Growth, Automation and the Long Run Share of Labor

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Abstract

We uncover a simple argument for long-term automation and decline in the labor share of national income, driven by capital accumulation rather than biased technical progress or rising markups. The argument hinges on a fundamental asymmetry across physical and human capital in modern economies. While physical capital can be scaled up for the same activity and accumulates in natural units, human capital accumulates principally via education and training that alters choice into higher-skilled occupations, but — from the vantage point of a household or individual — cannot scale up the quantity of labor within a given occupation to an unlimited degree. This one asymmetry implies that in any growing economy, occupations must be progressively automated. Under a singularity condition on the technology of the robot-producing sector, we argue that a basic “Kaldor fact” cannot hold: the share of capital in national income approaches 100%. At the same time, the displacement of human labor is gradual: the rising share of capital could coexist with rising real wages. Our main result holds for a world with no technical progress, and is robust to endogenous technical change provided that there are symmetric opportunities for technical progress in all inputs.

1. Introduction

The growing evidence worldwide for a declining labor share in national income (Karabarbounis and Neiman 2014, Piketty 2014) — thereby dismantling a central “Kaldor fact” — has generated active debate and research. Piketty’s “r > g explanation” has been severely criticized, given its compatibility with a variety of standard growth models exhibiting no decline in labor share (Acemoglu and Robinson 2015, Mankiw 2015, Ray 2015). Explanations in terms of capital-labor substitution along some aggregate CES production function have also been subject to controversy: e.g., standard theories require capital-labor substitution elasticities to exceed one, rejected by panel studies of industry level production functions (Chirinko and Mallick 2014). Other arguments include rapid globalization, whereby labor in developed countries are devastated by cheap-labor imports (Autor, Dorn and Hansen 2016), increasing selection into highly profitable and therefore higher-markup firms (Autor et al 2017), the rise of the gig economy and a general decline in firm competition and the bargaining power of labor, perhaps via greater product differentiation that thereby permits higher markups (Neary 2003, Gutiérrez and Philippon 2017, Azar and Vives 2018, and Eggertsson, Robbins, and Wold 2018), or simply the possibility of technical progress that favors automation (Santens 2016, Acemoglu and Restrepo 2019).

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These explanations have useful roles to play in explaining or at least in describing better the situation at hand, though none is fully satisfactory. For instance, globalization cannot explain why we see a similar decline in labor share or a greater awareness of “jobless growth” in countries such as China and India. Factors such as greater industrial concentration or a decline in bargaining power are in themselves endogenous outcomes, and are possibly better viewed as accompaniments to the overall situation rather than primitive explanations. As for technical progress — and while we extend our analysis to include it — Karabarbounis and Neiman (2014) show that a substantial fraction of the decline in labor share worldwide is explained by declining capital good prices, even after controlling for capital-augmenting technical progress, markup rates and the skill composition of the labor force.

In this paper we emphasize a distinct source of a declining labor share, arising from the growth process itself, coupled with a fundamental asymmetry between the accumulation of physical and human capital. Our approach stands in contrast to a long tradition (see Becker and Tomes 1979, Loury 1981, Lucas 1988 or Mankiw, Romer and Weil 1992), in which human and physical capital are imagined to be symmetric engines of accumulation. Under this view, both are driven by a dynamic process which can scale up individual endowments in “efficiency units” to arbitrarily large degrees. Long-term growth in these settings typically entails a balanced, stationary ratio of human to physical capital; see, e.g., Mankiw, Romer and Weil (1992).

This is a useful caricature for some macroeconomic exercises — particularly for those focused on steady states or balanced growth — but not others.\(^1\) By acquiring education, humans do not become unboundedly better in some given occupation, though to some extent this is possible. The bulk of human capital acquisition has to do with the acquisition of skills that are not perfect substitutes for less skill, but rather new skills that enable entry into a possibly related but distinct occupation or sector. Greater education allows us to obtain promotions, new jobs — that is, human capital accumulation allows us to move from occupation to occupation, perhaps over some unbounded set of occupations in the indefinite future. But an individual agent cannot turn an endowment of, say, one bank manager into two bank managers.\(^2\)

In contrast, machines and factories can be scaled up in natural units, in a way that is not possible with humans; this is mirrored in the ability of households to accumulate claims on physical capital to an unbounded degree. Like human capital, physical capital can move over a large number of different sectors, and could have sector-specific characteristics. But it can also scale up within a sector. An endowment of one machine can be grown into two. By “scaling-up,” we don’t intend to suggest that there will always be enough demand for the final output or that complementary inputs would be available. The point is just that capital can be replicated in a way that a labor endowment cannot be. In any growing economy, a consequence of this asymmetry is a widening imbalance between capital and humans in physical units. Any model

\(^1\)For instance, in Mookherjee and Ray (2002, 2003, 2010), we argue that by lumping different occupational choices into homogeneous efficiency units, much of this literature misses the endogenous movements in relative prices that lie at the heart of persistent inequality.

\(^2\)We are not referring here to technical progress that raises effective productivity of humans within a particular sector. That is different from an individual accumulating an ever-larger labor endowment in natural units for the same job.
that accommodates this asymmetry is open to progressive automation in the workplace, with or without technical progress. The implications of this asymmetry is the central focus of this paper.

Our baseline economy has a large (initially finite) number of final goods sectors, and intermediate sectors that respectively produce education, machine capital and an offshoot of capital services which we model separately as robots. Machine capital and robots are meant to capture two distinct aspects of “capital.” The first is complementarity: working with a machine enhances human productivity. The second is substitution: robots displace humans. In reality, most capital goods have a bit of machine and a bit of robot in their DNA. We separate them.

Specifically, every sector has a traditional production function defined on machine capital and (possibly sector-specific) labor. Nothing changes if we allow for many types of labor per sector, but it is just expositionally convenient to presume there is one occupation per sector. In each such occupation, there is scope for replacing humans by robots, though their relative productivities can vary in an arbitrary way across sectors. Barring this possibility of automation at the sectoral (or occupational) level, the model is completely classical, and quite general: many sectors and goods, arbitrary heterogeneous preferences over those goods, an arbitrary initial distribution of capital endowments across individuals, full intertemporal maximization of utility, perfectly competitive firms, and market clearing in all sectors.

The asymmetry between physical and human labor endowments, already discussed, means that ongoing per-capita economic growth and rising output within each sector which has not yet been automated puts upward pressure on wages (relative to rentals). This lowers the cost effectiveness of human labor relative to robots. The extent to which humans become less cost-effective depends on what is happening to the price of robot services at the same time. This depends on the costs of producing robot services. Since robot services are in turn created by machine capital and labor, the price of such services will be sandwiched between the return to machine capital and the wages of technicians, engineers and researchers active in the robot producing sector. A central question pertaining to downstream automation is then whether robot prices will be tied more closely to human wages or to the price of capital in the robot producing sector as capital continues to accumulate.

This leads to the second building block of our theory. We argue that the presence of a crucial singularity\textsuperscript{4} determines the answer to this question. It is connected to the eventual automation of the robot sector itself. If that condition — placed on the robot producing technology alone, involving the elasticity of substitution within that sector relative to the effectiveness of humans

\textsuperscript{3}As explained in more detail when we develop the model, we also presume that machines are durable, though robot services are not and must be produced every period. This apparent additional asymmetry involves no loss of generality, as the robot production function allows — as a special case — for robot services to be fully embedded in machines. Technically, all we do is move all the durability on to the “machine sector.”

\textsuperscript{4}The idea of such a singularity appears to have been first envisaged by von Neumann in his work on self-reproducing automata; see von Neumann and Burns (1966).
\textit{vis-a-vis} robots — is satisfied,\footnote{The singularity condition compares the elasticity of substitution of capital for operating tasks (or labor for short) performed by humans or their robot equivalents, and the efficiency of robots relative to humans in providing labor. The condition is automatically met if the former elasticity equals or exceeds 1, but is consistent with sub-unitary elasticities of substitution, the exact magnitude depending on the efficiency of robot-human substitution in the robot sector. Note that while the singularity condition has economy-wide ramifications, it is a condition on the robot sector alone — we place no restriction on substitution elasticities in any other sector.} the price of robots must eventually become tied to that of machine capital (Proposition 1). In turn, this causes sustained automation, not necessarily in every other sector, but in every “consequential sector” that grows along with national income. And the share of machine capital approaches 100\% in the long run (see Theorems 1 and 3 for fuller statements).

However, while such automation must happen, it cannot happen “too fast.” In that case, labor would be too quickly driven out of productive activities, and they could contest the robot intrusion. After all, while it is surely feasible to automate the production of hamburgers, a society with enough cheap labor will push back against this.

Might a symmetry between physical and human capital be restored by an infinite number of potential sectors, so that human capital can forever “accumulate” by moving over sectors or occupations? We explore this question by subsequently extending our model to cover an infinite set of final goods. Moreover, we allow for a sequence of such goods to be progressively human-friendly, even though each one has its own automation threshold. Now there is unbounded scope for human capital accumulation, but this power can only be exercised if the demand conditions are right. Theorem 2 shows that if preferences over final goods are homothetic, then again we recover Theorem 1: the share of labor converges to zero over time. Indeed, if a sequence of progressively human-friendly sectors exist, and can be reached via education, the results acquire a more paradoxical feel: human wages will rise to infinity in all sectors with positive employment, even as the human share in national income vanishes.

This result makes visible a second escape hatch for human workers, in addition to a possible failure of the singularity condition; see Section 6 for more on the latter. If preferences are non-homothetic, but not just that, they are non-homothetic in a \textit{particular} direction — one that lines up with ever greater “human-friendliness” in production — then it is possible to envisage a future where human labor perpetually retains a positive share of income. With growth, humans progressively move to these friendly sectors, and by the assumed non-homotheticity of preferences, they could find adequate demand for such products. We leave this possibility open. How broad or narrow a channel it might be is an empirical matter.

The preceding analysis show how an economy can drift towards progressive automation even in the absence of any technical change. Section 5 extends the model to allow for directed technical progress in capital, human and robot productivities. Technical progress is sector-specific, and operates on the intensive margin, raising productivity of different inputs within each sector. Now, it may well be that the physics of technical progress favors one sort of input or sector over another, but we have nothing to say on such matters: we presume that there are bounds — common across all inputs and sectors — on the speed of such progress. In particular, we impose complete symmetry on the costs and opportunities of directed technical progress. Then our earlier results survive: the long-run share of human labor must converge to zero, as before; see
Theorem 4. Asymmetries in the opportunities for technical progress which favor humans could provide another escape hatch for humans to survive automation, but we would then have to assume those asymmetries, and their relative impact would be an empirical question. The recent contribution by Acemoglu and Restrepo (2018) is a leading example of this approach, and we compare our approach to technical progress with theirs in Section 5.

In contrast to existing literature, ours is a simpler and more basic approach. The fall in the functional share of labor is rooted in a growth process that embodies the fundamental asymmetry between accumulation of capital and labor, and it is consistent with declining, constant or increasing absolute wages. The underlying logic is straightforward and transparent; it holds in a fully competitive setting without technical progress, and is augmented in the presence of technical progress. It can also be similarly extended to incorporate monopolistic competition and rising markups, which represent possibly supplementary causes of declining labor shares. At the same time, the baseline model highlights two new routes that could allow human labor to retain a positive share of national income: a failure of the singularity assumption in the robot production technology, and specific forms of non-homotheticity in household preferences.

2. Model

2.1. Production. There is a collection $I$ (initially finite) of final goods for consumption, indexed by $i$. In addition, there are three sectors for the production of robot services, education, and machine capital. The index $j$ serves as generic notation for any of this full collection of sectors. In any sector $j$, capital can have two roles. It is, first, a complementary input to labor. Think of such capital as “machine capital” $k_j$, used as input along with labor $\ell_j$:

$$y_j = f_j(k_j, \ell_j)$$

where $f_j$ is an increasing, smooth, strictly quasi-concave, linearly homogeneous production function with $f_j(k, 0) = f_j(0, \ell) = 0$, displaying unbounded steepness in each input at zero.

The second aspect of capital is its potential to displace human labor; think of such capital as “robot services.” That is, we view $\ell$ as labor input, representing tasks traditionally performed by humans ($h$), but which can be automated by the use of robot services ($r$). Specifically we presume that each unit of human labor $h$ is worth one unit of labor $\ell$ in sector $i$, while each unit of robot labor $r$ is equivalent to $\nu_j$ units of labor $\ell$ in that sector. (So $\nu_j$ is the efficiency of robot labor relative to human labor in sector $j$.) Without technical progress, $\nu_j$ is a parameter, fixed once and for all in any given sector. We then have $\ell_j = \nu_j r_j + h_j$. We assume that no sector can be freely automated or is fully protected; that is, $\nu_j \in (0, \infty)$ for all $j$.

While the assumption of linear substitutability of robots for humans simplifies the analysis, it is not essential. Our results extend with no change to specifications of less than perfect substitutability where one factor ceases to be used if its relative price is sufficiently high. Note also that humans may continue to work in automated sectors in our model, provided they are willing to work at a wage which renders employers indifferent between them and robots.\textsuperscript{6}

\textsuperscript{6}This will happen, of course, if workers have specific skills which tie them to particular sectors, and are limited in their ability (at least in the short run) to move to other sectors. But it must be more generally true of equilibrium outcomes with finitely many or countably many sectors.
In particular, humans could be displaced by robots in the production of robots. We will refer to such a phenomenon as the von Neumann singularity, after John von Neumann who imagined this possibility; more on the singularity below.

We employ the convention throughout that final goods, robot services, and education are produced within the period, while machine capital is produced for the “next” period.

2.2. Prices. Machine capital services will be our numeraire, so the rental price of $k$ will be set at 1. The collection $(w, w_r, w_e, w_k)$ is the wage system, where $w = \{w_i\}$ are wages in final goods sectors. With free mobility of labor, these wages would all be the same. The output prices are $(p, p_r, p_e, p_k)$, which is the price system for all final goods, robot services, education, and capital goods. By constant returns to scale and the assumption of a price-taking competitive economy, we know that all prices will be pinned down by unit costs of production.\footnote{It is easy to check that our results extend to a context of monopolistic competition with CES preferences, as that generates a constant markup of price over cost in all sectors. Of course, profits would appear in that setting, so national income would be the sum of returns to capital, to workers and profits. Our distributional results would continue to apply to the share of wages in national income.} That is, output and input prices are connected by:

\begin{equation}
\begin{aligned}
p_j &= c_j(1, \lambda_j), \\
\end{aligned}
\end{equation}

where 1 is the normalized return to machine capital, $\lambda_j \equiv \min\{w_j, \nu_j^{-1}p_r\}$ is the effective price of the labor input in sector $j$, and $c_j$ is the unit cost function, dual to the production function $f_j$. This cost function has standard properties, which we shall have occasion to invoke below.

In a dynamic competitive equilibrium, these prices will generally vary with time; we augment notation later to allow for this.

2.3. Factor Demands and Automation. In each sector $j$, the demand for machine capital is given by a familiar marginal product criterion:

\[ p_j \frac{\partial f_j(k_j, \ell_j)}{\partial k_j} = 1, \]

where we recall that the rental rate serves as numeraire. Likewise, labor demand (human or non-human) must satisfy

\[ p_j \frac{\partial f_j(k_j, \ell_j)}{\partial \ell_j} = \lambda_j \]

If $w_j > \nu_j^{-1}p_r = \lambda_j$, sector $j$ is “fully automated,” with $r_j = \ell_j$ and $h_j = 0$. If $\lambda_j = w_j < \nu_j^{-1}p_r$, sector $j$ is “non-automated,” with $h_j = \ell_j$ and $r_j = 0$. Finally, for a “partially automated sector” $j$, we have $\lambda_j = w_j = \nu_j^{-1}p_r$, and so $h_j + \nu_j r_j = \ell_j$, with firms indifferent across all such combinations of $h_j$ and $r_j$. 
2.4. Factor Endowments. At each date, the available machine capital and robot services move freely across sectors. (An extension of the model to accommodate sector-specific machine capital and robots services is tedious but straightforward.) Over time, the machine capital stock grows — or shrinks — depending on the rate of depreciation and the production function for capital. We assume that capital depreciates at rate $\delta \in [0, 1]$ and take this rate to be independent of sector.\(^8\) Then at every date $t$,

$$K(t + 1) = (1 - \delta)K(t) + y_k(t).$$

We take robot services to be fully depreciating within the period. But the alternative view that robots are also durable machines can easily be accommodated, by embodying robot services in machines. Those machines can be viewed as capital input to the robot-producing sectors, where they produce services again and again under the auspices of the robot production function $f_r$ (think of labor also being needed there for maintenance).

The stock of human labor is given in physical units. By constant returns to scale, it makes no difference if human labor grows exogenously; we will only need to re-normalize all variables by the rate of growth of labor. That said, we want to accommodate sector-specific skill acquisition. We allow humans to differ in the sector-specific skills they are born with. So we take as given some initial allocation of the population across sectors. There could be a place-holding “null sector” where individuals without any acquired skill can be initially placed, or can “drop out” to at zero educational cost. At the beginning of every period, an individual can move from one sector to another. To move from sector $j$ to $j'$ a person needs $e_{jj'}$ units of education at the price $p_e$, the latter in turn endogenously determined by the unit cost of education.\(^9\) We allow for the possibility of depreciation of human capital; that is, it is possible for $e_{jj}$ to be strictly positive for some or all $j$. We place no restrictions at all on the amount of education needed to switch occupations, so the model captures both completely inflexible sector-specificity (whereby it is not possible for individuals to move at all) at one extreme to complete flexibility (where no education is needed to switch sectors) at the other extreme, and everything in between.

2.5. Preferences. A continuum of infinitely lived individuals live on $[0, 1]$. Individual $m \in [0, 1]$ has a continuous one-period utility function $u_m(x)$ defined on the vector $x = \{x_i\}$ of final goods consumption, a discount factor $\beta_m \in (0, 1)$, and one unit of human labor. (These vary measurably with $m$.) At the start of any date, an individual has financial wealth and is currently located in some production sector. She makes an education decision that moves her to a possibly new sector. Within the sector, each individual with skills appropriate to that sector, supplies one unit of labor to that sector with no disutility. (The latter assumption of zero disutility of working is inessential, since all that matters is the existence of an upper bound on the amount of labor that can be supplied by any individual.) She participates in production and is paid the sectoral wage. Then, given her wealth and income, she decides how much to spend today and correspondingly

\(^8\)If depreciation rates are sector-dependent, then the net rate of return on capital is not defined by the equality of value of marginal products across sectors, and our normalization that the value of the marginal product of capital in every sector is set equal to 1 is no longer valid.

\(^9\)If initial skills are not sector-specific, then everyone is in the null sector to start with, and education requirements can be written only as a function of the “destination sector” $j'$. We can also accommodate “utility costs” of education that are nonlinear in baseline wages, so as to mimic imperfect or missing capital markets.
adjusts her holdings of financial wealth at a market rate of return to be endogenously determined. Then she allocates her current expenditure across final goods and services. This process repeats itself in each period.

For any total expenditure $z$ on final goods and any price vector $p$, the demand vector for goods will be determined by maximizing $u_m(x)$, subject to $px \leq z$. That generates a demand function $x_m(p, z)$. Denote by $v_m(z, p)$ the corresponding indirect utility function. We assume that $u_m$ satisfies what it needs to satisfy so that this indirect utility function is increasing, differentiable and strictly concave for every $p$, with unbounded steepness at zero. Given some dated price-wage system for all goods, capital, and occupations, our agent maximizes

$$
\sum_{t=0}^{\infty} \beta^t v_m(z(t), p(t)),
$$

subject to $F_m(0)$ given, by choosing occupation $j_m(t)$, education $e_m(t)$, current expenditure $z_m(t)$, and financial wealth $F_m(t+1)$ for every $t \geq 0$. Two feasibility conditions apply at every date. First,

$$
F_m(t) + w_{j_m(t)}(t) = z_m(t) + p_e(t) + \frac{F_m(t+1)}{\gamma(t)},
$$

where $\gamma(t)$ is the endogenously determined “return factor” on financial wealth at date $t$; that is, $1$ plus the implied rate of return. We know what this factor has to be — it is just the implied return on holding machine capital across a period:

$$
\gamma(t) = 1 + (1 - \delta) \frac{p_k(t+1)}{p_k(t)}.
$$

For one unit of machine capital bought at date $t$ at price $p_k(t)$ yields $1 + (1 - \delta) p_k(t + 1)$ tomorrow. This has to equal the rate of return on “financial wealth.”

The second feasibility condition describes the evolution of human capital. If our individual chooses a sequence of occupations $\{j_m(t)\}$, then for all $t$,

$$
e_m(t) = e_{j_m(t)j_m(t-1)}.
$$

We can think of initial sectoral location “$j_m(-1)$” as a real allocation or as some original null state from which she makes a move into the labor market.

Notice that by making education an intra-period choice we have suppressed issues of imperfect capital markets — the costs do not have to be paid upfront. We can easily introduce capital market imperfections with no changes to the results. Indeed, notice that there may be a “least-cost education path” going from sector $j$ to sector $j'$ which could entail “passing through” some low-paying additional sector(s) — “apprenticeships” — and such situations would effectively involve upfront payments. We could also just as easily allow or disallow for borrowing on future income to sustain consumption smoothing. It really does not matter, but for concreteness we take $F(t) \geq B_m$ for all $t$, where $-B_m \geq 0$ represents a borrowing limit, and we impose $\lim \inf_t F_t \geq 0$ to guarantee ultimate repayment of all debts.
We presume throughout that the above maximization problem is always well-defined. A quick way to ensure this is to assume that all utility functions are bounded. Of course there are well-known weaker conditions that can imposed, for instance, when utility functions have a well-defined asymptotic elasticity.

2.6. Equilibrium. Given an initial allocation of individuals to sectors, \( \{j_m(-1)\} \), and an initial allocation \( \{k_m(0)\} \) of the capital stock \( K(0) \) to all individuals, an equilibrium is a sequence of wages \( \{w(t), w_r(t), w_e(t), w_k(t)\} \), prices \( \{p(t), p_r(t), p_e(t), p_k(t)\} \) and quantities \( \{F_m(t), z_m(t), e_m(t), j_m(t), k_j(t), r_j(t), h_j(t), y_j(t)\} \) for every person and every sector such that:

A. All individuals maximize lifetime utility as described in (4)–(7), with \( F_m(0) = p_k(0)k_m(0) \) for all \( m \), and firms maximize per-period profits at every date.

B. The final goods markets clear: at every date, and for every final good \( i \):

\[
\int \limits_m x_i(z_m(t), p(t)) dm = y_i(t).
\]

C. The robot market clears; for each \( t \):

\[
y_r(t) = \sum \limits_i r_i(t) + r_r(t) + r_e(t) + r_k(t).
\]

D. The human labor market clears; for each \( t \) and each sector \( j \), labor demand equals supply:

\[
h_j(t) = \text{Measure}(m : j_m(t) = j).
\]

E. The capital market clears; for each \( t \), \( K(t) \) evolves as in (3), with:

\[
K(t) = \sum \limits_i k_i(t) + k_r(t) + k_e(t) + k_k(t),
\]

and the available capital stock is willingly held by all individuals at each \( t \):

\[
p_k(t)K(t) = \int \limits_m F_m(t) dm.
\]

F. The education market clears; that is, for every \( t \):

\[
y_e(t) = \int \limits_m e_m(t) dm, \text{ where } \{e_m(t)\} \text{ satisfies equation (7)}.
\]

Per-capita national income (gross) is given by the expenditure on all final goods, plus investment in new capital goods and education:\(^{10}\)

\[
Y(t) = \sum \limits_i p(t)y_i(t) + p_e(t)y_e(t) + p_k(t)y_k(t).
\]

\(^{10}\) We could analogously work with net national income by subtracting the depreciation of capital. Nothing changes.
3. LONG RUN GROWTH AND AUTOMATION

3.1. An Illustrative Example. Consider the following special case of our model:

EXAMPLE 1. There is a single final good sector, indexed by 1, with production function

\[ y_1 = k_1^{1/2} \ell_1^{1/2}, \]

a capital goods sector with production function

\[ y_k = k_k^{1/2} \ell_k^{1/2}, \]

and a robot sector that has a CES production function with elasticity 1/2:

\[ y_r = \left( \frac{1}{2} k_r^{-1} + \frac{1}{2} \ell_r^{-1} \right)^{-1}. \]

There is free mobility of labor across all three sectors, no education sector, and common robot productivity given by \( \nu = \nu_1 = \nu_k = \nu_r. \)

Let \( \lambda \) be the corresponding price of effective labor, common to all three sectors. Then \( \lambda \) is just the (common) wage \( w \) if there is no automation, and it is \( \nu^{-1} p_r \) if there is (partial or full) automation. Let’s record the unit cost functions. For the final good and capital sector, these are

\[ c_1(1, \lambda) = c_k(1, \lambda) = \sqrt{\lambda}, \]

and for the robot sector,

\[ c_r(1, \lambda) = \frac{1}{2} \left[ 1 + \sqrt{\lambda} \right]^2. \]

All individuals have the same one-period utility function \( u(x) = \ln(x) \) along with discount factor \( \beta \in (0, 1). \)

To see how equilibrium paths evolve, notice that at any date, robot prices must be given by

\[ (15) \quad p_r(t) = c_r(1, \lambda_r(t)) = \frac{1}{2} \left[ 1 + \sqrt{\lambda(t)} \right]^2. \]

**Case 1:** \( \nu \leq 1/2. \) Then we claim that automation cannot occur. For if it did at any date \( t \), then it must be the case that \( \lambda(t) = \nu^{-1} p_r(t). \) Substituting this into (15), we see that

\[ p_r(t) = \frac{1}{2} \left[ 1 + \nu^{-1} p_r(t) \right]^2 > \frac{1}{2} \nu^{-1} p_r(t), \]

which contradicts our premise that \( \nu \leq 1/2. \) So there is never any automation in any equilibrium. For \( \beta \) large enough, an equilibrium involves unbounded accumulation of capital, an ever-growing wage rate, and a constant share of labor income — 50% in this example — in the long run. This last observation follows from the fact that the production functions for final goods and capital are Cobb-Douglas, and the fact that the robot sector shuts down in equilibrium, though prices for robots are well-defined and rise in tandem with human wages.

**Case 2:** \( \nu > 1/2. \) Then we claim that if the economy exhibits sustained growth of per-capita income — as it indeed will if individuals are patient enough — then all sectors that grow must be fully automated in the long run. For suppose this assertion is false. Then the percentage of human labor used in at least one of these three growing sectors is bounded away from zero.
(perhaps along a subsequence of dates). But because the total amount of human labor is bounded, so must be the total labor input in that sector. Then sustained growth must imply that the machine capital stock is accumulated without bound, so human wages climb to infinity. If there is no full automation, then
\[
(1) \quad \nu^{-1}p_r(t) = w(t) \quad \text{along a subsequence of dates},
\]
and using (15),
\[

\frac{\nu^{-1}p_r(t)}{w(t)} = \frac{1}{2\nu} \left[ \frac{1}{\sqrt{w(t)}} + 1 \right]^2 \rightarrow \frac{1}{2\nu} < 1 \text{ as } t \rightarrow \infty,
\]
but this is a contradiction, as it implies that \( \nu^{-1}p_r(t) < w(t) \) for \( t \) far enough on the subsequence.

We note five additional points. First, automation pins down robot prices relative to the rental rate of capital. This is akin to a “non-substitution theorem” and would be quite generally true even with heterogeneous robots and multidimensional machine capital. In this specific case, automation of the robot sector ensures that
\[
(2) \quad p_r = p^*_r,
\]
where — by (15) — \( p^*_r \) solves the equation
\[

p^*_r = \frac{1}{2} \left[ 1 + \sqrt{\nu^{-1}p^*_r} \right]^2,
\]
and it can be checked that a solution to (2) is well-defined and unique, given \( \nu > 1/2 \).

Second, the pinning-down of robot prices as in (2) guarantees that with sustained growth, every sector that expands indefinitely must also become automated, no matter what its specific value of \( \nu \) is. (In our example \( \nu \) is the same for all three sectors, but this is irrelevant.) For if in some growing sector, automation did not occur, machine capital must grow without bound, and so human wages must be unbounded, showing that ultimately \( p_r(t)/w(t) = p^*_r/w(t) \) must decline to zero, thereby precipitating automation, a contradiction.

Third, and relatedly, Case 2 is predicated on there being long-run growth to begin with. If per-capita income is bounded, then automation may not happen — there will be no sectors that grow.

Fourth, full automation cannot be immediate, and indeed, it cannot occur at any finite time. The wage rate must equal \( \nu^{-1}p^*_r \) in any automated sector and all humans must be hired at any date — otherwise they would contest the robots by undercutting them.

And finally, despite the full employment of labor, the share of labor in national income must decline to zero. By the previous point, that decline is “gradual” but with sustained growth, it is inexorable. Also, if we were to extend this example to have many sectors, with different values of \( \nu \), the wage rate could progressively rise over time even as the share declines. Relative immiserization of the labor share is entirely consistent with absolute improvement of labor income.

However, while anticipating these insights, the example is more obscure about the deeper conditions that separate Case 1 from Case 2. We now investigate this issue in more detail.

\[\text{---11---}\]

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3.2. The Von-Neumann Singularity Condition. Return to the general model. Consider the robot sector. Provisionally fix the price of effective labor at $\lambda = 1$ and take the capital rental rate — call it $\eta$ — to zero; that is, look at

$$\lim_{\eta \to 0} c_r(\eta, 1).$$

In words, imagine what happens to the unit cost of production as the capital input becomes nearly free. In the CES class with

$$f_r(k, \ell) = \left[ak^{\frac{\sigma - 1}{\sigma}} + (1 - a)\ell^{\frac{\sigma - 1}{\sigma}}\right]^{\frac{\sigma}{\sigma - 1}},$$

with $a \in (0, 1)$ and the elasticity of substitution $\sigma \geq 0$, simple computation tells us that

$$c_r(\eta, 1) = \left[a^\sigma \eta^{1-\sigma} + (1 - a)\eta^\sigma\right]^{1/(1-\sigma)}.$$

So our limit of interest equals zero whenever $\sigma > 1$, which includes the Cobb-Douglas case (“enough” substitution is available). But the limit is positive when $\sigma < 1$. For instance, if the production function is “almost” Leontief, labor costs will enter unit cost no matter how cheap the capital input may be. It turns out that this limit unit cost is related to the automation of the robot sector itself, a singularity that was envisaged by John von Neumann.12

**Proposition 1.** Suppose that the following condition holds on the robot sector:

(17) \hspace{1cm} \nu_r > \lim_{\eta \to 0} c_r(\eta, 1).

Then there is a unique solution $p_r^*$ to the equation

(18) \hspace{1cm} p_r = c_r(1, \nu_r^{-1} p_r),

and in any equilibrium, $p_r(t) \leq p_r^*$ for all $t$; i.e., the price of robots is bounded relative to the rental rate on capital. If at any date, the robot sector is automated, then $p_r = p_r^*$.

To understand the proposition, we begin by parsing (17). If it holds, then in any equilibrium — noting that $p_r$ equals unit cost by the zero profit condition — we would have $\nu_r > p_r$ for $\eta$ small, or $\nu_r^{-1} p_r < 1$. That is, the effective price of robot services would fall below the labor price (fixed at 1) when $\eta$ is small, suggesting that robot production must be automated in such situations. That is why we call (17) a singularity condition, as it points towards automation of the robot sector, at least when capital is cheap relative to labor. In the CES case, observe that (17) must hold no matter how low the value of $\nu_r$ (or how inefficient robot services are relative to labor), provided the elasticity of substitution is 1 or more. When that elasticity falls below 1, then (17) does restrict the value of $\nu_r$. Specifically,

(19) \hspace{1cm} \text{Either } \sigma \geq 1, \text{ or } \sigma \in (0, 1) \text{ and } \nu_r > a_{\ell}^{\sigma / (1-\sigma)}.

In the example earlier, $\sigma = 1/2$ and $a_{\ell} = 1/2$, and the corresponding threshold for $\nu_r$ in (19) divides Case 1 from Case 2.

12See von Neumann (1966), *Theory of Self-Reproducing Automata*: “The accelerating progress of technology, and changes in the mode of human life, give the appearance of approaching some essential singularity in the history of the race beyond which human affairs, as we know them, can not continue.”
The result of Proposition 1 shows that under the singularity condition, the price of robots joins hands with the price of capital rather than the price of human labor, at least when human labor is relatively expensive. Again, this is related to the non-substitution theorem. It is always true that

\[ p_r = c_r(1, 0, 1) \leq c_r(1, \nu_r^{-1} p_r), \]

the inequality holding because \( \lambda_j \equiv \min\{w_r, \nu_r^{-1} p_r\} \leq \nu_r^{-1} p_r \). If singularity holds, then a little calculation using the linear homogeneity of \( c_r \) shows that \( c_r(1, \nu_r^{-1} p_r) \), viewed as a function of \( p_r \), must ultimately dip below the \( 45^0 \) line; see Panel A of Figure 1. It follows that \( p_r^* \), as described by (18) is well-defined and finite. (See Panel B to appreciate that \( p_r^* \) will not be finite if the singularity condition fails.) It follows that (20) is equivalent to the assertion of the Proposition that \( p_r(t) \leq p_r^* \) in any equilibrium and at every date. Moreover, if there is automation, then (20) holds with equality and so \( p_r(t) = p_r^* \). In this way, the von Neumann singularity condition implies that the price of robot services must be tied to the price of capital rather than labor, even though in principle, both inputs could be used to produce those services.

3.3. Inevitable Automation Under Long Run Growth: A Baseline Result. We begin with the case of a finite number of final goods. But finiteness is not a mere technicality, as we shall explain in a later section.

**Theorem 1.** Suppose there are a finite number of consumer goods, and the robot sector satisfies the von Neumann singularity condition (17). Consider any equilibrium with long-run growth in per-capita income (i.e., \( Y(t) \) grows to infinity as \( t \to \infty \)).

(i) There exists at least one sector \( j \) and a subsequence along which \( y_j(t) \) grows to infinity. Along any such subsequence for any such sector, it must get automated by some finite date, and the share of labor performed by robots must converge to one.
(ii) The wage rate is bounded, and the share of human labor in national income must converge to zero.

Proof. Part (i). As per capita income goes to $\infty$ in the long run, so must per capita consumption (via standard arguments). Next observe that given that the singularity condition holds, the price of every good $j$ is bounded above:

$$p_j = c_j(1, \lambda_j) \leq c_j(1, \nu_j^{-1} p_r) \leq c_j(1, \nu_j^{-1} p_r^*).$$ (21)

Hence there must exist at least one sector and a subsequence along which its output goes to $\infty$.

Next we argue that any such sector must eventually become automated along any subsequence where its output grows unboundedly. If this is false, there is a further subsequence of dates indexed by $\tau$ along which the sector is persistently non-automated, i.e., $w_j(\tau) \leq \nu_j^{-1} p_r(\tau)$ and $r_j(\tau) = 0$ for all $\tau$. Now $y_j(\tau) \to \infty$ along such a subsequence, but at the same time $h_j(\tau)$ is bounded above by the total labor endowment of the economy. Then along such a subsequence $k_j(\tau) \to \infty$, and $\frac{h_j(\tau)}{h_j(\tau)} \to \infty$. This in turn implies the marginal product of capital in sector goes to zero and the marginal product of labor goes to $\infty$ as $\tau \to \infty$. Let $\hat{k}_j(\tau)$ and $\hat{h}_j(\tau)$ denote amounts of capital and human labor used for one unit of output at date $\tau$. Then $\hat{k}_j(\tau)$ is bounded away from zero, while $\hat{h}_j(\tau) \to 0$ as $\tau \to \infty$. (The latter is obvious, the former owes to the fact that capital $k_j(\tau)$ must be growing faster in the long run than output $y_j(\tau)$ as labor input does not grow in the long run, and the marginal product of capital is going to zero). Therefore invoking (2) for this sector, we see that

$$p_j(\tau) = c_j(1, \lambda_j(\tau)) = c_j(1, w_j(\tau)) = \hat{k}_j(\tau) + w_j(\tau)\hat{h}_j(\tau),$$ (22)

is bounded away from zero. Using the end-point condition on $f_j$, it follows that

$$w_j(\tau) = p_j(\tau) \frac{\partial}{\partial h} f_j(k_j(\tau), h_j(\tau)) \to \infty \text{ as } \tau \to \infty.$$ (23)

This contradicts the hypothesis that the sector is never automated, i.e., $w_j(\tau) \leq \nu_j^{-1} p_r(\tau) \leq \nu_j^{-1} p_r^*$. Hence the sector must become automated by some finite date.

Moreover, we claim that the share of robot labor in total labor in this sector must converge to one along this subsequence. Otherwise there exists some $\epsilon > 0$ such that for all $\tau$:

$$\epsilon \leq \Psi_j(\tau) \equiv \frac{h_j(\tau)}{\nu_j(\tau)r_j(\tau) + h_j(\tau)},$$

Then the total effective labor in that sector is bounded:

$$\ell_j(\tau) = \nu_j(\tau)r_j(\tau) + h_j(\tau) \leq \frac{h_j(\tau)}{\epsilon} \leq \frac{1}{\epsilon},$$

where recall that “1” is the total human labor endowment. Now we can write a version of (22):

$$p_j(\tau) = c_j(1, \lambda_j(\tau)) = \hat{k}_j(\tau) + \lambda_j(\tau)\hat{\ell}_j(\tau),$$ (24)

and then observe — analogous to the way we derived (23) — that

$$\lambda_j(\tau) = p_j(\tau) \frac{\partial}{\partial \ell} f_j(k_j(\tau), \ell_j(\tau)) \to \infty \text{ as } \tau \to \infty.$$ (25)
But (25) contradicts \( \lambda_j(\tau) = \nu_j^{-1} p_r(\tau) \), because Proposition 1 tells us that \( p_r(\tau) \) is bounded above by \( p_r^* < \infty \). This concludes the proof of (i).

Part (ii). In what follows, we drop the index \( t \) for notational ease. Recall (14) to write out national income (at any date):

\[
Y = \sum_i p_i y_i + p_c y_e + p_k y_k.
\]

Given constant returns to scale, this can be expressed as the sum of (machine) capital and human income. Formally,

\[
Y = \sum_i p_i y_i + p_c y_e + p_k y_k = \sum_{j \neq r} [k_j + p_r r_j + w_j h_j]
\]

\[
= \sum_{j \neq r} [k_j + w_j h_j] + p_r [y_r - r_r] = \sum_{j \neq r} [k_j + w_j h_j] + [k_r + w_r h_r]
\]

\[
= \sum_j [k_j + w_j h_j] = K + \sum_j w_j h_j.
\]

At the same time, defining \( \nu \equiv \min_j \nu_j > 0 \), we know that for every sector \( j \) with \( h_j > 0 \),

\[
w_j \leq \nu_j^{-1} p_r \leq \nu_j^{-1} p_r \leq \nu_j^{-1} p_r^*
\]

by Proposition 1. Therefore wage income in the economy \( \sum_j w_j h_j \) is bounded. Now \( Y \) goes to infinity as \( t \to \infty \), since \( Y \) at any date is at least as large as per capita consumption expenditure. Hence the result follows from (27).

The underlying logic for the distributional result (iii) is quite simple. The singularity condition implies the feasibility of producing robots using only machine capital and robots, thereby dispensing with humans altogether. This generates a finite upper bound to the unit cost of producing robots, and hence the cost of robots. In turn this creates an upper bound to wages of human labor in every sector, owing to the threat (i.e., the ever-present option) of automation. Hence wage income is bounded, implying the share of humans in national income must go to zero as income grows unboundedly in the long run. Part (i) states in addition that in every sector that is “consequential” in the long run (in the sense of having unbounded output) must eventually become automated, with human labor asymptotically insignificant relative to robots. Otherwise, the capital-labor ratio in that sector must grow to infinity, resulting in ever-growing human wages — something prevented by the threat of automation.

Observe also that by the same argument, the share of human labor in value added within any given consequential sector must also be converging to zero. A standard model without “micro-level” automation will not generate this result, unless the elasticity of substitution between capital and labor exceeds one (which runs counter to the empirical evidence within sectors). Our theory therefore can explain a rising capital share, without any assumptions on elasticity of substitution between machine capital and labor within any sector (with the exception of the robot sector in the form of the singularity condition). Automation is an indirect way by which capital substitutes for human labor in the long run within each industry, by displacing them with robots which does not depend on humans to build (thanks to singularity).
Theorem 1 is stated in substantial generality and makes no assumptions about the structure of preferences. Nor does it require any assumptions on labor specificity, or determinants of human capital investments (education costs, or household borrowing constraints). The fundamental driving force is the unbounded growth of capital relative to human labor. Absent automation, the resulting rise in labor scarcity pushes wages upward without limit — this motivates the production and use of robots as a substitute for humans. Singularity implies the production and cost of robots can be divorced from humans altogether, thereby bounding the wages of humans.

But the logic does use the assumption of a finite number of sectors, in at least two important ways. First, it implies a uniform upper limit to the effectiveness of humans relative to robots across all sectors, i.e., a limit to the extent to which humans are protected from automation anywhere in the economy. This may seem hard to believe as a description of the world, where there exist tasks and sectors where the potential effectiveness of robots in replacing humans is arbitrarily small, and accordingly the prospect of automation is arbitrarily distant. Second, the assumption limits the scope for humans to respond to automation in sectors they have been traditionally been working in, by investing in skills needed to enter sectors that are yet to be automated. Every sector eventually gets automated at some date; hence a finite number of sectors implies that by some finite date every sector is automated, and there are no sectors left for humans to escape to.

3.4. **An Extension to an Unbounded Number of Sectors.** We now examine how our results extend to an infinite number of sectors. Without loss of generality arrange the goods so that $\nu_i \geq \nu_{i+1}$ for all $i$. The next result does not impose any assumption on $\lim_{i} \nu_i$, thereby allowing this limit to be 0. All other assumptions on the production side of the economy remain unchanged.

An infinity of goods allows the effectiveness of robots relative to humans to be vanishingly low in parts of the economy, while keeping the possibility of automation open in each sector. Therefore, at any finite date, there will exist a positive measure of sectors that are yet to be automated. The wages in such sectors might also be rising progressively, without bound. It is then conceivable that humans may withstand the progress of automation in the long run by constantly moving to such sectors, and thereby retain a positive fraction of national income in the long run.

But intersectoral migration of humans needs market acquiescence on the demand side. Below we show that the extent of migration is inadequate to prevent the slide in human labor share, if household preferences are homothetic. As explained below, this inadequacy persists if homotheticity were relaxed to apply only to preferences of “arbitrarily affluent” households. Homotheticity restrictions prevent households from skewing the pattern of their expenditures progressively in favor of “human friendly” sectors as they become arbitrarily affluent.

That said, the conclusion regarding the boundedness of wages in Theorem 1 no longer applies. We show below that if education costs are bounded, and if relative robot efficiency becomes vanishingly small along the sequence of sectors, humans enjoy unboundedly large wages in the long run, even as the share of those wages in national income declines to zero.

Turn now to the details. Each one-period utility indicator $u_m$ is now defined on infinite-dimensional vectors of final goods. For technical reasons, we suppose that there are only finitely many distinct utility functions and discount factors indexed by $m$. We assume that for every $m$, $u_m$ is homothetic, and that a demand function $x_m(p, z)$ is well-defined for every strictly positive price.
vector $p$. By homotheticity, that demand function can be factorized as follows:

$$x_m(p, z) = d_m(p)z$$

for some function $d_m$. We place mild restrictions on this demand function, but do so explicitly given that we are dealing with an infinite-dimensional commodity space:

[Well-Behaved Demand] For each $m$, $d_m(p)$ is pointwise continuous in $p$ at every price vector $p \gg 0$. Moreover, if for a given sequence of price vectors $p^n \gg 0$, $I$ the set of goods $i$ for which $p_i^n \to 0$ is non-empty, then $\lim \inf_n d_{jm}(p^n) > 0$ for at least one index $j \in I$.

Observe that Dixit-Stiglitz preference aggregators with elasticity of substitution no less than 1 satisfy this condition, as well as the additional restrictions on demand invoked (only) in the last part of the next Theorem.

**Theorem 2.** Suppose there are an infinite number of consumer goods, and:

(a) The robot sector satisfies the von Neumann singularity condition (17).

(b) Preferences are homothetic, generating well-behaved demand.

Consider any equilibrium with long-run growth in per-capita income, as defined in Theorem 1. Then:

(i) Part (i) of Theorem 1 continues to be true.

(ii) The share of human labor in national income converges to zero as $t \to \infty$.

(iii) If, moreover, the following additional assumptions hold, the wage earned by every human goes to infinity as $t \to \infty$:

[Positive Demand] For each price $p \gg 0$, $d_{mi}(p) > 0$ for infinitely many goods $i$.

[Protection] There is a sequence of sectors progressively friendly to humans: $\nu_t \to 0$.

[Bounded Education Costs] Let $E_{j_1 j_2}$ denote education costs along the least-cost path from $j_1$ to $j_2$. Then $\sup_{j_1 j_2} |E_{j_1'} - E_{j_2''}| < \infty$.

**Proof.** We start by showing that prices must be bounded away from zero:

**Lemma 1.** Let $p(t)$ be a corresponding sequence of equilibrium prices for any equilibrium with $Y(t) \to \infty$. Then for each good $i$, $\lim \inf_t p_i(t) > 0$.

**Proof.** See Appendix.

Now return to the main proof.

Part (i). Call a sector consequential if its output goes to $\infty$ along a subsequence. We claim that there is a consequential final-goods sector. By the singularity condition and Proposition 1, all

\[\text{As discussed earlier, there may be a “least-cost education path” which passes through some low-paying additional sector(s). } E_{j_1 j_2} \text{ is obtained by simply adding educational requirements for every step of this path.}\]
equilibrium prices are bounded above:

\[ p_j(t) = c_j(1, \lambda_j(t)) \leq c_j(1, \nu_j^{-1} r_j(t)) \leq c_j(1, \nu_j^{-1} p_r(t)). \]

Using a diagonal argument, extract a convergent subsequence of equilibrium prices that converges pointwise (but retain original index \( t \)) to some price vector \( p^* \). By Lemma 1, \( p^* \gg 0 \), and so, applying Well-Behaved Demand, we have \( d_m(p(t)) \) converging pointwise to \( d_m(p^*) \). It is easy to see that per-capita consumption expenditures on final goods must go to \( \infty \). Pick any type \( m \) for whom \( z_m(t) \to \infty \). Obviously, \( d_{im}(p^*) > 0 \) for some good \( i \). Combining all this information, we must conclude that for at least one final good sector, demand must go to infinity along this subsequence. The rest of the proof is identical to that of part (i) in Theorem 1.

Part (ii). Notice that

\[ \Psi_j(t) = \frac{h_j(t)}{v_j r_j(t) + h_j(t)} = \frac{w_j(t) h_j(t)}{p_r(t) r_j(t) + w_j(t) h_j(t)} \]

for every sector \( j \) and every date \( t \). If \( \sigma(t) \) denotes the share of human labor in national income, it follows that

\[ \sigma(t) = \frac{\sum_j w_j(t) h_j(t)}{Y(t)} = \frac{\sum_j \Psi_j(t)[p_r(t) r_j(t) + w_j(t) h_j(t)]}{Y(t)} \leq \frac{\sum_j \Psi_j(t) p_j(t) y_j(t)}{Y(t)} \]

at every date \( t \), where \( Z(t) \) stands for economy-wide per-capita current expenditures on final goods. Because there are finitely many homothetic preference groups indexed by \( m \), we can write for every final good \( i \) and date \( t \):

\[ \frac{p_i(t) y_i(t)}{Z(t)} = \sum_m \zeta_m(t) s_{mi}(t) \]

where \( \zeta_m(t) \) is the expenditure share of preference type \( m \) in total expenditure, and \( s_{mi}(t) \) is the corresponding expenditure share of good \( i \) for preference group \( m \). Because each of the terms \( \zeta_m(t) \) are bounded above by 1, we may combine (30) and (31) to write:

\[ \sigma(t) \leq Q(t) \equiv \frac{Z(t)}{Y(t)} \sum_{i=1}^{\infty} \Psi_i(t) \left[ \sum_m \zeta_m(t) s_{mi}(t) \right] + \sum_{j=e,r,k} \Psi_j(t) p_j(t) y_j(t) \frac{Z(t)}{Y(t)} \]

for all \( t \). We will show that the right hand side of (32) — defined to be \( Q(t) \) — converges to 0 as \( t \to \infty \). To this end, pick any subsequence of dates (but retain original notation) so that \( Q(t) \) converges. Exploiting the fact that the number of sectors is countable, using a diagonal argument to extract a further subsequence (again retain notation) so that each of the sequences \( \Psi_j(t), \zeta_m(t), s_{mi}(t), p_j(t), Z(t)/Y(t) \) and \([p_j(t)y_j(t)]/Y(t) \) also converge. The last term in \( Q(t) \) pertains only to three sectors: \( e, r \) and \( k \). For any of these sectors, call it \( j \), \( \Psi_j(t) \to 0 \) along any subsequence for which \( j \) is consequential, and on any other subsequence \( p_j(t)y_j(t) \) must be bounded. Putting these observations together with \( Y(t) \to \infty \), we must conclude that the last term in \( Q(t) \) converges to 0.

\[ \text{14This is trivial when either } r_j(t) \text{ or } h_j(t) \text{ equals zero; when both are positive, we know that } p_r(t) = \nu_j w_j(t). \]
Now for the first term. If $Z(t)/Y(t) \to 0$, we are done, so assume in what follows that $Z(t)/Y(t)$ has a strictly positive limit. Let $M$ be the set of all indices for which $\lim_{t} \zeta_{m}(t) > 0$ for the subsequence under consideration. Then, using the fact that the interchange of a finite and infinite sum is always valid, we have

$$
\sum_{i=1}^{\infty} \Psi_{i}(t) \left[ \sum_{m} \zeta_{m}(t) s_{mi}(t) \right] = \sum_{m} \zeta_{m}(t) \left[ \sum_{i=1}^{\infty} \Psi_{i}(t) s_{mi}(t) \right]
$$

$$
= \sum_{m \in M} \zeta_{m}(t) \left[ \sum_{i=1}^{\infty} \Psi_{i}(t) s_{mi}(t) \right] + \sum_{m \not\in M} \zeta_{m}(t) \left[ \sum_{i=1}^{\infty} \Psi_{i}(t) s_{mi}(t) \right]
$$

(33)

Because $\zeta_{m}(t) \to 0$ for all $m \not\in M$, the second term on the right hand side of this equation converges to 0. It remains to show that same is true of the first term. It will suffice to show that for each $m \in M$,

$$
\sum_{i=1}^{\infty} \Psi_{i}(t) s_{mi}(t) \to 0
$$

(34)

as $t \to \infty$ along our chosen subsequence. Because $\lim_{t} \zeta_{m}(t) > 0$ for $m \in M$, and given that $Z(t) \to \infty$ (after all, $Z(t)/Y(t)$ has a positive limit and $Y(t) \to \infty$) it follows that expenditures diverge to infinity for a positive measure of individuals of type $m$. Let $Z_{m}(t)$ be the aggregate income of type $m$ and $x_{mi}(t)$ the aggregate demand for good $i$ by this type. Homotheticity guarantees that

$$
\hat{s}_{mi} \equiv \lim_{t} s_{mi}(t) = \lim_{t} \frac{p_{i}(t)x_{mi}(t)}{Y_{m}(t)} = \lim_{t} \frac{p_{i}(t)d_{i}^{m}(p(t))Y_{m}(t)}{Y_{m}(t)} = \lim_{t} p_{i}(t)d_{i}^{m}(p(t)),
$$

(35)

By Lemma 1 and Well-Behaved Demand, it follows that $\hat{s}_{mi}$ forms a “bonafide share vector” with $\sum_{i} \hat{s}_{mi} = 1$. So the conditions in Observation 1 in the Appendix are satisfied (ignore index $m$). Therefore that Observation applies, which implies (34). Hence the income share of human labor must converge to zero. Because (27) holds unchanged, the income share of capital converges to 1.

Part (iii). Let $\{p(t)\}$ be the sequence of equilibrium prices. Given the bound (29), we can use a diagonal argument to extract a convergent subsequence $\{p(t_{s})\}$ which converges pointwise to some price sequence $p^{*}$ as $s \to \infty$. By Lemma 1, $p^{*} \gg m$, and so applying Well-Behaved Demand, we see that for each type $m$, $d_{m}(p(t)) \to d_{m}(p^{*})$ pointwise along this subsequence. By Positive Demand, $d_{mi}(p^{*}) > 0$ for an infinite number of indices $i$. Putting all these arguments together and invoking homotheticity, we see that an infinite number of final goods sectors must be consequential.

We now claim that for any number $W$, however large, there exists a time $T$ such that for all dates $t \geq T$, $w_{j}(t) \geq W$ for every sector $j$ for which $h_{j}(t) > 0$. For suppose that this claim is false; then there exists some sector $q$ and a subsequence of dates (retain original notation $t$) such that $\sup_{t} w_{q}(t) \equiv W_{q} < \infty$, but $h_{q}(t) > 0$. Next, pick a consequential final goods sector $\ell$ with the
property that

$$\nu_t^{-1} p_t^* > W_q + \bar{p}_e \sup_{j'} |E_{j'\ell} - E_{j'q}|,$$

where $\bar{p}_e$ is the upper bound on education prices given by (29). The existence of such a sector is guaranteed, first, by recalling that there is an infinite number of consequential sectors, then invoking Increasing Protection ($\nu_i \to 0$), and finally by Bounded Education Costs, which guarantees that the supremum in (36) is finite. Because sector $\ell$ is consequential, it is automated after some finite date so there exists $T'$ such that for all $t \geq T'$ along the subsequence identified above, we have $w_\ell(t) \geq \nu_t^{-1} p_t^*$. Combining this information with (36), we must conclude that for every sector $j$, and for $t \geq T'$,

$$w_\ell(t) \geq \nu_t^{-1} p_t^* > W_q + \bar{p}_e \sup_{j'} |E_{j'\ell} - E_{j'q}| \geq w_q(t) + p_e(t)[E_{j'\ell} - E_{j'q}],$$

so that

$$w_\ell(t) - p_e(t)E_{j'\ell} > w_q(t) - p_e(t)E_{j'q}.$$ 

But this inequality proves that while sector $\ell$ might lie along a least-cost educational path, no individual can ever want to enter sector $q$ at large dates, which implies that $h_q(t') = 0$ for all large enough $t'$ along the subsequence, a contradiction.

The model now allows existence of sectors where the comparative effectiveness of robots vis-à-vis humans is arbitrarily low. On the other hand, every consequential sector will eventually become automated. If both these are true, it must be the case that at any date, no matter how distant, there will exist sectors that are yet to be automated. And as time goes by, more sectors become automated, and in these the share of human wages will shrink to zero. Since the human share will be positive in yet-to-be-automated sectors, the aggregate share of human wages in national income can be positive in the long run only if the expenditure share of these sectors is bounded away from zero. The proof shows this cannot happen if preferences are homothetic and generate well-behaved demand. The argument is based on showing first (in Lemma 1) that any long run limit point of prices is strictly positive. The continuity of demand at this limit point ensures that the corresponding limit point of expenditure shares is a ‘bona-fide’ set of shares that adds up to unity, and in particular where the aggregate share of the set of automated sectors must converge to one (since this set eventually includes all sectors). Hence the share of the yet-to-be-automated sectors must shrink to zero. Intuitively, in the long run, market demand cannot generate sufficient entry into the ever-shrinking set of ‘niche’ sectors where humans are yet to be displaced.

At the same time, notice how part (iii) of the theorem permits the absolute return to human labor to grow without bound, at the same time that their return relative to that of capital converges to zero. Part (iii) is not applicable generally — it rests on the assumption that education costs are bounded across all pairs of occupations — but it is nevertheless of interest to see that absolute doom can be warded off even as relative doom is inevitable.

Homotheticity is sufficient for Theorem 2 but not necessary. For instance, a perusal of its proof will reveal that it would survive under “asymptotic homotheticity;” i.e., if (28) were assumed
to be approximately true only for large values of income. It would also remain true even in the absence of any homotheticity restriction on preferences, as long as with rising income, those preferences favor subsets of goods with the ratio of robot to human productivity bounded away from zero. There appears to be only one way to break the stranglehold imposed by the theorem: if preferences are non-homothetic and move progressively in favor of goods where humans have the largest productivity advantage over robots. We illustrate this possibility with an example.

**Example 2.** There is a countable infinity of final goods with production function \( f_i(k, \ell) = k^{1/2} \ell^{1/2} \) for all \( i \). There is a robot sector with production function \( f_r(k, \ell) = k^{1/2} \ell^{1/2} \), so the singularity condition is met. The capital goods producing sector produces non-depreciating capital with \( f_k(k, \ell) = ak \). There is free mobility of a fixed, unchanging amount of labor across all sectors, normalized to 1, and there is no education sector. All agents have the same utility function given by

\[
    u(x) = \ln \left( \sum_{i=1}^{\infty} \prod_{\ell=1}^{i} \min\{x, 1\} \right)
\]

and the same discount factor \( \beta > 1 + a \). The initial stock of capital is given.

These preferences are defined so that all available expenditure is spent on one unit each of the final goods, starting from \( i = 1 \) and progressing to higher indices until the budget is exhausted. This formulation mimics the lexicographic non-homotheticity pattern described in Baland and Ray (1991) and Matsuyama (2002).

With \( \beta > 1 + a \), this economy will grow without bound, and as it grows, the common human wage rate will climb. However, assuming robot productivity \( \nu_j \) falls “fast enough” with \( j \), and given the non-homothetic structure of preferences, humans will populate precisely those sectors that robots cannot compete in. Automation will indeed occur, but in the lower index sectors. While automation will spread over the entire space of goods with time, the highest index sectors will always be populated by humans. This intuitive argument can be formalized to show that the share of human labor in national income can remain bounded away from zero, even if there is unbounded automation in the long run.

The dismal conclusion of Theorems 1 and 2 can therefore be avoided, but only through a narrow escape hatch. Preferences must be non-homothetic, and the direction of that non-homotheticity must be precisely match the direction along which robot become progressively non-competitive compared to humans.

4. **What Guarantees Long Run Growth?**

Theorems 1 and 2 could be viewed as unsatisfactory in that they are predicated on the assumption of long-run growth, which isn’t a testable restriction on the primitives of the model. A theorem of the following kind might be preferable: “There exists a threshold for the discount factor (large enough) such that if some discount factors in the economy exceed this threshold, then the economy exhibits long-run growth.” We believe such a theorem to be true, but under the current

\footnote{Equation (35) in the proof of Theorem 2 remains true in the presence of asymptotic homotheticity.}
generality of our model we do not know (yet) how to prove it. However, we can make substantial progress by assuming that the capital goods producing sector only uses machines to produce capital. That is, in what follows, we will assume that
\[ f_k(k, \ell) = ak, \]
where \( a > 0 \). It is then obvious that along any equilibrium path, the price of capital is just the rental rate times \( 1/a \); so given our normalization, \( p_k(t) = 1/a \) for all \( t \). Therefore, invoking (6), the rate of return on financial capital is given by
\[ \gamma(t) = \gamma = 1 + (1 - \delta)(1/a) \]
\[ = a + (1 - \delta). \]
for all dates \( t \).

**Theorem 3.** Assume the singularity condition (17), and that
\[ \beta_m [a + (1 - \delta)] > 1 \]
for a positive measure of individuals. Then the economy exhibits long-run growth, and Theorems 1 and 2 must apply.

**Proof.** Consider the following infinite-horizon problem. Suppose that an individual has a constant discount factor \( \beta \) and a (possibly time-varying) sequence of smooth, concave one-period utility indicators \( u_t(z) \), each defined on a single consumption good \( z \) with unbounded steepness at zero. Suppose that the individual has \( F_0 \) units of a financial asset at date 0, receives a (fully anticipated) sequence of strictly positive incomes \( \{y(t)\} \), and faces a constant return factor \( \gamma \) on financial holdings. She chooses a sequence \( \{z(t), F(t)\} \) of consumptions and financial assets to maximize lifetime utility
\[ \sum_{t=0}^{\infty} \beta^t u_t(z(t)) \]
subject to the constraint that for every \( t \geq 0 \), \( F(t) \geq B \) (a credit limit), \( \lim \inf F(t) \geq 0 \), and
\[ y(t) + F(t) = z(t) + [F(t + 1)/\gamma]. \]
Assume that an optimum is well-defined, and assume \( \beta \gamma > 1 \). Then we claim that
\[ \lim_{t \to \infty} u'_t(z(t)) = 0. \]
The proof follows on observing that the Euler equation holds (with an appropriate inequality if there is a borrowing constraint):
\[ u'_t(z(t)) \geq \beta \gamma u'_{t+1}(z(t + 1)) \]
for all \( t \).\(^{16}\) Because \( \beta \gamma > 1 \) by assumption, it follows that \( u'_t(z(t)) \) must decline (geometrically in fact) to zero.

\(^{16}\)The opposite inequality cannot hold because the utility indicators have unbounded steepness at zero.
We can transplant this problem easily to our setting, provided we view incomes here as wages net of education costs. Identifying $u_t(z)$ with $v_m(z, p(t))$ for each individual $m$, it follows that in any equilibrium, if $\beta_m \gamma > 1$ (where $\gamma$ is defined in (37)), we have

\begin{equation}
\lim_{t \to 0} v_m'(z_m(t), p(t)) = 0.
\end{equation}

But $v_m$ is strictly increasing and concave for every $p$. Moreover, every final goods price is bounded above; see (29). Therefore (40) can only hold if $z_m(t) \to \infty$ as $t \to \infty$, and by our assumption, this must occur for a positive measure of individuals. With a bounded credit limit on the rest, it is easy to conclude that per-capita income $Y(t)$ as defined in (26) must go to infinity.

5. Technical Progress

We extend the theory to incorporate directed technical progress. "Directedness" means that technical progress is geared to input scarcity. The key assumption we make is that opportunities for such technical progress are symmetric over all inputs, and across all sectors of the economy. This is not to deny the possibility that the nature of science and technology might generate biases in certain directions. But the effect of such exogenous biases would be rather obvious and unsurprising. If they were to favor unbridled automation, our results would be a foregone conclusion. If they favored the augmentation of human quality over robots or machine capital, that would raise the share of humans in national income instead.

The two possibilities combined appear to point to a long run “balanced-growth” view of technical progress, a view developed by Acemoglu and Restrepo (2018, 2019), with antecedents in a literature that includes Hicks (1932), Salter (1966), Galor and Maov (2000), and Acemoglu (1998, 2002), among many others. Acemoglu and Restrepo (2018) generate balanced growth by assuming that newly developed tasks lie in the human domain, and humans enjoy temporary protection in such tasks from the robot invasion. But the latter is also hard at work, automating existing tasks and perennially chasing the moving human frontier created by new tasks. In the end, balance is achieved by an equilibrium across these two forces. This is a potentially fruitful approach, but one that is — perhaps unavoidably — laden with questions. Why are new tasks biased in favor of humans; mightn’t new tasks involving robots also appear as an outcome of technical progress? Or (the flip side): why cannot humans recover their edge in old tasks?

In this section, we take a more agnostic view, which is not to say that we adopt a more general specification than the one just described, but a more “symmetric” one. We consider a fixed number of sectors and tasks, and technical progress occurs on the intensive rather than extensive margin. There is no a priori restriction on sectors in which humans and robots can be active: the sectors in which a factor is used are determined endogenously. In contrast, Acemoglu and Restrepo study the extensive margin, where the task space is extended in the asymmetric way already described. In particular, within each sector/occupation, we allow for technical progress (on the intensive margin) in (non-robot) machine capital, on par with the possibility of technical progress in human labor, and in robot capital. As we will see, technical progress in non-robot capital will play an important role in our analysis.\footnote{Acemoglu and Restrepo abstract from non-robot machine capital in their model.}
The sole asymmetry we retain — and one that we have already developed — is the difference between the scaling-up of physical and human capital. It will turn out that this asymmetry is augmented in the presence of endogenous and symmetric — but directed — technical progress. To us, this is the natural theoretical benchmark. If there are stronger opportunities for technical progress that heighten the productivity of machine capital or robots relative to humans, our results will a fortiori be reinforced. And in the opposite case they will be moderated.

One other assumption will play an important role: productivity growth in any factor in any given sector will spill over to the use of the same factor in other sectors. There is empirical evidence of such spillovers (see, e.g., Bernstein and Nadiri 1988 and Johnson 2018). Such spillovers will ensure that the rate of productivity growth of any factor cannot diverge across sectors in the long run. The importance of this assumption goes beyond the simplification that it affords, as we explain in further detail below.

In summary, here is our setting: there is a fixed, finite number of sectors. Technical progress is symmetric but directed, and the productivities of all inputs are potentially affected in a way we make precise below. We study equilibria in which capital grows unboundedly in natural units, while human labor in natural units is stationary (or grows exogenously). With technical progress this is stronger than the preceding assumption of long run growth in per capita income: it requires the degree of patience of households to be high relative to a threshold that depends on the maximum feasible rate of technical progress. The resulting formalities are a generalization of Theorem 3 that we do not pursue here, though we expect the details to be straightforward.

5.1. Framework. We consider the baseline model with a finite number of final goods sectors. To allow for changes in the productivity of every input, we attach coefficients to physical capital and human labor in addition to the already-noted robot productivity $\nu$. So:

$$y_j = f_j(\theta_j k_j, \mu_j h_j + \nu_j r_j),$$

where the same assumptions are made on $f_j$ as before, and $(\theta_j, \mu_j, \nu_j) \gg 0$ are productivities that can be changed by deliberate technical progress.

We assume that the singularity condition holds on the robot-sector cost function, evaluated at the starting robot productivity $\nu_r(0)$ at date 0:

$$\nu_r(0) > \lim_{\eta \to 0} c_r(\eta, 1).$$

The later robot productivities will all be endogenous and so we impose no restrictions involving these, but it is easy to see that (41) could be further weakened. All our results hold for any competitive equilibrium for which (41) holds at some date $t$ along the equilibrium path.

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18 As emphasized in the Introduction, the facts (see, e.g., Karabarbounis and Neiman (2014)) regarding such asymmetry are quite clear, and they hold irrespective of whether controls are included for technical progress or the skill composition of the labor force.

19 This is not a consideration in Acemoglu and Restrepo (2019). There is a single sector in their model, so they do not have to deal with the complications arising from multiple sectors (and the related issue of inter-sectoral spillovers). However, we do not consider this last item to be a fundamental source of difference in the results.
5.2. **Endogenous Technical Progress.** We presume that each factor-sector pair is serviced by a short-lived, sector-specific inventor whose activities and returns are external to the economy in question.\(^{20}\) She can generate an increase in the sectoral productivity of that factor. In particular, assume that the productivity \(\pi_j(t)\) of a typical factor in sector \(j\) at \(t\) can be increased at some rate \(\rho_j(t)\) through R&D investment. A fixed proportion \(\gamma > 0\) of this growth spills over to the same factor in other sectors, subject to a maximum rate \(\bar{\rho}\) of productivity growth in any sector. So for any sector \(j\):

\[
\pi_j(t+1) - \pi_j(t) = \max \left\{ \rho_j(t)\pi_j(t) + \gamma \sum_{j' \neq j} \rho_{j'}(t)\pi_{j'}(t), \quad \bar{\rho}\pi_j(t) \right\}.
\]

The associated R&D cost of the inventor is \(\kappa(\rho)\), where \(\kappa'(0) = 0, \kappa''(\rho) > 0\) and \(\kappa''(\rho) > 0\) for all \(\rho\). View R&D investment as a game played across sectors and factors by inventors. Then for any \(i\), \(\rho_j(t)\) can be positive only if the first term on the right-hand side of (42) binds. So our inventor is either inactive, or fully owns (and so can license) the proprietary advance \(\rho_j(t)\pi_j(t)\) to firms operating in sector \(j\) in period \(t + 1\). The license fee is levied per (natural) unit of the factor employed by the firm at \(t + 1\). On the other hand, the spillovers are in the public domain, and are utilized freely by all firms.

Each inventor takes prices as given, as in competitive innovation models of Grossman and Hart (1979) and Makowski (1980). The maximum license fee \(L_j(t + 1)\) that can be charged by a date-\(t\) inventor:

\[
L_j(t + 1) = q_j(t + 1)\rho_j(t)
\]

where \(q_j(t)\) is the relevant factor price at \(t\). This is because the one efficiency unit of the factor costs \(\frac{q_j(t+1)}{\pi_j(t)}\) for someone without access to the technical advance, while it costs \(\frac{q_j(t+1)}{(1+\rho_j(t))\pi_j(t)}\) for someone who has access to it. The difference in unit cost is precisely \(\frac{q_j(t+1)\rho_j(t)}{\pi_j(t)(1+\rho_j(t))}\), so this can be sucked out as a license fee per efficiency unit. Multiplying by the number of efficiency units \(\pi_j(t)(1+\rho_j(t))\) made possible by the advance, we obtain expression (43). Intuitively, the “effective factor price” for licensees must rise by exactly the same proportion as the proprietary productivity advance.

It follows that if \(x_j(t)\) is the scale of employment of that factor in sector \(j\) at date \(t\), the total return earned by the inventor equals \(L_j(t + 1)x_j(t + 1) = \rho_j(t)E_j(t + 1)\), where \(E_j(t + 1) \equiv q_j(t + 1)x_j(t + 1)\) is the total bill for that factor in sector \(j\). Therefore, given the R&D choices \(\{\rho_j(t)\}_{j'\neq j}\) by inventors in other sectors, and given that our inventor is specific to the sector at hand, it is obvious that she will invest until \(\kappa'(\rho_j(t)) = E_j(t + 1)\) — provided \(\kappa'^{-1}(E_j(t + 1))\pi_j(t) + \gamma \sum_{j' \neq j} \rho_{j'}(t)\pi_{j'}(t) < \bar{\rho}\) — and \(\max\{0, \bar{\rho} - \gamma \sum_{j' \neq i} \rho_{j'}(t)\pi_{j'}(t)\}\) otherwise.

\(^{20}\)We can fully integrate the inventor into the economy by providing her with a production technology that again depends on machine capital and human/robot labor. We avoid that recursive extension here. However, one difference that will arise is that unlike all other sectors, ‘firms’ in the R&D sector will not be perfectly competitive, and will earn profits which will typically constitute a positive fraction of national income. The extent to which humans can be replaced by robots in the R&D sector then is an additional determinant of the share of humans in national income. This will be driven by the logic of cost minimization in the production of R&D, in a manner similar to that in other sectors. Hence it is possible to extend the model in this direction as well, without any qualitative change.
So the Nash equilibrium rate of productivity increase must satisfy:

\[
\frac{\pi_j(t + 1) - \pi_j(t)}{\pi_j(t)} = \max \left\{ \lambda^{-1} \left( E_j(t + 1) \right) + \frac{1}{\pi_j(t)} \sum_{j' \neq j} \rho_{j'}(t) \pi_{j'}(t), \bar{\rho} \right\}
\]

5.3. Equilibrium. With this setup, an equilibrium looks just like the competitive equilibrium in Section 2.6. That is because licensees give up all their surplus to the inventor, so that they effectively use the old technology until the new innovations enter the public domain. With the rental rate on capital once again chosen as numeraire, an equilibrium is a sequence of \( w(t), w_r(t), w_e(t), w_k(t) \), \( p(t), p_r(t), p_e(t), p_k(t) \) and associated quantities \( F_m(t), z_m(t), e_m(t), j_m(t), k_j(t), r_j(t), h_j(t), y_j(t) \) for every person and every sector, along with productivity coefficients \( \theta_j(t), \mu_j(t), \nu_j(t) \) for every input, sector, and date, such that:

(a) Given the sequence of productivities, the remaining sequence of outcomes constitutes a competitive equilibrium; and

(b) At every date, and for every sector and factor, given the equilibrium prices and quantities, productivity changes are the outcome of a Nash equilibrium as described earlier in this section.

5.4. Automation and the Vanishing Labor Share with Technical Progress. To state the main result of this section, we invoke two additional conditions. First, we assume that consumer expenditure shares on each final good are bounded away from zero:

[E] For any individual type \( m \) and good \( i \), \( \inf_{(p,z) \geq 0} p_i x_{mi}(p, z) / z > 0 \).

As there are a finite number of goods, this is fairly innocuous. Next, we make an additional assumption on the production technology:

[F] There is some final good \( i \) for which \( \lim_{\zeta \to 0} c_i(1, \zeta) > 0 \).

It is easy to verify that Condition F holds as long as there is at least one final good sector with constant elasticity of substitution strictly smaller than 1.

Theorem 4. Assume the singularity condition (41), and Conditions E and F. Then in any equilibrium which exhibits unbounded accumulation of machine capital, the wage earned by human labor in every sector is bounded, and the income share of human labor in the economy as a whole (excluding the R&D sector) must converge to zero as \( t \to \infty \).

Theorem 4 resurrects the dismal prediction on human labor shares under symmetric assumptions on the scope of technical progress. It goes without saying that the theorem is strengthened if there is greater scope for technical progress in automation rather technical progress in human productivity. It continues to highlight the effects of the main asymmetry across the nature of human and capital input, with regard to their growth in endowments in natural units.

The theorem makes, again, an assumption on growth, but this time the assumption is stronger: it is stated in a setting where machine capital is accumulated without bound. In a world without technical progress, such an assumption is equivalent to per-capita income growth, but in the
current situation, it is obviously stronger. It is possible to provide sufficient conditions for machine capital growth along the lines of Theorem 3. In that theorem, the discount factor needed to be high enough to encourage accumulation of capital at stationary prices. Now, technical progress induces a downward drift on prices (relative to incomes), which is an “automatic” — albeit endogenous — source of real income growth. For machine capital to be willingly accumulated despite this drift, the degree of patience must clear a higher threshold (which depends on the maximal rate $\bar{\rho}$ of technical progress), so that per-capita growth at this maximal rate is not enough to deter households from saving and investing at a positive rate.

Before proceeding to the proof, we remark on the additional conditions E and F. We believe that both conditions are technical. Condition E is possibly needed for the result, but is mild. As for Condition F, we don’t know if it can be dropped free of charge. We do know that it can be replaced by other conditions. For instance, for any sector $j$, define the intensive-form function $g_j(e) \equiv f_j(e, 1)$. Then the following condition substitutes for F:

[F'] There is some sector $j$ such that the infimum elasticity of the intensive form is positive:

$$\inf_{e>0} \frac{g_j'(e)e}{g_j(e)} > 0.$$ 

Conditions F and F' are not nested. A CES production function with elasticity of substitution smaller than 1 satisfies F but not F'. A Cobb-Douglas production function satisfies F' but not F.

Now turn to the proof. We outline the main steps, relegating technical details to the Appendix.

**Lemma 2.** (a) For each factor and sector $j$, there is a threshold $M$, such that factor productivity in that sector grows at the rate $\bar{\rho}$ if the factor bill $q_jx_j$ for that factor exceeds $M$.

(b) If the spillover rate $\gamma$ is positive, the relative productivity $\pi_j(t)/\pi_{j'}(t)$ of any factor between any pair of sectors $j$ and $j'$ is bounded.

Part (a) is straightforward, following from (44). If the factor bill $E_j(t+1)$ is large enough, the right-hand-side of (44) will equal the maximal rate $\bar{\rho}$ of technical progress. This reflects the force of the intensive margin in motivating technical progress in sector $j$. Part (b) ensures that benefits of technical progress in any sector spills over to other sectors, thereby preventing inter-sectoral divergence of factor productivity.

**Lemma 3.** Assume (41). For any sector $j$, there is $B_j < \infty$ such that in any equilibrium and at any date $t$,

$$\theta_j(t)\frac{p_c(t)}{\nu_c(t)} \leq B_j.$$ 

The von Neumann singularity property (41) implies a bound on robot prices $\frac{p_c(t)}{\nu_c(t)}$ in efficiency units, relative to the capital rental rate $\frac{1}{\nu_c(t)}$ in the robot sector. Spillovers across sectors then imply the same is true in every other sector in the long run, as a consequence of part (b) of Lemma 2. That gives us the bound in (45).
Next, note that unbounded accumulation of capital in natural units in the economy as a whole is likely to be spread across sectors, so there will exist sectors in which the use of capital in natural units will grow unboundedly. This can be verified to be true under conditions E and F:

**Lemma 4.** If E and F hold, in any equilibrium with \( K(t) \to \infty \), it must be the case that \( k_i(t) \to \infty \) for some sector \( i \).

The reasoning is as follows. Condition E ensures that household expenditure shares of different sectors cannot fluctuate ‘too much’ across dates, and the expenditure on each good must grow unboundedly in the long run. In turn this implies the value of output in any final good sector cannot fluctuate excessively, and must grow unboundedly. Condition F implies the ratio of value of output to capital is bounded, at least in some sectors. In such sectors, it follows that the amount of capital allocated used (in natural units) must grow without bound.

These three steps are combined as follows. We see from Lemma 4 that there is some sector in which the capital factor bill grows without bound (recall that capital in natural units is the numeraire). Lemma 2 then implies that the rate of technical progress in capital must eventually be at the maximal rate \( \bar{\rho} \) in this sector, and hence spills over to every other sector. In other words, the rate of technical progress in capital cannot fall behind the rate of technical progress in human labor in the long run. The asymmetric growth in endowments in natural units between capital and human labor thereby generates a bias (at least weakly) in technical progress in favor of capital. This compounds the distributional shift in favor of capital; hence our preceding results concerning the long run income share of human labor continue to hold in the presence of directed technical progress.

To formalize these arguments a bit more, notice that in any sector \( j \) that employs human labor at any date \( t \), humans must be cost-effective relative to robots:

\[
\frac{w_j(t)}{\mu_j(t)} \leq \frac{p_r(t)}{\nu_j(t)} \leq \frac{B_j}{\theta_j(t)\nu_j(t)},
\]

where the second inequality follows from Lemma 3. This implies that the wage earned by humans is bounded in every sector in which humans are employed, since (46) implies that

\[
w_j(t) \leq \frac{\mu_j(t)}{\theta_j(t)} \frac{B_j}{\nu_j(t)},
\]

and \( \frac{\mu_j(t)}{\theta_j(t)} \) is bounded owing to capital productivity growing at least as fast as human productivity in every sector in the long run, while \( \nu_j(t) \) is non-decreasing in \( t \).

Hence human wages continue to be bounded, just as in the absence of technical progress, implying that the share of human labor income \( \sum_j h_j(t)w_j(t) \) in national income \( K(t) + \sum_j h_j(t)w_j(t) \) must converge to 0 in the long run, given that \( K_t \to \infty \) as \( t \to \infty \).

The proof illustrates the role of the underlying asymmetry between growth of capital and human labor in natural units. Unbounded capital accumulation relative to human labor implies a corresponding asymmetry in factor bills, and thereby in technical progress on the intensive margin. That ensures a parallel asymmetry in rates of technical progress, assuming that the opportunities for technical progress are symmetric to begin with.
What is less transparent is the importance of inter-sectoral spillovers in this argument. We have constructed an example (available on request) where in the absence of spillovers, Theorem 4 does not hold despite all other assumptions remaining the same. In this example there is a single final good sector in which productivity of machine capital and humans grow owing to technical progress, which does not spill over to the robot sector. The robot sector is not productive enough to start with, resulting in no robots being produced initially. That state of affairs continues to persist in the absence of spillovers, and the robot sector remains perennially uncompetitive with humans in the final good sector. So the final good sector is never automated: all humans are employed in this sector, and the growing scarcity of labor implies that wages grow in unbounded fashion, thereby ensuring the benefits of growth are spread evenly between capital owners and human workers. In contrast, the presence of spillovers of technical progress from the final good sector to the robot sector ensures that the robot sector will eventually become competitive with humans, and displace them progressively. What Theorem 4 shows is that will inevitably happen, no matter how low the spillover rate is.

6. Concluding Remarks

We study long-term automation and decline in the labor share of national income, driven by capital accumulation rather than biased technical progress or rising markups. Our argument relies on a fundamental asymmetry across physical and human capital in modern economies. While physical capital can be scaled up for the same activity and accumulates in natural units, human capital accumulates principally via education and training that alters choice into higher-skilled occupations, but — from the vantage point of a household or individual — cannot scale up the quantity of labor for a given occupation to an unlimited degree. Under a singularity condition on the technology of the robot-producing sector, we argue that a basic “Kaldor fact” cannot hold: the share of capital in national income approaches 100%.

The singularity condition is central to our findings. So the condition itself merits greater scrutiny. There is greater recognition that the “production of robots by means of robots” is not merely a hypothetical possibility. The Technology section of the New York Times (November 5, 2017) reports:

“They are a dream of researchers but perhaps a nightmare for highly skilled computer programmers: artificially intelligent machines that can build other artificially intelligent machines … Jeff Dean, one of Google’s leading engineers, spotlighted a Google project called AutoML. ML is short for machine learning, referring to computer algorithms that can learn to perform particular tasks on their own by analyzing data. AutoML, in turn, is a machine-learning algorithm that learns to build other machine-learning algorithms. With it, Google may soon find a way to create A.I. technology that can partly take the humans out of building the A.I. systems that many believe are the future of the technology industry.”

The singularity condition is placed on the robot sector alone, but it has economy-wide ramifications. It compares two objects: the elasticity of substitution between capital and labor in the robot sector, and the relative effectiveness of robots and humans in the performance of labor tasks. This condition is automatically satisfied whenever the elasticity of substitution is at least
one (that includes Cobb-Douglas production). If this condition fails, humans can never be displaced in the production of robots, and are always guaranteed a positive share of national income. For either some sectors of the economy are automated, in which case the robot sector becomes active, and humans must obtain a positive share of value added in the robot sector. Or there is no automation anywhere, in which case humans are not displaced in any sector, and the standard theory applies. Therefore, in the absence of singularity, there is a route by which the benefits of physical capital accumulation could persistently flow to humans.

Our paper also takes note of a second escape from the dismal conclusion of an ever-falling labor share. This is the possibility that the number of final goods sectors is unbounded, and that there are sequences of sectors which are progressively more friendly to humans. Theorem 2 showed, however, that this condition is not enough. Human-friendliness has to be supplemented with adequate demand for the products of human-friendly sectors. When preferences are homothetic, that demand is not adequate, and the result of a declining labor share continues to hold. That raises the subtle and deep question of finding a complete characterization of preference and technology patterns that would permit an escape hatch. Non-homotheticity would be necessary, but still not sufficient. The condition in question would need to place lower bounds on the rate at which human friendliness increases along the direction of non-homotheticity. This is a fertile avenue for future research.

A third possibility is that technical progress will step in to create new sectors, tasks or occupations that only humans can participate in, at least initially. Then safe spaces can always be carved out for humans, though each such space is ultimately threatened. This is the “race of man against machine” emphasized by Acemoglu and Restrepo (2018). We agree that it is always possible to specify the laws of technical progress in ways that could equally save human labor from extinction, or eliminate it entirely. We “symmetrize” this problem by presuming that every input — machine, human, robot — stands to benefit to the same degree from directed technical progress. This symmetry assumption is shown to lead to the same conclusion — that the share of human labor must dwindle over time. An escape hatch via technical progress must rely on some asymmetric specification of technical change that benefits humans.

A final remark. Our emphasis throughout is on the functional distribution of income between capital and labor, rather than the distribution of personal incomes (which obviously depends on household investments in different kinds of capital). Whether any given household will be able to earn incomes on par with the rest of the economy will depend on the extent to which they either become capital owners, or move into sectors where humans are more efficient relative to robots. We take no stand on this question here, except to expose the extreme importance of financial education for all individuals, and to emphasize the need to establish financial claims on non-human endowments, so as to preserve sanity in personal income distributions over the long run. The simplicity and tractability of the model may allow it to be useful in analyzing effects of fiscal policies such as capital taxes, education subsidies, universal basic income or other interventions to address the distributional consequences of automation.
REFERENCES


**APPENDIX**

*Proof of Lemma 1.* Since input prices are always positive, \( p(t) \gg 0 \) for every \( t \), so demands are well-defined. If the Lemma is false, then by Well-Behaved Demand, there is a subsequence of dates (retain original index \( t \)) and a good \( i \) with \( p_i(t) \to 0 \) and \( \lim_{t} d_{im}(p(t)) > 0 \) for every \( m \).

Because \( Y(t) \to \infty \), it is easy to see that expenditures on final goods for some type \( m \) must also grow to infinity. Therefore total demand for good \( i \), which is bounded below by \( d_{im}(p(t))z_m(t) \), must grow to infinity. Moreover, because the price of this good converges to zero, the labor input price \( \lambda_i(t) \to 0 \). It follows that labor-capital ratio in this sector must be bounded away from 0 in \( t \) (it goes to infinity, in fact). Because the amount of human labor is finite, it follows that robot use in this sector goes to infinity. In particular the production of robots goes to infinity. By (17), labor input prices in the robot sector are bounded above, so the labor input in the robot sector
along a subsequence of dates. We first claim that for any
Part (b). Suppose, on the contrary, that for some pair of sectors
Proof of Lemma 2. Let $S$ be the set of all infinite-dimensional nonnegative vectors $s \equiv (s_1, s_2, \ldots)$, with components in $[0, 1]$ and with $\sum_{j=1}^{\infty} s_j = 1$. Let $s(t)$ be a sequence in $S$ indexed by $t$, and suppose that there is $\hat{s} \in S$ such that $s(t)$ converges pointwise to $\hat{s} = (\hat{s}_j)$. Let $\Psi(t)$ be a corresponding convergent sequence with components $(\Psi_1(t), \Psi_2(t), \ldots)$, where $\Psi_j(t) \in [0, 1]$ for every $j$ and $t$, with $\Psi_j(t) \to 0$ as $t \to \infty$ for every $j$ with $\hat{s}_j > 0$. Then
\begin{equation}
\lim_{t \to \infty} \sum_{j=1}^{\infty} \Psi_j(t) \hat{s}_j(t) = 0.
\end{equation}

Proof. For any $n$, let $J$ be some positive integer such that for $\hat{s}$ in the statement of the lemma, $\sum_{j=1}^{J} \hat{s}_j \geq 1 - (1/2)^{n+2}$. Then there exists $T_1(n)$ such that along the sequence \{s(t)\},
\[\sum_{i=1}^{J} s_i(t) \geq 1 - (1/2)^{n+2} - (1/2)^{n+2} = 1 - (1/2)^{n+1}\]
for $t \geq T_1(n)$, using pointwise convergence on the finite set \{1, \ldots, J\}. Because $\Psi_i(t) \in [0, 1]$ for all $i$ and $t$ and $\sum_{j} s_j(t) = 1$ for every $t$, it follows that for $t \geq T_1(n)$,
\begin{equation}
\sum_{j=J+1}^{\infty} \Psi_j(t) s_j(t) \leq \sum_{j=J+1}^{\infty} s_j(t) < (1/2)^{n+1}.
\end{equation}
Because $\Psi_j(t) \to 0$ as $t \to \infty$ for every $j$ with $\hat{s}_j > 0$, we know that $\Psi_j(t) s_j(t) \to 0$. Therefore there exists $T(n) \geq T_1(n)$ so that in addition to (48),
\begin{equation}
\sum_{i=1}^{J} \Psi_i(t) s_i(t) \leq (1/2)^{n+1}
\end{equation}
for $t \geq T(n)$. Combining (48) and (49), we must conclude that
\[\sum_{j=1}^{\infty} \Psi_j(t) s_j(t) < (1/2)^{n}.
\]
for $t \geq T(n)$. Because $n$ can be made arbitrarily large, the proof is complete. \hfill \blacksquare

Proof of Lemma 2. Part (a). This follows immediately on inspecting equation (44).

Part (b). Suppose, on the contrary, that for some pair of sectors $i$ and $j$, $R(t) \equiv \pi_i(t)/\pi_j(t) \to \infty$ along a subsequence of dates. We first claim that for any $m < \infty$, there is a date $t_m$ such that
\begin{equation}
R(t_m + 1) > R(t_m) > m
\end{equation}
To show this, note that there certainly exists a subsequence \( \{ t_s \} \to \infty \) such that \( R(t_s + 1) > m \) for all \( k \) and \( R(t_s + 1) \to \infty \). If the claim is false, \( R(t_s) \leq m \) for all \( k \). Then, in particular,

\[
R(t_s + 1) - R(t_s) \to \infty.
\]

On the other hand, because the difference in productivity growth rates of the factor in the two sectors cannot exceed \( \bar{\rho} \), we know that

\[
\frac{\pi_i(t_s + 1)}{\pi_i(t_s)} - \frac{\pi_j(t_s + 1)}{\pi_j(t_s)} \leq \bar{\rho},
\]

but this implies that for all \( k \),

\[
R(t_s + 1) - R(t_s) \leq \bar{\rho} \frac{\pi_i(t_s)}{\pi_j(t_s + 1)} = \bar{\rho} \left\{ \frac{\pi_i(t_s)}{\pi_j(t_s)} \right\} \frac{\pi_j(t_s)}{\pi_j(t_s + 1)} \leq \bar{\rho} m,
\]

given the contrary presumption that \( R(t_s) \leq m \) and given that \( \pi_j(t_s) \leq \pi_j(t_s + 1) \). This contradicts (51) and establishes the claim in (50).

Returning to the main proof, let \( \alpha_i(t) \equiv \rho_i(t) \pi_i(t) \) be the absolute productivity advance in any sector \( i \), and, for some fixed pair of sectors \( i \) and \( j \), let \( A(t) \equiv \sum_{k \neq i,j} a_k(t) \) be the aggregate productivity advance in all sectors apart from \( i \) and \( j \). Consider first the case in which

\[
\frac{1}{\pi_i(t_m)} \gamma [a_j(t_m) + A(t_m)] \geq \bar{\rho}.
\]

Then \( \rho_i(t_m) = \bar{\rho} \). But \( a_j(t_m) + \gamma A(t_m) > \gamma [a_j(t_m) + A(t_m)] \geq \bar{\rho} \) and \( \pi_i(t_m) \geq m \pi_j(t_m) > \pi_j(t_m) \) by (50). Therefore (52) implies that

\[
\frac{1}{\pi_j(t_m)} [a_j(t_m) + \gamma A(t_m)] = \rho_j(t_m) + \gamma \frac{1}{\pi_j(t_m)} A(t_m) \geq \bar{\rho}
\]

so that productivity growth rate in sector \( j \) must also equal \( \bar{\rho} \), which contradicts (50). So (52) cannot hold, and therefore

\[
\frac{\pi_i(t_m + 1)}{\pi_i(t_m)} - 1 = \rho_i(t_m) + \frac{1}{\pi_i(t_m)} \gamma [a_j(t_m) + A(t_m)].
\]

Moreover, the rate of growth of productivity in sector \( j \) must be smaller than \( \bar{\rho} \), otherwise (50) is contradicted again. So (53) also holds for sector \( j \):

\[
\frac{\pi_j(t_m + 1)}{\pi_j(t_m)} - 1 = \rho_j(t_m) + \frac{1}{\pi_j(t_m)} \gamma [a_j(t_m) + A(t_m)],
\]

and moreover, the right hand side of (54) must be smaller than the right hand side of (53); i.e.:

\[
\rho_i(t_m) + \frac{1}{\pi_i(t_m)} \gamma [a_j(t_m) + A(t_m)] > \rho_j(t_m) + \frac{1}{\pi_j(t_m)} \gamma [a_j(t_m) + A(t_m)].
\]

By (50) and \( m > 1 \), and also recalling the definition of \( \alpha_i(t) \), this inequality implies:

\[
\rho_i(t_m) \left[ 1 - \frac{\pi_i(t_m)}{\pi_j(t_m)} \right] > \rho_j(t_m) \left[ 1 - \frac{\pi_i(t_m)}{\pi_j(t_m)} \right]
\]

But (55) cannot hold for all \( m > 1 \): its right hand side is positive while its left hand side is negative for all \( m > 1/\gamma \). This contradiction proves the lemma.
Proof of Lemma 3. The proof relies on the following claim, which is parallel to Proposition 1. For any \((\theta_r, \nu_r)\) with \(\nu_r \geq \nu_r(0)\), there is a unique solution \(p^*_r(\theta_r, \nu_r)\) to the equation
\[
 p_r = c_r \left( \frac{1}{\theta_r} \frac{p_r}{\nu_r} \right),
\]
and in any equilibrium, \(p_r(t) \leq p^*_r(\theta_r(t), \nu_r(t))\) for all \(t\).

No change in the proof of Proposition 1 is needed to prove this claim. We only note that (41) implies the singularity condition (17) for every \(\nu_r \geq \nu_r(0)\).

With this claim in hand, we first establish (45) for \(j = r\). Write \(p^*_r(t) \equiv p^*_r(\theta_r(t), \nu_r(t))\). For any \(t\), (56) tells us that
\[
 p^*_r(t) = c_r \left( \frac{1}{\theta_r(t)} \frac{p_r(t)}{\nu_r(t)} \right).
\]
Define \(\eta(t) \equiv \nu_r(t)/p^*_r(t)\theta_r(t)\) for all \(t\). Then, using the linear homogeneity of \(c_r\), the above equality can be rewritten as
\[
 \nu_r(t) = c_r(\eta(t), 1),
\]
for all \(t\). It follows that
\[
 \nu_r(t + 1) - \nu_r(t) = c_r(\eta(t + 1), 1) - c_r(\eta(t + 1), 1) \leq c^1_r(\eta(t + 1), 1)[\eta(t + 1) - \eta(t)],
\]
where \(c^1_r\) stands for the partial derivative of \(c_r\) with respect to its first argument, and the inequality above follows from the concavity of the unit cost function \(c_r\). Using (57) again, it is easy to see that for every \(t\),
\[
 \frac{\eta(t + 1) - \eta(t)}{\eta(t)} \geq \frac{\nu_r(t + 1) - \nu_r(t)}{\nu_r(t)} \geq \frac{c_r(\eta(t), 1)}{c^1_r(\eta(t), 1)\eta(t)} \geq \frac{\nu_r(t + 1) - \nu_r(t)}{\nu_r(t)},
\]
where the last inequality is a consequence of the fact that \(c_r\) is concave. Notice also that (41) and (57) together guarantee that \(\eta_r(0) > 0\). Putting all this information together, there exists \(b_r > 0\) such that \(\eta_r(t) \geq b_r\nu_r(t)\) for all \(t\). Inverting this inequality, defining \(B_r \equiv b_r^{-1} < \infty\), recalling the definition of \(\eta(t)\), and noting that \(p_r(t) \leq p^*_r(t)\) by the claim, we must conclude that
\[
 \theta_r(t)p_r(t) \leq \theta_r(t)p^*_r(t) \leq B_r < \infty
\]
for all \(t\), which establishes (45) for \(j = r\). To complete the proof for all \(j\), we recall part (b) of Lemma 2, which bounds the productivity ratios of each factor across sectors. That means we can replace \(\theta_r(t)\) by \(\theta_j(t)\) in the inequality (58) — adjusting \(B_r\) suitably to a new bound \(B_j\) — without jeopardizing boundedness.

Proof of Lemma 4. Pick some sector \(i\) satisfying Condition F (or F'). We know that in any competitive equilibrium and at any date \(t\)
\[
 p_i(t)\theta_i(t)\frac{\partial f_i}{\partial \theta_i k_i}(t) = p_i(t)\theta_i(t)g^i_e(e_i(t)) = 1,
\]
where we recall the intensive form \(g_i(e) = f_i(e, 1)\) already defined. It follows that:
\[
 p_i(t)y_i(t) = \frac{y_i(t)}{\theta_i(t)g^i_e(e_i(t))} = \left[ \frac{y_i(t)}{\theta_i(t)k_i(t)g^i_e(e_i(t))} \right] k_i(t).
\]
Dividing above and below in this expression by the second input (human or robotic), we see that
\[ p_i(t) y_i(t) = \left[ \frac{g_i(e_i(t))}{e_i(t) g_i'(e_i(t))} \right] k_i(t). \]

We claim next that under condition F, the sequence of input ratios \( \{e_i(t)\} \) is bounded above and below by strictly positive, finite numbers. To this end, recall that
\[ p_i(t) = c_i(\theta_i^{-1}(t), \lambda_i^*(t)), \]
where — following earlier notation — we let \( \lambda_i^* \) stand for the price of the second input in efficiency units. Therefore
\[ p_i(t) \theta_i(t) = c_i(1, \theta_i(t) \lambda_i^*(t)) \geq \epsilon \]
for some \( \epsilon > 0 \), by F. It follows from (59) that \( e_i(t) \) must be bounded below by a strictly positive number.

Moreover, using (61) and recalling that \( \lambda_i^*(t) \leq p_r(t) \nu_i^{-1}(t) \), we also see that
\[ p_i(t) \theta_i(t) \leq c_i(1, \theta_i(t) p_r(t) \nu_i^{-1}(t)) \leq c_i(1, p_r(t) \theta_i(t) \nu_i^{-1}(0)) \leq c_i(1, B \nu_i^{-1}(0)), \]
where the very last inequality uses Lemma 3. Therefore \( p_i(t) \theta_i(t) \) is bounded above in \( t \). It follows from (59) that \( e_i(t) \) must also be bounded above, establishing the claim.

It follows from this claim that the elasticity \( e_i(t) g_i'(e_i(t))/g_i(e_i(t)) \) is bounded below by some strictly positive number; call it \( a \). (Notice that this is true by assumption if Condition F’ is satisfied, without any need for the derived bounds on \( e_i(t) \).) Using this information in (60), we must conclude that for all \( t \),
\[ k_i(t) \geq a p_i(t) y_i(t). \]

Given our growth assumption, per-capita income goes to infinity. It follows from Condition E that the right hand side of (63) goes to infinity as well. So \( k_i(t) \to \infty \), as claimed. \( \blacksquare \)