“Contractual Structure and Wealth Accumulation

Detailed Appendix

In what follows, assumptions [α] and [β] are maintained throughout, though they do not apply to all the lemmas or propositions. The normalization $u(0) = 0$ is also employed without comment. Finally, recall that v^* is the payoff to the agent under the one-shot P-optimal contract, and $V^* \equiv (1-\delta)^{-1}v^*$.

Lemma **1** *In any P-equilibrium, if $y \leq y'$, then $B(y') - B(y) \geq u(y') - u(y)$.*

Proof. Letting $c(y)$ denote A's optimal consumption at wealth y, we have $B(y) = u(c(y)) + \delta V(w(y))$, while $B(y') \geq u(c(y) + y' - y) + \delta V(w(y))$, implying $B(y') - B(y) \geq u(c(y) + y' - y) - u(c(y)) \geq u(y') - u(y)$, the last inequality resulting from the concavity of u and $c(y) \leq y$. ■

Lemma **2** *In any P-equilibrium, $V(0) > B(0) > 0$.*

Proof. Suppose that P selects $x = 0$ and $y = y^*(> 0)$ from the static optimal contract. In response to this, A will respond with $e \geq e^*$, since $B(y) - B(0) \geq u(y) - u(0)$ for all y (by Lemma 1). This will generate a profit for P no smaller than that of the static optimal contract, so assumption [α] implies that there does exist a feasible and profitable contract at $w = 0$. But any such contract *must* yield the agent strictly positive return (because either x or y or both must be strictly positive). So $V(0) > 0$.

But $B(0) = u(0) + \delta V(0) = \delta V(0)$. Because $V(0) > 0$, we conclude that $V(0) > B(0)$. ■

Lemma **3** *Consider a (deterministic) Ramsey problem with exit option V^*.*

(i) *If*

$$\lim_{c\to\infty}[u(c) - cu'(c)] > (1-\delta)V^*, \tag{11}$$

is violated, then the solution is (for all z):

$$c(z) = z, \quad B^*(z) = u(z) + \delta V^* \tag{12}$$

(ii) *If, on the other hand, (11) is satisfied, the solution is described as follows. Let $T(z)$ equal the optimal exit time for an agent with initial wealth z, with $T(z) = \infty$ if the agent never exits. There exists an infinite sequence $\{z_k\}_{k=0}^{\infty}$, with $0 = z_0 < z_1 < z_2 < \cdots$, and $z^* \equiv \lim_{k\to\infty} z_k < \infty$, such that $T(z) = k$ for $z \in [z_{k-1}, z_k]$ (with indifference holding for adjacent values of T at the endpoints), and $T(z) = \infty$ for all $z \geq z^*$. Moreover, under [β], we have $R < z_1$.*

The associated value function in case (ii) is

$$\begin{aligned} B^*(z) &= \frac{1-\delta^k}{1-\delta} u\left(\frac{(1-\delta)z}{1-\delta^k}\right) + \delta^k V^* \text{ for all } z \in [z_{k-1}, z_k] \\ &= \frac{1}{1-\delta} u\left((1-\delta)z\right) \text{ for all } z \geq z^*. \end{aligned} \tag{13}$$

Proof. Consider the related problem of selecting a real number $x \in [1, \frac{1}{1-\delta}]$ to maximize $\psi(x; z) \equiv xu(\frac{z}{x}) + [1-(1-\delta)x]V^*$.

(Here x corresponds to $\frac{1-\delta^k}{1-\delta}$, where the exit date k is treated as a continuous variable in $[1, \infty)$. Since it is optimal to smooth consumption perfectly until the exit date, A will consume uptil the exit date at the steady level of $c(z) = \frac{z(1-\delta)}{(1-\delta^k)}$, thereby running down wealth to 0 at k, and exiting with V^*. This generates the value function $B(z) = \frac{1-\delta^k}{1-\delta} u(\frac{z(1-\delta)}{1-\delta^k}) + \delta^k V^*$.)

Notice that

$$\begin{aligned} \psi_x = & \; u(\tfrac{z}{x}) - xu'(\tfrac{z}{x})\tfrac{z}{x^2} - (1-\delta)V^* \\ = & \; u(\tfrac{z}{x}) - \tfrac{z}{x}u'(\tfrac{z}{x}) - (1-\delta)V^* \end{aligned}$$

so the concavity of u implies that ψ is concave in x, for any z.

Note, moreover, that if (11) fails with a strict inequality, then $\psi_x(x; z) < 0$ for all x, so then the optimal value of $x = 1$, i.e., $k = 1$. Then A consumes all current wealth and exits at the next date, implying (i). [It is easily checked the same is the case when (11) holds as an equality.]

If (11) is satisfied, there exists m such that $u(m) - mu'(m) = (1-\delta)V^*$. Define $\bar{z} = \frac{m}{1-\delta}$. Then $z \geq \bar{z}$ implies $\frac{z}{x} \geq \bar{z}(1-\delta) = m$ for all $x \in [1, \frac{1}{1-\delta}]$. Hence $\psi_x(x; z) \geq 0$ for all x, so optimal $x = \frac{1}{1-\delta}$, or $k = \infty$. Conversely, $z < \bar{z}$ implies that $\psi_x(\frac{1}{1-\delta}; z) < 0$, so it is optimal for the agent to exit at some finite date. In particular, $z \leq m$ implies that optimal $x = 1$ (so that $k = 1$). And $z \in (m, \frac{m}{1-\delta})$ implies that the agent must exit at some date $k > 1$.

To calculate the exact exit date, the concavity of ψ implies that it suffices to look at the two integer solutions for k generating values of $\frac{z}{x}$ closest to m. Every finite k will therefore be an optimal exit date for some z, and the exact switch points can be calculated by the condition of indifference between adjacent exit dates.

Finally, note from [β] that for all $z \leq R$,

$$\frac{1}{\delta}[(1+\delta)u\left(\frac{z}{1+\delta}\right) - u(z)] < v^*,$$

because $(1+\delta)u\left(\frac{c}{1+\delta}\right) - u(c)$ is nondecreasing in c. Rearranging this inequality and recalling that $V^* = (1-\delta)^{-1}v^*$, we see that

$$u(z) + \delta V^* > (1+\delta)u\left(\frac{z}{1+\delta}\right) + \delta^2 V^*,$$

which means that exit at date 1 is strictly preferred to exit at date 2. By the concavity of ψ, it follows that exit at date 1 is uniquely optimal, proving that $R < z_1$. ■

The reader is reminded that in what follows, a *continuous* P-equilibrium refers to a P-equilibrium with continuous *ex post* value function B.

LEMMA **4** *For any continuous P-equilibrium, w^n be a sequence of wealth levels converging to $\hat{w}$, with corresponding contracts (x^n, y^n, e^n) converging to $(\hat{x}, \hat{y}, \hat{e})$. Then $(\hat{x}, \hat{y}, \hat{e})$ is an optimal contract for P at $\hat{w}$.*

Proof. Recalling that B and D are continuous, this follows from a simple application of the maximum theorem to the principal's (constrained) optimization problem. ■

LEMMA **5** *For a continuous P-equilibrium and wealth w, suppose that a non-null contract is offered and $V(w) = B(w)$. Then $x(w) < w < y(w)$. Moreover, for any compact interval $[0, \bar{w}]$, there exists $\eta > 0$ such that (a) $x(w) < w - \eta$ and (ii) $y(w) > w + \eta$, whenever $w \in [0, \bar{w}]$ and a non-null contract is offered with $V(w) = B(w)$.*

Proof. We are given that $B(w) = V(w) = e(w)B(y(w)) + (1-e(w))B(x(w)) - D(e(w))$. We first claim that $e(w) > 0$ (and hence $y(w) > x(w)$). Otherwise $e(w) = 0$, and P's expected profit is $w - f - x(w)$. Moreover, $B(w) = B(x(w))$, implying $x(w) = w$. So P earns a negative expected profit, a contradiction.

Next we claim that $y(w) > w$. Otherwise $x(w) < y(w) \leq w$, implying $V(w) = e(w)B(y(w)) + (1-e(w))B(x(w)) - D(e(w)) < B(w)$, contradicting (APC).

Finally, suppose $x(w) \geq w$. Then $y(w) > x(w) \geq w$, and A would obtain an expected present value utility of at least $B(w)$ upon selecting $e = 0$. Since $D(.)$ is strictly convex and $e(w) > 0$, A must end up with a present value utility that strictly exceeds $B(w)$, contradicting the property that $V(w) = B(w)$.

Now suppose that the uniform bound does not obtain, say, for part (a). Then we can find a sequence w_n in $[0, \bar{w}]$ such that $w_n - x(w_n) \to 0$. Note that (PPC) implies $\{x_n = x(w_n), y_n = y(w_n)\}$ must be bounded, since $x_n \leq x_n + e_n(y_n - x_n) \leq w_n - f + R \leq \bar{w} - f + R$, and $y_n \to \infty$ would imply $e_n \to 1$ and hence $x_n + e_n(y_n - x_n) \to \infty$. Construct a subsequence along which (x_n, y_n, e_n, w_n) converges, say, to $(\hat{x}, \hat{y}, \hat{e}, \hat{w})$. Then by Lemma 4, the contract $(\hat{x}, \hat{y}, \hat{e})$ is optimal for P at $\hat{w}$, and $\hat{x} = \hat{w}$, contradicting what we have already established in the first part of this lemma. A similar argument establishes (b). ■

Lemma 6 *For any continuous P-equilibrium and any compact interval $[0, \bar{w}]$, there is $\epsilon > 0$ such that whenever $w \in [0, \bar{w}]$ and a non-null contract is offered, we have $e(w) \in (\epsilon, 1-\epsilon)$.*

Proof. Consider any sequence $w^n \in [0, \bar{w}]$ with associated contracts $\{x_n, y_n, e_n\}$. By the same argument as in the proof of Lemma 5, $\{x_n, y_n\}$ must be bounded. Consequently, we can extract a subsequence (retaining notation) such that $w_n \to w \in [0, \bar{w}]$ and $\{x_n, y_n, e_n\} \to \{x, y, e\}$. By Lemma 4, $\{x, y, e\}$ is P-optimal at w.

We claim that e cannot be zero. For if it were, it follows from (APC) that $x \geq w$. But this means that (PPC) is violated, a contradiction. Morever, since y and x are bounded, e cannot be 1, given our assumptions on effort disutility D. Thus limit points with e equal to 0 or 1 are ruled out, establishing the lemma. ■

Lemma 7 *Consider a P-optimal contract for some continuous B. If (APC) is not binding at some w, so that $V(w) > B(w)$, then the P-optimal contract has $x(w) = 0$ and $y(w) < R$.*

Proof. Suppose that $V(w) > B(w)$. We first claim that the principal's return conditional on success must be higher than his return conditional on failure; that is,

$$w + R - y(w) > w - x(w). \tag{14}$$

For suppose this is false; then P can lower $y(w)$ by a small amount to y'. Because $V(w) > B(w)$ and because B is continuous, the participation constraint for the agent does not bite after this (small) change. Moreover, $e(w) > 0$ otherwise (PPC) would be violated. Hence the change can be made small enough so that the resulting effort e' satisfies $e(w) \geq e' > 0$. Consequently,

$$\begin{aligned} e'[w + R - y'] + (1-e')[w - x(w)] &> e'[w + R - y(w)] + [1-e'][w - x(w)] \\ &\geq e(w)[w + R - y(w)] + [1 - e(w)][w - x(w)], \end{aligned}$$

which contradicts the optimality of the contract $\{x(w), y(w), e(w)\}$ for the principal.

With (14) in hand, we now claim that $x(w) = 0$. For suppose $x(w) > 0$. Then lower $x(w)$ by a small amount to x'. Because $V(w) > B(w)$ and because B is continuous, the participation constraint for the agent still does not bite after this change. Moreover, e' — the effort after this change — is no lower but is still less than 1, as $y(w)$ is finite. Consequently, using (14) and $e' < 1$,

$$\begin{aligned} e'[w + R - y(w)] + (1-e')[w - x'] &> e'[w + R - y(w)] + [1-e'][w - x(w)] \\ &\geq e(w)[w + R - y(w)] + [1 - e(w)][w - x(w)], \end{aligned}$$

a contradiction.

That $y(w) < R$ now follows from another application of (14) and the fact that $x(w) = 0$. ■

Lemma 8 *If $w \geq R$, then $V(w) = B(w)$.*

Proof. Suppose not; then $V(w) > B(w)$. By Lemma 7, $x(w) = 0$ and $y(w) < R \leq w$. But then

$$V(w) = e(w)B(y(w)) + [1 - e(w)]B(x(w)) - D(e) < B(y(w)) < B(R) \leq B(w),$$

a contradiction to (APC). ■

Lemma 9 *If $V(w) > B(w)$, then $V(w) \geq V^*$.*

Proof. By Lemma 7, if $V(w) > B(w)$, then $x(w) = 0$. So $\{e(w), y(w)\} = (e, y)$ is chosen to maximize

$$e(R - y) \tag{15}$$

subject to

$$D'(e) = B(y) - B(0) \tag{16}$$

and (APC). Define $g(y) \equiv B(y) - B(0)$ and $g^*(y) \equiv u(y) - u(0)$, and then $h(e) \equiv g^{-1}(D'(e))$ and $h^*(e) \equiv g^{*-1}(D'(e))$. Note that h and h^* are well-defined, because B and B^* are strictly monotone and continuous. Applying Lemma 1:

$$h(e) \leq h^*(e) \tag{17}$$

and

$$h(e') - h(e) \leq h^*(e') - h^*(e) \tag{18}$$

for all $e < e'$. Moreover, the maximization problem (15) above can be rewritten equivalently as

$$\max_e e[R - h(e)] \tag{19}$$

subject to (APC): $eD'(e) - D(e) \geq B(w)$, where it is understood that the corresponding value of y will be read off from (16), given e. In this context, recall that the one-shot P-optimal contract can be identified with the unconstrained solution to

$$\max_e e[R - h^*(e)]. \tag{20}$$

Denote by e^* the solution to this problem.

Let $e(w)$ be a solution to (19). We claim that $e(w) \geq e^*$.

Suppose this is false and $e(w) < e^*$. Setting $e = e(w)$ and noting that $e^* > e$ implies that e^* satisfies (APC) (since e does), it follows that e^* is feasible in P's problem of designing an optimal contract for an agent with w. Hence

$$e[R - h(e)] \geq e^*[R - h(e^*)].$$

At the same time e^* is the unconstrained maximizer of (20), which is a strictly concave function of e. Given that $e^* > e$ it follows that

$$e^*[R - h^*(e^*)] > e[R - h^*(e)].$$

Combining these two inequalities, we see that

$$e^*[h^*(e^*) - h(e^*)] < e[h^*(e) - h(e)]. \tag{21}$$

Combine (17) and (21) to conclude that

$$h^*(e^*) - h^*(e) < h(e^*) - h(e),$$

but this inequality, together with the assumed $e < e^*$, contradicts (18). So $e(w) \geq e^*$, as asserted.

Now we can solve out for $V(w)$. Notice that

$$\begin{aligned} V(w) &= e(w)B(y(w)) + [1 - e(w)]B(0) - D(e(w)) \\ &= B(0) + [e(w)D'(e(w)) - D(e(w))] \\ &= \delta V(0) + [e(w)D'(e(w)) - D(e(w))], \end{aligned} \tag{22}$$

where the second equality uses (16) and the last equality follows simply from the definition of B. Setting $w = 0$ in (22), using the fact that $V(0) > B(0)$ (Lemma 2) so that $e(0) \geq e^*$ (as claimed and shown above), and noting that $eD'(e) - D(e)$ is increasing in e, we conclude that

$$V(0) = (1 - \delta)^{-1}[e(0)D'(e(0)) - D(e(0))] \geq (1 - \delta)^{-1}[e^*D'(e^*) - D(e^*)] = V^*. \tag{23}$$

Now consider any w such that $V(w) > B(w)$. With (23) in mind and again using $e(w) \geq e^*$, we may conclude that

$$V(w) = \delta V(0) + [e(w)D'(e(w)) - D(e(w))] \geq \delta V^* + [e^*D'(e^*) - D(e^*)] = V^*,$$

Therefore $V(w) \geq V^*$ for all w with $V(w) > B(w)$, as claimed. ■

In what follows, we consider another Ramsey problem with exit, this one more general than the one described in Lemma 3. Recall that in any equilibrium, we have

$$B(z) \equiv \max_{0 \leq \delta w \leq z} [u(z - \delta w) + \delta \max\{V(w), B(w)\}]. \tag{24}$$

This induces the following exit problem: for each initial wealth level z, choose an *exit date* $T(z) \geq 1$ and an *exit wealth* $\iota(z)$ such that after $T(z)$ periods of consumption and saving, the agent takes the 'outside option' $V(\iota(z)) > B(\iota(z))$. If no wealth w with $V(w) > B(w)$ is ever reached, set $T(z)$ equal to infinity.

Since u is concave and $\delta = \frac{1}{1+r}$ it is optimal to smooth consumption till the exit date. This means that if $T \leq \infty$ denotes the exit date, an optimal sequence of wealths until exit at wealth w is given by the difference equation

$$w_{t+1} = (1/\delta)\left[w_t - \frac{(z - \delta^T w)(1-\delta)}{1-\delta^T}\right], \tag{25}$$

where w_0 is just z. Note that this is the uniquely optimal policy if u is strictly concave.

Lemma **10** *There exists a finite integer M such that for every $z \in [0, R]$, exit occurs in the generalized Ramsey problem at some date $T(z) \leq M$.*

Proof. Consider the special Ramsey problem with exit as described in Lemma 3, with the exit option set equal to V^*. Given Lemma 9, it follows that if it is optimal to exit in the special Ramsey problem, then it will also be optimal to exit in the generalized Ramsey problem as well (otherwise it is optimal to smooth consumption in perpetuity, which is dominated by exiting at the next date with a payoff of V^*). The last statement in part (ii) of Lemma 3 assures us that under assumption [β], it is optimal to exit in the special Ramsey problem at date 1, provided $z \in [0, R]$. Hence $T(z) < \infty$ in the generalized Ramsey problem.

To obtain a uniform bound, let $\epsilon(z) > 0$ denote the payoff difference between date 1 exit and indefinite maintenance. Suppose, on the contrary, that $T(z^k) \to \infty$ along a sequence of initial wealths $z^k \in [0, R]$. Choose any wealth z^k in this sequence with the property that (defining $T = T(z^k)$ and $\epsilon = \epsilon(z^k)$),

$$\delta^T \bar{V} < \epsilon,$$

where $\bar{V}$ is the (finite) supremum of V over $[0, R]$.[1]

The agent can always choose an exit time of date 1 by consuming all wealth immediately. This gives a return of $u(z) + \delta V(0)$. Using Lemma 9,

$$\begin{aligned} u(z) + \delta V(0) &\geq u(z) + \delta V^* \\ &\geq \text{Payoff from indefinite maintenance } + \epsilon \\ &\geq \text{Payoff from choosing presumed optimal exit date } T \; - \delta^T \bar{V} + \epsilon \\ &> \text{Payoff from choosing presumed optimal exit date } T \;, \end{aligned}$$

a contradiction. ■

[1]This is finite because whenever $V(w) > B(w)$, we know from (22) that $V(w) = \delta V(0) + [e(w)D'(e(w)) - D(e(w))]$ and this is uniformly finite on $[0, R]$. Whenever $V(w) = B(w)$, it is uniformly finite on any compact set by the assumed continuity of B.

LEMMA **11** *If exit from $z \in [0, R]$ occurs in the generalized Ramsey problem at some wealth $\iota(z) \le z$, then exit must occur at date 1.*

Proof. Let exit occur at some wealth $z' = \iota(z)$, where $z' \le z$. It will suffice to prove that for all $T \ge 2$,

$$\delta(1-\delta^{T-1})V(z') > \frac{1-\delta^T}{1-\delta} u\left(\frac{(z-\delta^T z')(1-\delta)}{1-\delta^T}\right) - u(z-\delta z'), \tag{26}$$

since it is optimal to hold consumption steady till exit, so that the condition (26) above is just a sufficient condition for exit at $T = 1$.

Suppose, on the contrary, that (26) is false. Then

$$\frac{1-\delta^T}{1-\delta} u\left(\frac{(z-\delta^T z')(1-\delta)}{1-\delta^T}\right) - u(z-\delta z') \ge \delta(1-\delta^{T-1})V(z'). \tag{27}$$

for some $T \ge 2$ at which exit is optimal. Moreover, since the agent could have spent these T periods running his wealth down to zero and taking $V(0)$ instead, it must be the case that

$$\frac{1-\delta^T}{1-\delta} u\left(\frac{(z-\delta^T z')(1-\delta)}{1-\delta^T}\right) + \delta^T V(z') \ge \frac{1-\delta^T}{1-\delta} u\left(\frac{z(1-\delta)}{1-\delta^T}\right) + \delta^T V^*, \tag{28}$$

where we use Lemma 9 to replace $V(0)$ by V^*.

We may combine (27) and (28) to eliminate $V(z')$. Doing so, we see that

$$\begin{aligned} & \frac{1-\delta^T}{1-\delta} u\left(\frac{(z-\delta^T z')(1-\delta)}{1-\delta^T}\right) - u(z-\delta z') \\ & -\frac{(1-\delta^T)(1-\delta^{T-1})}{(1-\delta)\delta^{T-1}} \left[u\left(\frac{z(1-\delta)}{1-\delta^T}\right) - u\left(\frac{(z-\delta^T z')(1-\delta)}{1-\delta^T}\right)\right] \\ \ge\ & \delta(1-\delta^{T-1})V^*. \end{aligned} \tag{29}$$

Differentiation of the LHS of (29) with respect to z' yields the expression

$$-\delta u'\left(\frac{(z-\delta^T z')(1-\delta)}{1-\delta^T}\right) + \delta u'(z-\delta z'),$$

and this expression is easily checked to be non-positive as long as $z' \le z$. Thus if we set $z' = 0$ in (29), we may conclude that

$$\frac{1-\delta^T}{1-\delta} u\left(\frac{z(1-\delta)}{1-\delta^T}\right) - u(z) \ge \delta(1-\delta^{T-1})V^*.$$

But this contradicts Assumption [β]. ■

Lemma 12 *If exit from z occurs at some wealth level $\iota(z) > z$ in the generalized Ramsey problem, then for all intervening wealth levels w until exit, both $x(w)$ and $y(w)$ lie in $[0, R]$ (assuming a non-null contract is offered at those wealths).*

Proof. If u is strictly concave, then (25) holds, and it is easy to check from (25) that

$$\text{If } z < z' \equiv \iota(z), \text{ then } w_t < z' \text{ for all } 0 < t < T. \tag{30}$$

A stronger claim — which we make now — is that (30) holds even if u is weakly concave. To this end, suppose on the contrary that $w_{t+1} \geq z'$ for some date $t + 1 < T$, where T is the exit date. Because $z < z'$, we may without loss of generality take $t = 0$ (simply take the initial condition to be w_t). Then keep in mind that $T > 1$. Now total payoff in the Ramsey problem with exit is

$$U \equiv \sum_{s=0}^{T-1} \delta^s u(c_s) + \delta^T V(z').$$

Now consider an alternative wealth policy $\{w'_s\}$, in which $w'_0 = w_0$, and $w'_s = z'$ for $s = 1, \ldots, T$. This is clearly feasible because w_1, which is no less than z', was feasible from w_0. It is easy to see that the resulting payoff under this policy — call it U' — satisfies

$$U' \equiv u(w_0 - \delta z') + \delta \left[\sum_{s=1}^{T-1} \delta^{s-1} u([1-\delta]z') + \delta^{T-1} V(z') \right] \tag{31}$$

which is at least as large as U owing to the smoothing of the intermediate consumption stream.

However — because $T > 1$ — the term inside the square brackets of (31) cannot exceed $B(z')$. On the other hand, a move to z' followed by exit in one step is *also* feasible, and would yield the payoff

$$u(w_0 - \delta z') + \delta V(z')$$

which is strictly higher than U' (because $V(z') > B(z')$) and hence also larger than U. This contradicts the optimality of the original policy.

It follows that every intervening wealth satisfies $w < z' = \iota(z)$ and $V(w) = B(w)$. Now $V(z') > B(z')$, so by the monotonicity of B,

$$V(z') > B(z') \geq B(w) = V(w). \tag{32}$$

On the other hand, we know that $x(z') = 0$, that $y(z') \leq R$, and that $x(w) \geq 0$ (assuming that a contract is offered at w). So the only way in which (32) can be satisfied is by having both $x(w)$ and $y(w)$ lie in $[0, R]$. ∎

In what follows, we focus on the actual evolution of wealths rather than the optimal saving strategy. Hence we move away from the Ramsey problem, as it does not provide information concerning the former. Indeed, it is artificial insofar as it refers to planned wealth levels which may diverge from the actual evolution of wealth in the future owing to uncertainty in project returns. Every reference to a process of wealths is now (unless otherwise stated) to the *actual* wealth evolution of the agent in the P-equilibrium. Notice, however, that for every end-of-period wealth z, the beginning wealth in the succeeding period is given exactly by some policy function induced by the artificial Ramsey problem.

LEMMA **13** *If end-of-period wealth* $z \in [0, R]$*, then the stochastic process of all subsequent wealths in any continuous* P*-equilibrium must lie in* $[0, R]$ *almost surely.*

Proof. Consider any $z \in [0, R]$. Let $\hat{z}$ denote the random variable that describes end-of-period wealth next period, conditional on z today. Recall that $\hat{z}$ is determined by the conjunction of two processes. First, a choice is made for next period's starting wealth; call it w_1. This is done by following *exactly* some first-period solution to the artificial Ramsey problem. Next, a contract may be offered at w_1, leading to the random variable $\hat{z}$ that describes next period's end-of-period wealth. If a contract is not offered, then $\hat{z}$ is simply equal to w_1.

Suppose, first, that exit takes place at date 1. Then $V(w_1) > B(w_1)$. By Lemma 8, it must be that $w_1 < R$. Moreover, by Lemma 7, we have $x(w_1) = 0$ and $y(w_1) < R$, so that $\hat{z} \in [0, R]$ almost surely in this case.

Suppose, next, that exit takes place at some finite time greater than 1 (these are the only two possibilities: by Lemma 10, finite exit *must* occur). Then, by Lemma 11, it must be the case that $w_1 < \iota(z)$, so that by Lemma 12, both $x(w_1)$ and $y(w_1)$ lie in $[0, R]$, if a non-null contract is offered. So $\hat{z} \in [0, R]$ once again. Moreover, since $w(z) < R$ (see Lemma 8), $w_1 \in [0, R]$ as well.

Finally, if no contract is offered at w_1, then $\hat{z} = w_1$ and by the same argument as in the previous paragraph, $\hat{z} \in [0, R]$. ■

LEMMA **14** *¿From any end-of-period wealth* $z \in [0, R]$*, agent wealth must visit 0 infinitely often (with probability one) in any continuous* P*-equilibrium.*

Proof. For any initial end-of-period wealth z, recall that the policy function (or any selection from the policy correspondence) of the artificial Ramsey problem gives us next period's beginning wealth — say w_1 — after which end-of-period wealth $\hat{z}$ is either w_1 (if no contract is offered), or (if a contract is offered) the random variable which takes values $y(w)$ with probability $e(w)$ and $x(w)$ with probability $1 - e(w)$.

Define S to be the smallest integer greater than $M \times R/\eta$, where M is given by Lemma 10 and η is given by Lemma 5. Pick some agent with initial wealth $z \in [0, R]$.

Consider a sample path in which — over the next S periods — there are successes for this agent whenever contracts are offered. We claim that the agent's wealth must visit some value w for which $V(w) > B(w)$ within these S periods.

Suppose that the claim is false. Then it must be that at any end-of-period wealth z_t along this sample path ($0 \le t < S$), beginning wealth next period must satisfy $w_{t+1} \ge z_t$. [This follows from Lemma 11. If $w_{t+1} < z_t$, then $V(w_{t+1}) > B(w_{t+1})$.] It follows that S cannot contain a subset of more than R/η periods for which a contract is offered. [If it did, then, by Lemma 5 and the assumption that only successes occur, wealth would wander beyond $[0, R]$, a contradiction to Lemma 13.] But then, by the definition of S, it must contain more than a set of M *consecutive* periods for which the null contract is optimal.

In this case, wealth simply follows the sequence in the artificial Ramsey problem. But Lemma 10 assures us that within M periods, a wealth level w *will* be reached for which $V(w) > B(w)$. This proves the claim.

Let $\iota(z)$ denote the first wealth level for which — following the path described above for S periods — $V(\iota(z)) > B(\iota(z))$. Let $C(z)$ denote the number of times a contract is offered until the wealth level $\iota(z)$ is reached. Consider the event $E(z)$ in which successes are obtained each time, and the first failure occurs at $\iota(z)$. Define $q(z) \equiv \epsilon^{C(z)+1}$, where ϵ is given by Lemma 6. Then the probability of the event $E(z)$ is bounded below by $q(z)$. But $E(z)$ is clearly contained in the event that wealth hits zero starting from z. We have therefore shown that the probability of this latter event is bounded away from zero in z (because $q(z) \ge \epsilon^S$).

The lemma follows from this last observation. ■

LEMMA **15** *In any continuous P-equilibrium with strictly concave u, $w(z_1) = z_1$ for some $z_1 > R$ implies $w(z) = z$ for all $z > z_1$.*

Proof. Otherwise there exists $z_2 > z_1$ such that exiting at some future date is optimal at z_2, while wealth maintenance is optimal at z_1. Hence there exists integer T and $z' \le R$ such that

$$B(z_2) = \frac{1-\delta^T}{1-\delta} u\left(\frac{(z_2 - \delta^T z')(1-\delta)}{1-\delta^T}\right) + \delta^T V(z') \ge \frac{u(z_2(1-\delta))}{1-\delta} \tag{33}$$

while $z' < R < z_1$ implies

$$\frac{1-\delta^T}{1-\delta} u\left(\frac{(z_1 - \delta^T z')(1-\delta)}{1-\delta^T}\right) + \delta^T V(z') \le \frac{u(z_1(1-\delta))}{1-\delta}. \tag{34}$$

This contradicts the fact that $\frac{1-\delta^T}{1-\delta} u\left(\frac{(z-\delta^T z')(1-\delta)}{1-\delta^T}\right) - \frac{u(z(1-\delta))}{1-\delta}$ is strictly decreasing in z, by the strict concavity of u. ■

LEMMA **16** *In any continuous P-equilibrium with strictly concave u, $w(z) \leq z$ for all $z > R$.*

Proof. We know that either an agent will smooth wealth indefinitely or (by virtue of Lemma 8) will plan to exit at some wealth not exceeding R. In the latter case, strict concavity of u implies that (25) holds, and wealth approaches the exit wealth monotonically over time, so $w(z) < z$. ■

LEMMA **17** *Assume that u is strictly concave and*

$$\lim_{c\to\infty}[u(c) - cu'(c))] > u(R). \tag{35}$$

Then, in any continuous P-equilibrium, there exists $z^* \in (R, \infty)$ such that $z > z^*$ implies $w(z) = z$ and $z \in [R, z^*)$ implies $w(z) < z$.

Proof. The optimality of nonmaintenance of wealth at $z > R$ implies the existence of T and exit wealth $z' \leq R$ such that

$$\frac{1-\delta^T}{1-\delta}u(\frac{(z-\delta^T z')(1-\delta)}{1-\delta^T}) - \frac{u(z(1-\delta))}{1-\delta} + \delta^T V(z') \geq 0. \tag{36}$$

Since $z' \geq 0$, this requires

$$\frac{1}{1-\delta}[(1-\delta^T)u(\frac{z(1-\delta)}{1-\delta^T}) - u(z(1-\delta))] + \delta^T V(z') \geq 0. \tag{37}$$

But the concavity of u implies that

$$(1-\delta^T)u(\frac{z(1-\delta)}{1-\delta^T}) - u(z(1-\delta)) \leq \delta^T[z(1-\delta)u'(z(1-\delta)) - u(z(1-\delta)] \tag{38}$$

Combining (37) and (38), we conclude that

$$(1-\delta)V(z') \geq u(z(1-\delta)) - z(1-\delta)u'(z(1-\delta)). \tag{39}$$

Now we know that $V(z') > B(z')$ (by the definition of an exit wealth), and for such wealths w we know that $x(w) = 0$ and $y(w) \leq R$ (Lemma 7). Consequently, $V(z')$ cannot exceed $(1-\delta)^{-1}u(R)$. Using this information in (39), it follows that a *necessary* condition for nonmaintenance is

$$u(R) \geq u(z(1-\delta)) - z(1-\delta)u'(z(1-\delta)).$$

Using (35) and the fact that $u(c) - cu'(c)$ is increasing in c, define z^{**} such that equality holds in the expression above. Then $w(z) = z$ for all $z > z^{**}$. On the other hand,

$w(z) < z$ for z in a (right) neighborhood of R, since assumption $[\beta]$ ensures that exit is optimal in the Ramsey problem for such a neighborhood of R. Hence there exists $z^* \in (R, z^{**}]$ defined by $\inf\{z|w(z) = z\}$ such that it is optimal for the agent to maintain wealth above z^* and decumulate below, this very last observation being a consequence of Lemma 15. ■

Proof of Proposition 1. Part (i) follows from Lemmas 2 and 8. Part (ii) follows from Lemmas 15, 16 and 17. Part (iii) follows from Lemmas 13 and 14. ■

LEMMA **18** *Consider any continuous P-equilibrium and any $\bar{w} < \infty$. Then there exists $\kappa > 0$ such that for any initial wealth $w_0 \in [R, \bar{w})$ not in $\mathcal{W}$, w_t enters $[0, R]$ or the set $\mathcal{W}$ at some date with probability at least κ. In particular, if $w_0 < z^*$ then, w_t enters $[0, R]$ at some date t with probability at least κ.*

Proof. Since w_0 is not in $\mathcal{W}$, either a contract is offered at w_0, or $w_0 < z^*$ and a contract is not offered at w_0. Consider the event that a failure results whenever a contract is offered and the agent's wealth exceeds R. Note that in such an event the agent's wealth falls monotonically (conditional on wealth exceeding R and not entering $\mathcal{W}$), by virtue of the description of the agent's saving behavior (if a contract is not offered then wealth must lie below z^*, in which case the agent's starting wealth in the next period is smaller) and given Lemmas 5 and 8 (in case a contract is offered failure implies the agent's wealth drops from the beginning to end of the period). Moreover, in this event wealth falls by at least η every time a contract is offered. If the number of times a contract is offered exceeds $K \equiv \frac{\bar{w}-R}{\eta}+1$, the agent's wealth must — at some time — fall below R. Note also that if a contract is not offered at some intervening wealth less than z^*, his wealth falls deterministically until such time that it either drops below R or arrives at a level where a contract is offered. Hence with probability at least $\kappa \equiv \epsilon^K$, where ϵ is the uniform lower bound on the probability of failure given by Lemma 6, the agent's wealth must eventually either fall below R or enter $\mathcal{W}$.

Finally, note that if $w_0 < z^*$, notice that exactly the same event can be constructed to yield the second part of the lemma. ■

LEMMA **19** *Consider any continuous P-equilibrium and an unbounded sequence of wealth levels $w_n \to \infty$ for each of which a feasible contract exists. Suppose also that u is strictly concave and $z^* < \infty$. Then there exists $\bar{W} > z^*$ and a number $\tau > 0$ such that for any $w > \bar{W}$:*

(i) $x(w) > z^*$

(ii) $e(w) \geq \underline{e} \equiv \frac{R}{f}$

(iii) $e(w)y(w) + [1 - e(w)]x(w) \geq w + \tau$.

Proof. If (i) is false, we can construct a sequence $w^n \to \infty$ such that $x^n = x(w^n) \leq z^*$. Then $y^n \to \infty$, otherwise (APC) will be violated for large enough n. So $D'(e^n) = B(y^n) - B(x^n) \to \infty$, and $e^n \to 1$. Let $z^n \equiv e^n y^n + (1 - e^n)x^n$. Then for large n, $z^n > z^*$, and

$$\begin{aligned}(1 - e^n)[B(z^n) - B(x^n)] + e^n[B(z^n) - B(y^n)] &\geq (1 - e^n)[B(z^n) - B(z^*)] + e^n[B(z^n) - B(y^n)] \\ &\geq (1 - e^n)B'(z^n)(z^n - z^*) + e^n B'(z^n)(z^n - y^n) \\ &= B'(z^n)[(1 - e^n)(z^n - z^*) + e^n(z^n - y^n)]\end{aligned}$$

the second inequality utilising the fact that the agent maintains wealth above z^* so B is concave over this range. Now note that

$$\lim_n [(1 - e^n)(z^n - z^*) + e^n(z^n - y^n)] = -\lim_n (1 - e^n)(z^* - x^n) = 0$$

since $e^n \to 1$. So

$$\lim_n (1 - e^n)[B(z^n) - B(x^n)] + e^n[B(z^n) - B(y^n)] \geq 0.$$

Since $(1 - e^n)B(x^n) + e^n)B(y^n) = B(w^n) + D(e^n)$, it follows that

$$\lim_n [B(z^n) - B(w^n) - D(e^n)] \geq 0.$$

Moreover, $\lim_n D(e^n) = \infty$, so

$$\lim_n [B(z^n) - B(w^n)] = \infty,$$

implying

$$\lim_n [z^n - w^n] = \infty.$$

Then P's expected profit which is bounded above by $w_n - z^n - f + e^n R$, must converge to $-\infty$, a contradiction.

To establish (ii), utilize (i) and the strict concavity of B above z^* to infer that

$$\begin{aligned}B(e(w)y(w) + (1 - e(w))x(w)) &> e(w)B(y(w)) + (1 - e(w))B(x(w)) \\ &= B(w) + D(e(w)) \\ &\geq B(w)\end{aligned}$$

so that

$$e(w)y(w) + (1 - e(w))x(w) > w.$$

Hence $e(w) < \underline{e}$ implies that P's profit

$$w - e(w)y(w) - (1 - e(w))x(w) - f + e(w)R < 0.$$

Finally, (iii) follows from the fact that

$$B(e(w)y(w) + (1 - e(w))x(w)) \geq B(w) + D(\underline{e})$$

which completes the proof. ■

Proof of Proposition 2. By the first part of Lemma 18, the probability of entering either $\mathcal{W}$ or the poverty trap is bounded away from zero over any compact interval of initial wealths of the form $[0, \bar{w}]$. By the second part of that lemma, the probability of entering the poverty trap is bounded away from zero if initial wealth is less than z^*.

To complete the proof of the proposition, it suffices to prove that the only remaining limit event is one in which wealth goes to infinity, and that this has positive probability whenever initial wealth w exceeds $\sup \mathcal{W}$.

To this end, we make the following claim: there exists $w^* > 0$ and $\theta > 0$ such that if $w \geq w^*$, then $\text{Prob}(w_t \to \infty) \geq \theta$.

To prove this claim, pick any $w \geq \bar{w}$, where $\bar{w}$ is given by Lemma 19, and consider the events

$$\Lambda(w) \equiv \{w_0 = w; \quad w_t \geq \bar{w} \text{ for all } t\}$$

and

$$\Gamma(w) \equiv \{w_0 = w; \quad w_t \to \infty\}.$$

Using Lemma 19, it is easy to verify that

$$\text{Prob}[\Gamma(w)|\Lambda(w)] = 1 \text{ for all } w > \bar{w}. \tag{40}$$

The reason for this is that to the right of $\bar{w}$, the process is akin to a Markov process in which, for every w, there are two possible continuation values for end-of-period wealth. The expected value exceeds w by some amount bounded away from zero (see Lemma 19, part (iii)). Moreover, the probability of the success wealth is also bounded away from zero (see part (ii) of that Lemma). Finally, since all wealths lie above z^* (part (i) of Lemma 19), next period's starting wealth equals this period's ending wealth. Standard arguments then show that conditional on staying above $\bar{w}$, the process must converge to infinity almost surely, which is exactly (40).

The same argument actually reveals something stronger:that if w is large enough (and sufficiently larger than $\bar{w}$), the process will stay above $\bar{w}$ forever with probability that is bounded away from zero.[2] In other words, there exists $w^* > 0$ and $\epsilon > 0$ such that for all $w \geq w^*$,

$$\text{Prob}[\Lambda(w)] \geq \epsilon. \tag{41}$$

[2]The formal proof of this result requires a simple coupling argument which we omit.

Combining (40) and (41), we may conclude that for all $w \geq w^*$,

$$\text{Prob}[\Gamma(w)] \geq \epsilon. \tag{42}$$

Next, observe that starting from any wealth $w > \sup \mathcal{W}$, it is possible to hit a starting wealth that exceeds w^* with probability bounded away from zero (take η as given in Lemma 5, and simply look at the event in which K successes occur, where K is the smallest integer exceeding $[w^* - \sup \mathcal{W}]/\eta$). It follows (applying (42)) that there exists $\epsilon' > 0$ such that for all $w > \sup \mathcal{W}$,

$$\text{Prob}[w_0 = w; \quad w_t \to \infty] \geq \epsilon'.$$

Combining this observation with the fact that over any compact interval, the probability of entering $\mathcal{W}$ or the poverty trap is bounded away from zero, the proof of the proposition is complete. ∎

LEMMA **20** *If [β] holds, then $e(0) = e^*$ and $y(0) = y^*$ maximizes $e[R - y]$ subject to $D'(e) = \max\{B^*(y) - B^*(0), 0\}$ and $y \geq 0$, where B^* denotes the ex post value function in the Ramsey problem with exit option V^*.*

Proof. Note that we can restrict the range of feasible values of y to $[0, R]$, since any $y > R$ is strictly dominated by $y = R$. By Lemma 3, part (ii), it follows that $y < z_1$. That is, $B^*(y) = u(y) + \delta V(0) = u(y) + B^*(0)$. So $B^*(y) - B^*(0) = u(y)$, and the problem reduces to the static optimal contracting problem. ∎

Proof of Proposition 3. Lemma 3 describes the solution B^* to the Ramsey problem with exit option V^*. Assuming for the moment that B^* will be the *ex post* value function under the constructed P-equilibrium, it should be clear — from Lemma 5 itself — that (ii) and (iii) of the proposition are immediately satisfied.

By definition, B^* satisfies the functional equation

$$B^*(z) = \max_{0 \leq \delta w \leq z} [u(z) + \delta \max\{V^*, B^*(w)\}].$$

Thus — if $B^*(z)$ is to be an equilibrium — it must be that the *ex ante* value function satisfies

$$V(w) = \max\{V^*, B^*(w)\}. \tag{43}$$

Part (i) of the proposition would follow right away from (43), and part (iv) would only require the additional assistance of Lemma 7.

So all that remains to be proved is that if the principal does take the value function B^* as given to solve his constrained optimization problem, then indeed, the resulting *ex ante* value function satisfies (43).

To this end, define w^* by $B^*(w^*) \equiv V^*$. It is easy to see that w^* is well-defined and unique. First, consider $w \leq w^*$. We claim that neither (APC) nor (PPC) binds in the principal's problem. To see this, drop these two constraints in the maximization problem. Then the problem reduces to that considered in Lemma 20 with $x = 0$. Moreover, the resulting solution automatically satisfies both (APC) and (PPC). This proves the claim. So invoking Lemma 20 yet again, we see that $V(w) = V^*$ for all $w \leq w^*$.

Next, consider $w > w^*$. If there is no feasible non-null contract, then $V(w) = B^*(w)$. So suppose that there is a feasible non-null contract. We claim that (APC) must bind in any optimal contract.

Suppose not, so that $V(w) > B^*(w)$ for some $w > w^*$. Then by Lemma 7, $x(w) = 0$ and $y(w) < R$. Moreover, given that (APC) does not bind, this is exactly the problem described in Lemma 20. But the solution to this will violate the APC, as $w > w^*$ by hypothesis. Hence it must be the case that $V(w) = B^*(w)$ for all $w > w^*$. This establishes the claim, and completes the proof of the proposition. ■

Lemma **21** *In any A-equilibrium, PPC binds at every wealth level* w.

Proof. Suppose this is false at w. Then a non-null contract is offered there. If B is continuous either at $y(w)$ or at $x(w)$, reduce the transfer to P at whichever state happens to be a continuity point of B. For a sufficiently small transfer, the corresponding effort change induced will be small, so the breakeven constraint will be preserved. The variation will increase A's present value utility, a contradiction.

Next, suppose that B is not continuous at either y or x. Then there are three possibilities: (a) $R > y(w) - x(w)$, (b) $R = y(w) - x(w)$, and (c) $R < y(w) - x(w)$. If (a) holds, increase y slightly in a way that preserves the inequality $R > y(w) - x(w)$. The agent's effort cannot fall in response, so (PPC) will continue to be respected. Then A's present value utility increases, a contradiction. In case (c) the reverse argument works: the transfer in the unsuccessful state can be reduced slightly to effect an improvement. In case (b), both y and x can be raised in step, so as to preserve the equality of R with $y - x$. Then any change in effort does not affect (PPC), while A is rendered better off. ■

Lemma **22** *In any A-equilibrium* V *is strictly increasing.*

Proof. Obviously V is nondecreasing. Suppose there exist w_1, w_2 with $w_2 > w_1$ such that $V(w_2) = V(w_1)$. Then $\{x(w_1), y(w_1), e(w_1)\}$ is feasible at w_2, and hence must also be optimal at w_2. But here (PPC) does not bind, which contradicts Lemma 21. ■

Proof of Proposition 4. Consider part (i). We divide the proof into three cases: (a) $R > y - x$; (b) $R < y - x$; (c) $R = y - x$.

Case (a): Take any positive $\epsilon < e[R - (y - x)]$, where e denotes the effort assigned at w. Construct a contract $(\tilde{y} = y + \frac{\epsilon}{e}, \tilde{x} = x)$, and let $\tilde{e}$ denote the associated effort response. Then by construction PPC is satisfied at wealth $w + \epsilon$ by the new contract $(\tilde{y}, \tilde{x})$, if the agent were to continue to select effort e. Since B is strictly increasing, the agent's optimal effort response $\tilde{e} \geq e$. Given $R > \tilde{y} - \tilde{x}$, PPC must continue to be satisfied at $\tilde{e}$. Hence the new contract is feasible at wealth $w + \epsilon$. Since the effort e is still available to the agent,

$$V(w + \epsilon) \geq eB(\tilde{y}) + (1 - e)B(\tilde{x}) - D(e) \tag{44}$$

Note that $B(\tilde{y}) - B(y) \geq u(c_s + \frac{\epsilon}{e}) - u(c_s)$ if $c_s \equiv c(y)$ since it is always feasible for the agent to entirely consume any increment in end-of-period wealth. Hence

$$\begin{aligned} V(w + \epsilon) - V(w) &\geq e[u(c_s + \frac{\epsilon}{e}) - u(c_s)] \\ &\geq \epsilon u'(c_s + \frac{\epsilon}{e}) \\ &\geq \epsilon u'(\max\{c_s, c_f\} + \frac{\epsilon}{e}) \end{aligned} \tag{45}$$

where (c_s, c_f) denotes $(c(y), c(x))$. The result then follows upon dividing through and taking limits with respect to ϵ.

Case (b): Reverse the argument of the preceding case, and distribute a wealth increase entirely to the unsuccessful rather than successful state to obtain the same conclusion.

Case (c): For arbitrary positive ϵ, construct the new contract $(\tilde{y} = y + \epsilon, \tilde{x} = x + \epsilon)$, which will be feasible at wealth $w + \epsilon$ if the agent continues to select e. It will be feasible even if he were to change his effort, since $R = y - x$. Using similar reasoning to that used in case (a) above:

$$\begin{aligned} V(w + \epsilon) - V(w) &\geq [e\{u(c_s + \epsilon) - u(c_s)\} + (1 - e)\{u(c_f + \epsilon) - u(c_f) \\ &\geq \epsilon[eu'(c_s + \epsilon) + (1 - e)u'(c_f + \epsilon)] \\ &\geq \epsilon u'(\max\{c_s, c_f\} + \epsilon). \end{aligned} \tag{46}$$

Now turn to part (ii). By assumption,

$$V(w) = eB(y) + (1 - e)B(x) - D(e) \tag{47}$$

where B is right-continuous at $y = y(w)$ and $x = x(w)$, with e denoting $e(w)$. Then for small enough $\epsilon > 0$, there exist positive incremental payments $\Delta_y(\epsilon), \Delta_x(\epsilon)$ that solve the following two equations:

$$\begin{aligned} e\Delta_y + (1 - e)\Delta_x &= \epsilon \\ B(y + \Delta_y) - B(y) &= B(x + \Delta_x) - B(x) \end{aligned}$$

Moreover, $\Delta_y(\epsilon)$ and $\Delta_x(\epsilon)$ both tend to 0 as $\epsilon \to 0+$, and so does $\psi(\epsilon) \equiv B(y+\Delta_y(\epsilon)) - B(y)$. By construction, the contract $y + \Delta_y(\epsilon), x + \Delta_x(\epsilon)$ elicits the same effort response e as the previous contract; hence it is feasible at wealth $w + \epsilon$. Therefore:

$$V(w+\epsilon) - V(w) \geq \psi(\epsilon). \tag{48}$$

Since $B(z+\Delta) - B(z) \geq u(c(z)+\Delta) - u(c(z))$, it follows that for $z = y, x$:

$$\lim_{\epsilon \to 0+} [\psi(\epsilon) - \Delta_z(\epsilon) u'(c(z))] \geq 0. \tag{49}$$

Weighting inequality (49) by the probability of the corresponding outcomes and adding across the two states, it follows that

$$\lim_{\epsilon \to 0+} [\psi(\epsilon) - \theta^{-1}\epsilon] \geq 0 \tag{50}$$

where θ denotes $[e\frac{1}{u'(c(y(w(z))))} + (1-e)\frac{1}{u'(c(x(w(z))))}]$. The first inequality in part (ii) of the proposition then follows from combining (48) and (50).

Finally to establish the second inequality in (ii), note that if this result is false at some z, $c(z)$ must be positive, so it is feasible for the agent to consume a little bit ($\epsilon > 0$) less. This would cause wealth at the following date to be $w(z) + \frac{\epsilon}{\delta}$ instead of $w(z)$. It must therefore be the case that for every small $\epsilon > 0$:

$$u(c(z)) - u(c(z) - \epsilon) \geq \delta[V(w(z) + \frac{\epsilon}{\delta}) - V(w(z))] \tag{51}$$

Taking limits with respect to ϵ, and using the first inequality in (ii) (which we have already established), we obtain a contradiction. ■

Proof of Proposition 5. Suppose there is an equilibrium with a strict poverty trap, which requires that $w(y(0)) = 0 = w(x(0))$. If u is strictly concave, a standard revealed preference argument implies that $w(.)$ must be nondecreasing. We also know that $y^* \equiv y(0) > x^* \equiv x(0)$ is necessary for the agent to exert effort and thus satisfy PPC. Hence $w(z) = 0$ and $B(z) = u(z) + \delta V(0)$ for all $z \in [0, y^*]$. So B is differentiable at x^*.

If B is right-differentiable at y^*, then using $u'(y^*)$ to denote the right-derivative at y^*, (10) of Proposition 4 implies that $\frac{1}{u'(y^*)} \leq [e(0)\frac{1}{u'(y^*)} + (1-e(0))\frac{1}{u'(x^*)}]$ upon using the hypothesis that $c(y^*) - y^* = c(x^*) - x^* = 0$. Since $\frac{1}{u'}$ is strictly increasing, and e less than 1 (otherwise (PPC) will be violated), this inequality contradicts $y^* > x^*$.

If B is not right-differentiable at y^*, then note that the rate of increase of B at y^* is bounded below by $u'(y^*)$, since it is feasible for the agent to entirely consume all incremental wealth. Now modify the proof of Proposition 4 to infer that the inequality in (10) must be strict at y^*, which will again generate a contradiction.[3] ■

LEMMA **23** *In any A-equilibrium, let* Ω *denote the set of wealth levels for which a nonnull contract is offered. Then*

(i) *if either* Ω *is bounded or* $u'(\infty) > 0$, $\inf_{w\in\Omega} e(w) > 0$.

(ii) *if* Ω *is bounded,* $\sup_{w\in\Omega} e(w) < 1$.

Proof. Suppose that (i) is false. Then there is a sequence w_n and corresponding nonnull contracts (x_n, y_n, e_n) with $e_n \to 0$. First we claim that $y_n - x_n \to 0$. To see this, use the (EIC) to obtain

$$D'(e_n) = B(y_n) - B(x_n) \geq u(y_n) - u(x_n) \geq u'(y_n)(y_n - x_n),$$

and now observe that in either of the hypotheses underlying (i) of the lemma, $\inf_n u'(y_n) > 0$. The claim now follows from the fact that $D'(e) \to 0$ as $e \to 0$.

Now, (PPC) tells us that for all n, $w_n - x_n \geq f + e_n(y_n - x_n - R)$. Because $y_n - x_n \to 0$, it follows that there exists $\theta > 0$ such that for all n sufficiently large, $w_n - x_n \geq \theta > 0$. Once again using the fact that $y_n - x_n \to 0$, we may conclude that $w_n - y_n > 0$ for all n sufficiently large. That is, *both* x_n and y_n are strictly less than w_n for n large, which must violate (APC).

If (ii) is false we can find a sequence of wealths $w_n \to w$ and corresponding feasible contracts (y_n, x_n, e_n) in which PPC binds at every n, with $e_n \to 1$, so $D'(e_n) = B(y_n) - B(x_n) \to \infty$, which requires $\lim_n y_n = \infty$. Since PPC binds for each n (Lemma 21), we have $w_n - f + e_n[R - y_n] - (1 - e_n)x_n = 0$. This implies that $\lim_n w_n \geq f - R + \lim_n y_n = \infty$, contradicting the boundedness of Ω. ■

Proof of Proposition 6. Since $u'(\infty) > 0$, Proposition 4 implies that $\frac{1}{u'(c_t)}$ forms a submartingale in any right-continuous A-equilibrium, where c_t denotes the consumption of the agent at date t. Hence $\frac{1}{u'(c_t)}$ converges almost surely. Since u is strictly concave, this implies that c_t converges almost surely to a (possibly infinite valued) random variable $\tilde{c}$.

[3] If $R \leq y^* - x^*$ then note that the reasoning employed in the proof of Proposition 4 implies that the rate of increase of V at zero wealth is strictly greater than $u'(x^*)$, since at least part of the incremental wealth can be distributed to the agent in the unsuccessful state. And if $R > y^* - x^*$, the incremental distributions Δ_y, Δ_x constructed in the proof of part (ii) of Proposition 4 on the assumption that B is right-continuous at y^* with a slope bounded below by $u'(y^*)$, will be feasible even if B is not right-continuous at y^*. The reason is that the rate of increase of B is then even higher, so in case it is optimal for the agent to change his effort, he would increase it. Since $R > y^* - x^*$ the increased effort cannot jeopardize (PPC).

We claim that almost surely $\tilde{c} = \infty$, implying that $z_t \to \infty$, and hence that $w_t \to \infty$.[4]

Because $u'(\infty) > 0$ by assumption, part (i) of Lemma 23 applies. Define $m \equiv D'(\inf_{w\in\Omega} e(w))$. Then $m > 0$. Pick any $Z < \infty$ and integer $T > \frac{1}{m}[B(Z) - B(0)] - 1$. Also select any nonnegative integer q. Define the event

$$\mathcal{B}_q(Z) \equiv \{\tilde{c} \in [q, q+1) \quad \text{and} \quad z_t < Z \quad \text{for all} \quad t\}$$

Note that $z_{t-1} < Z$ implies that $w_t < W \equiv \frac{Z}{\delta}$. Applying both parts of Lemma 23, there are bounds $\underline{e}, \bar{e} \in (0,1)$ for effort levels arising in any contract corresponding to wealth in $[0, W]$.

Next, select $\eta \in (0, (1-\delta)m)$. Note that since $u(0) = 0$, continuity and concavity of u imply that u is uniformly continuous. Hence we can find $\epsilon > 0$ such that $|u(c) - u(c')| < \eta$ whenever $|c - c'| < \epsilon$.

Conditional on wealth w at the beginning of date 0, define for any positive integer T the T-step ahead possible realizations of z, c and w under the given A-equilibrium in the following manner. Let $n_t \in \{s, f\}$ denote the outcome of the project t dates ahead, and let n^t denote the history of project outcomes $(n_t, n_{t-1}, \ldots, n_0)$ between dates 0 and t. The contract itself specifies $z^0(n_0, w) = y(w)$ if $n_0 = s$ and $x(w)$ otherwise. Using the equilibrium consumption strategy this enables us to work out $c^0(n_0, w) = c(z^0(n_0, w))$, and the wealth at the beginning of next date: $w^1(n_0, w) = \delta^{-1}[z^0(n_0, w) - c^0(n_0, w)]$. Proceeding in this fashion we can find the T-step-ahead-realizations as functions of the history of the outcomes of the project between 0 and T: $z^T(n^T, w), c^T(n^T, w), w^T(n^{T-1}, w)$.

Define $\mathcal{C}(w, T) = \{c | c = c^t(n^t, w), t \leq T\}$, the set of possible realizations of consumptions T-steps ahead.

Then define the event

$$\mathcal{A}(Z) \equiv \{e_t \in (\underline{e}, \bar{e}) \quad \text{for all} \quad t \quad \text{and} \quad \lim_{t\to\infty} \text{diam}\, \mathcal{C}(w_t, T) = 0\}$$

Lemma **24** *For any Z and any integer q:*

$$\text{Prob}[\mathcal{A}(Z) | \mathcal{B}_q(Z)] = 1. \tag{52}$$

Proof. Define the event $\mathcal{A}_1(Z) \equiv \{e_t \in (\underline{e}, \bar{e}) \quad \text{for all} \quad t\}$. Note that $\text{Prob}[\mathcal{A}_1(Z)|\mathcal{B}_q(Z)] = 1$, implying that $\text{Prob}[\mathcal{A}(Z)|\mathcal{B}_q(Z)] = \text{Prob}[\mathcal{A}(Z)|\mathcal{A}_1(Z) \cap \mathcal{B}_q(Z)]$. We claim that this latter probability equals one.

If this is false then the event $\mathcal{E}$ defined by the following conditions (i) - (iv) has positive probability: (i) diam $\mathcal{C}(w_t, T)$ does not converge to 0 as $t \to \infty$; (ii) $e_t \in (\underline{e}, \bar{e}) \quad \text{for all} \quad t$; (iii) z_t and w_t are bounded above by Z and W respectively; and (iv) $\tilde{c} \in [q, q+1)$.

[4] It is readily verified that z_t is bounded if and only if w_t is bounded.

Now (i) implies that there exists $\zeta > 0$ and dates $t_k, k = 1, 2, \ldots$ such that diam $\mathcal{C}(w_{t_k}, T) > \zeta$ for all k. Take any integer $\kappa > \max\{\frac{3}{\zeta}, 1\}$, and define $\theta = \frac{1}{\kappa}$. Then partition $[q, q+1)$ into intervals $I_p \equiv [q + p\theta, q + (p+1)\theta)$, for $p = 0, 1, 2, \ldots, \kappa - 1$. Define $\mathcal{E}_p$ to be the event where (i)-(iv) holds, and in addition $\tilde{c} \in I_p$.

The proof is completed by showing that for every p, the event $\mathcal{E}_p$ has zero probability. Condition (i) implies that for any k there exists integer $l(k,i) \leq T$ and outcomes $n_i^{l(k,i)}, i = 1, 2$ for the next $l(k,i)$-steps such that the corresponding consumptions $c_i \equiv c^{l(k,i)}(n_i^{l(k,i)}, w_{t_k}), i = 1, 2$ are at least ζ apart. This implies in turn that $\inf_{x \in I_p} |c_i - x| \geq \frac{\zeta}{3}$ for at least one i, since $\theta < \frac{\zeta}{3}$. Since e_t is bounded away from zero and one by (ii), every T-step-ahead-outcome sequence has probability bounded away from zero. Hence conditional on event $\mathcal{E}_p$, almost surely there will be an infinite number of values of k for which $c_{t_k + l(k,i)}$ will equal c_i which is at least $\frac{\zeta}{3}$ distant from I_p. This implies that the event $\mathcal{E}_p$ has zero probability. ■

Lemma **25** *For any Z and any q:*

$$Prob[\mathcal{A}(Z) \cap \mathcal{B}_q(Z)] = 0. \tag{53}$$

Proof. The proof rests on the following claim.
Claim. Let s^t (resp. f^t) denote t-step-ahead histories in which the project results in a success (resp. failure) in every period. Then for any Z and any q:

$$\text{Prob}[B(z^T(s^T, w_t)) - B(z^T(f^T, w_t)) > m(T+1) \quad \text{for all} \quad t > t^* \quad \text{for some} \quad t^* | \mathcal{A}(Z) \cap \mathcal{B}_q(Z)] = 1 \tag{54}$$

To prove this claim, consider any path in $\mathcal{A}(Z) \cap \mathcal{B}_q(Z)$, and select t^* such that diam $\mathcal{C}(w_t, T) < \frac{\epsilon}{2}$ for all $t > t^*$. Note that along any such path, $e_t > \underline{e}$ at all t; hence the effort incentive constraint implies $B(z^0(s^0, w)) - B(z^0(f^0, w)) > m$ for all w. So the inequality

$$B(z^{\tilde{T}}(s^{\tilde{T}}, w_t)) - B(z^{\tilde{T}}(f^{\tilde{T}}, w_t)) > m(\tilde{T} + 1) \tag{55}$$

holds for $\tilde{T} = 0$ for all t. We shall show that if it holds for $\tilde{T} - 1$ it holds for $\tilde{T}$ as well.

Use z^S and z^{SS} to denote $z^{\tilde{T}-1}(s^{\tilde{T}-1}, w_t), z^{\tilde{T}}(s^{\tilde{T}}, w_t)$ respectively. Similarly use z^F and z^{FF} to denote $z^{\tilde{T}-1}(f^{\tilde{T}-1}, w_t), z^{\tilde{T}}(f^{\tilde{T}}, w_t)$ respectively. And use z^{FS}, z^{SF} to denote $z^{\tilde{T}}(f, s^{\tilde{T}-1}, w_t)$ and $z^{\tilde{T}}(s, f^{\tilde{T}-1}, w_t)$ respectively.

Next note that

$$V(w(z^S)) - V(w(z^F)) = \frac{1}{\delta}[B(z^S) - B(z^F)] - \frac{1}{\delta}[u(c(z^S)) - u(c(z^F))]$$

By the induction hypothesis $B(z^S) - B(z^F) > \tilde{T}m$. Moreover, since $t > t^*$, we have ensured by construction that $\frac{1}{\delta}[u(c(z^S)) - u(c(z^F))] < \frac{1}{\delta}\eta$. Since $\eta < (1-\delta)m$, it follows that $V(w(z^S)) - V(w(z^F)) \geq \frac{1}{\delta}[\tilde{T}m - \eta] > \tilde{T}m$. Since

$$\begin{aligned} V(w(z^S)) &= \max_e [eB(z^{SS}) + (1-e)B(z^{FS}) - D(e)] \\ V(w(z^F)) &= \max_e [eB(z^{SF}) + (1-e)B(z^{FF}) - D(e)] \end{aligned}$$

it is evident that $V(w(z^S)) - V(w(z^F)) > \tilde{T}m$ implies that $\max\{B(z^{SS}) - B(z^{SF}), B(z^{FS}) - B(z^{FF})\} > \tilde{T}m$. Since $B(z^{SS}) - B(z^{FS}) \geq m$ and $B(z^{SF}) - B(z^{FF}) \geq m$, it then follows that $B(z^{SS}) - B(z^{FF}) > (\tilde{T}+1)m$, establishing the Claim.

We are now in a position to prove Lemma 25. Consider the event $\mathcal{B}_q(Z)$. Since z_t is bounded above by Z, $B(z_t)$ is bounded above by $B(Z)$. Recall that we selected $T > \frac{1}{m}[B(Z) - B(0)] - 1$. The Claim above implies then that

$$B(z^T(s^T, w_t)) > m(T+1) + B(z^T(f^T, w_t)) \geq m(T+1) + B(0) > B(Z)$$

i.e., that $z^T(s^T, w_t)$ exceeds the upper bound Z for all $t > t^*$. Since given event $\mathcal{A}(Z)$, a string of T successive successes will almost surely occur infinitely often, it follows that the event $\mathcal{A}(Z) \cap \mathcal{B}_q(Z)$ has zero probability, and Lemma 25 follows.

Combining the results of Lemmas 24 and 25 it follows that $\text{Prob}[\mathcal{B}_q(Z)] = 0$ for any Z and q. If we define the event $\mathcal{B}_q \equiv \cup_{k=1}^{\infty} \mathcal{B}_q(k) = \lim_k \mathcal{B}_q(k)$ that z_t is bounded while $\tilde{c}$ lies in $[q, q+1)$, this implies that $\text{Prob}[\mathcal{B}_q] = 0$. Hence if consumption converges to a limit in $[q, q+1)$, almost surely $z_t \to \infty$; i.e., $w_t \to \infty$. But in this event Proposition 4 implies that the asymptotic rate of increase of V will be bounded below by $u'(q+1)$. On the other hand V is bounded above by the value function $\tilde{V}$ corresponding to the case where effort disutility function is identically zero, whose asymptotic rate of increase equals $u'(\infty) < u'(q+1)$, and we obtain a contradiction. Hence consumption converges to a limit between q and $q+1$ with zero probability. Since this is true for all integers q, it follows that almost surely consumption will converge to ∞. ■

Proof of Proposition 7. Proceed in a manner analogous to that in the proof of the previous proposition. If the result is false, then wealth and consumption is bounded with probability one, which ensures that $\frac{1}{u'(c_t)}$ again forms a submartingale, so c_t converges almost surely. Lemma 23 ensures that effort is bounded away from zero and one, so all finite step histories will occur infinitely often with probability one. If the agent receives a contract at all dates, then the agents wealth must be unbounded with probability one in order to provide necessary effort incentives at all dates, and we obtain a contradiction. ■