is always positive, but \( x_- - \theta \) can take either sign. Forget about the term \( \frac{1}{\sigma^2} \) on the left: we will find the biggest possible value of the term inside the square brackets, which will be a number, say \( \kappa \). Then we ask for \( c''(\sigma) \geq \frac{\kappa}{\sigma^2} \) \( \forall \sigma \). Let us plot the \( \phi(z) z (2 - z^2) \) function in Figure A.7. In order to find the critical values of this function, compute the first-order derivative and set it equal to zero:

\[
\frac{\partial}{\partial z} \phi(z) z (2 - z^2) = \phi(z) (z^4 - 5z^2 + 2) = 0.
\]
We have 4 values of \( z \) that satisfy the condition:

\[
z = \pm \sqrt{\frac{5}{2}} \pm \sqrt{\frac{17}{4}} \Rightarrow z = \{-2.14, -0.66, 0.66, 2.14\}.
\]

Finally, to find the maximum value of \( c''(\sigma) \geq \frac{\kappa}{\sigma^2} \forall \sigma \)
where

\[
\kappa = \phi(z_2) z_2 (2 - z_2^2) - \phi(z_1) z_1 (2 - z_1^2) \\
\approx 0.662594.
\]