Growth, Automation, and the Long-Run Share of Labor

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Supplementary Appendix: Not for Publication

Here, we provide omitted proofs of some results from the main text. All numbered references for figures, equations, lemmas, etc. refer to the main text. References that start with "a" refer to corresponding objects in this Appendix.

Proof of Lemma 2: If the robot sector is active at date t,

$$p_r(t) = c_r\left(1, \boldsymbol{\omega}_r(t)\right) = c_r(1, \{c^o(p_r(t), w^o(t))\}) \tag{a.1}$$

where the occupation index o runs over elements of O_r. Observe that $w^0(t)$ is bounded below by the minimum wage $\underline{w} > 0$. We claim first that there cannot be an equilibrium where $p_r(t) = 0$. For this would require $c_r(1, \{c^o(0, \underline{w})\}) = 0$, and this unit cost cannot be achieved by any combination of finite factor demands (given the essentiality of capital in production of robot services).

Next, we claim that the collection of equilibrium robot prices is bounded away from zero. Suppose otherwise and there exists a sequence of equilibria with $p_r \to 0$. Because tasks can be produced by robots alone in every $o \in O_r$ and the human wage is bounded away from zero, production of tasks in o will be asymptotically automated as $p_r \to 0$. That means $\partial c^o(p_r, \underline{w})/\partial p_r \to \nu_r^o$ as $p_r \to 0$. As for the unit production of robots, notice that it is also the case that $c^o(p_r, \underline{w}) \to 0$ for each $o \in O_r$ as $p_r \to 0$. This implies that the use of robots in this sector will asymptote to infinity, while machine capital use goes to 0 (as $p_r \to 0$). Putting all this together, we see that

$$\lim_{p_r \to 0} \frac{\partial c_r(1, \{c^o(p_r, \underline{w})\})}{\partial p_r} = \sum_{o \in O_r} \frac{\partial c^o}{\partial p_r} \ell^o = \sum_{o \in O_r} \frac{1}{\nu_r^o} \ell^o = \sum_{o \in O_r} r^o = \infty.$$

In particular, there exists $\epsilon > 0$ such that $c_r(1, \{c^o(p_r, \underline{w})\}) > p_r$ for all $p_r \in (0, \epsilon)$. It follows that no equilibrium robot price $p_r(t)$ can lie in $(0, \epsilon)$ at any date t. For if it did, along with $w^o(t) \geq \underline{w}$ for $o \in O_r$, (a.1) would imply that $p_r(t) = c_r(1, \{c^o(p_r(t), w^o(t))\}) \geq c_r(1, \{c^o(p_r(t), \underline{w})\}) > p_r(t)$, a contradiction.

Therefore the price of robot services and of human labor is bounded away from zero (uniformly in t), implying that the unit cost of producing any good j (and hence its price) is also bounded away from zero (uniformly in t). This establishes (37).

If self-replication holds, Proposition 1 bounds (above) the price of every good j:

$$p_j(t) \leq c_j(1, \boldsymbol{\omega}_j(t)) \leq c_j(1, \{(\nu^o)^{-1} p_r(t)\}) \leq c_j(1, \{(\nu^o)^{-1} \sup P^*\}) < \infty,$$

which establishes (38). ■

Proof of Lemma 4: For any n, let J be an integer such that for $\hat{\mathbf{s}}$ in the statement of the lemma, $\sum_{j=1}^{J} \hat{s}_j \geq 1 - (1/2)^{n+2}$. Then there is $T_1(n)$ such that along the sequence $\{\mathbf{s}(t)\}$,

$$\sum_{i=1}^{J} s_i(t) \geq 1 - (1/2)^{n+2} - (1/2)^{n+2} = 1 - (1/2)^{n+1}$$

for $t \geq T_1(n)$, using pointwise convergence on the finite set $\{1, \ldots, J\}$. Because $\Psi_i(t) \in [0,1]$ for all i and t and $\sum_j s_j(t) = 1$ for every t, it follows that for $t \geq T_1(n)$,

$$\text{(a.2)} \qquad \sum_{j=J+1}^{\infty} \Psi_j(t) s_j(t) \leq \sum_{j=J+1}^{\infty} s_j(t) < (1/2)^{n+1}.$$

Because $\Psi_j(t) \to 0$ as $t \to \infty$ for every j with $\hat{s}_j > 0$, we know that $\Psi_j(t) s_j(t) \to 0$. Therefore there exists $T(n) \geq T_1(n)$ so that in addition to (a.2),

$$\text{(a.3)} \qquad \sum_{i=1}^{J} \Psi_i(t) s_i(t) \leq (1/2)^{n+1}$$

for $t \geq T(n)$. Combining (a.2) and (a.3), we must conclude that

$$\sum_{j=1}^{\infty} \Psi_j(t) s_j(t) < (1/2)^n.$$

for $t \geq T(n)$. Because n can be made arbitrarily large, the proof is complete. ■