INFORMATION AGGREGATION WITH HETEROGENEOUS TRADERS

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Note. After the main theorem in this paper was proved, we came across Hellwig (1980), which is motivated by exactly the same considerations and proves the same result as our Theorem 2, except that our model allows for general signal structures with arbitrary covariances and asymmetries. Despite the significant additional generality, we fully appreciate Hellwig's contribution and do not intend to publish these notes. We simply put up the results and proofs here in the hope that the approach here (which is quite distinct) will be useful to others working in the field. We intend to take these methods in other directions as well.

1. INTRODUCTION

We consider a setting in which a single security with unknown fundamental is traded by a continuum of agents. Each agent belongs to one of a finite number of "informational groups." Each individual in each group obtains an idiosyncratically noisy signal which is built from a group-level aggregate signal plus iid individual noise. There is an aggregate signal for every informational group. In additional to this signal for each individual, each person also observes the price and can make inferences about the fundamental using that price.

Groups differ in size. They also differ in their attitudes to risk. There is risk heterogeneity both within and across groups. There is a common prior on the fundamental.

The setting is multivariate normal, with arbitrary correlations across fundamental and signals.

If a super-agent (with the same common prior) were to observe each of the aggregate signals, she would have a best prediction of the fundamental, which is a linear function of the signal vector. Any price function which is the same linear function (up to an intercept term) would aggregate information perfectly. Agents observing such a price function would entirely ignore their own signal. Their demand for the asset would also be insensitive to the price. The intercept term can be adjusted so that aggregate demand equals supply. This forms the basis for Grossman's remarkable observation (1976): that there is an "equilibrium" that is fully revealing.

Because no trader uses any of his information once the price aggregates all information efficiently, this notion of equilibrium is deeply problematic. Indeed, the "equilibrium" cannot be justified even by resorting to indifference on the part of the trader to use (or not use) his private information, not even when the cost of information acquisition is zero. A trader would *strictly* prefer to *not* use any of his information, whether or not it is freely available. In other words, even the redundancy that tolerates some degree of mixing and allows information to seep in via indifference, is not to be

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had. Obviously Grossman himself is aware of this point: "If all traders ignore their information how does the information get into the price?" (p. 582).

We discuss Grossman's philosophy in more detail below. Our approach here is different. We demand that the equilibrium price function be built from information that is collectively and actively used by all the traders. This can be achieved by permitting some noise trade, which we do. The problem is that once there is noise trade, the equilibrium price function. cannot be fully revealing. Indeed, it fails to be fully revealing in two distinct ways. One is, of course, the trivial way: noise creates additional variation in the price that a full observation of all signals would have avoided. The second, deeper problem arises because different signals now acquire different weights: if a group is large or close to risk neutral, it will react more sensitively to a change in signal. Other smaller groups will carry less weight. In other words, group-level characteristics will affect price formation, and the coefficients on each signal will vary accordingly.

Nevertheless, our main result (Theorem 2) is a vindication of Grossman: as the variance on the noise goes to zero, any corresponding sequence of (linear) equilibrium price functions must converge (modulo an intercept term) to the fully revealing predictor used by a super-agent who observes all signals.

2. Some Remarks

Grossman is. of course, fully aware of the conceptual problem we raise here. His response is to view full aggregation as the limit of some hypothetical iterative process. Grossman motivates his solution concept thus: the equilibrium price function "can be interpreted as a stationary point of the following process. Suppose traders initially begin in a naive way, thinking of [price] as a number and conditioning only on [their own signals]. Let an auctioneer call out prices until the market clears ... After many repetitions traders can tabulate the empirical distribution of [price-fundamental] pairs ... After this joint distribution is learned, traders will have an incentive to change their bids just as the market is about to clear ... This changes their demands and thus the market will not clear at [the old price]." In contrast, continues Grossman, the full-information price function "is a self fulfilling expectations equilibrium: when all traders think prices are generated by [that function], they will act in such a way that the market clears ... "

But viewing full information aggregation as a stationary point of this iteration is not enough, precisely because at this stationary point, traders *strictly* do not want to condition on their information. Matters would be more convincing if, for instance, information aggregation could be viewed not just as a stationary point but as an *attractor* of the above iteration. In that case, the incentives to use individual information would be gradually eroded, but not fast enough so that the required information fails to seep into the price function "in the limit." But to the best of our knowledge, no such result is provided in Grossman (1976) or elsewhere in the competitive framework.[1]

We interpret Grossman as acknowledging this lacuna when he concludes: "[S]uch economies need not be stable because prices are revealing so much information that incentives for the collection of information are removed. The price system can be maintained only when it is noisy enough so that traders who collect information can hide that information from other traders." At the same time, he

[1]We note that it is not enough to prove that no other price function can be a stationary point of the iterative system. It is entirely possible, in principle, for any iterative sequence to cycle without converging anywhere at all.

correctly notes that the full information result "will not hold if there is noise in the price system." In fact, in the presence of noisy trades, not only will that noise carry over to the price function, it will also distort that price function when there are heterogeneous traders, in the sense of placing different *relative* weights on each signal relative to the relative weights under full information aggregation.

The main result of this paper can be viewed as a contribution to the "stability" of full information aggregation. As the variance of noise vanishes, any corresponding sequence of equilibrium price functions must generate a limit situation in which each trader behaves as if he is perfectly informed. That is, not only does the noise vanish, but the distortion of relative weights generated by heterogenous trading groups (in their sizes or attitudes to risk) must vanish as well.

3. Model

3.1. **Information.**

There is a single risky asset and a single trading period. The asset has a fixed supply $X \in \mathbb{R}$, and a fundamental value θ. The fundamental isn't directly observed, but it generates signals given by $\mathrm{s} \in \mathbb{R}^n$, where $n \geq 1$. Specifically, assume that (θ, s) is multivariate normal with mean $\mathbf{0}$ and covariance matrix Σ.

We will use the following notation throughout: $\mathrm{Var}(x)$ will stand for the variance of a random variable x, and $\mathrm{Cov}(x_i, x_j)$ — or where there is no risk of confusion, $\mathrm{Cov}(i, j)$ — will stand for the covariance of two random variables x_i and x_j. For a linear combination $\boldsymbol{\lambda}$ over the signals, we will use $\mathrm{Var}(\boldsymbol{\lambda})$ and $\mathrm{Cov}(\boldsymbol{\lambda}, x_i)$ (or $\mathrm{Cov}(\boldsymbol{\lambda}, i)$), with the understanding that in this case $\boldsymbol{\lambda}$ is being identified with the random variable $\sum_i \lambda_i s_i$. Finally, $\mathrm{Cov}(\theta, \mathrm{s})$ (resp. $\mathrm{Cov}(\mathrm{s}, \theta)$) will denote the row (resp. column) vector of covariances $(\mathrm{Cov}(\theta, s_1), \ldots, \mathrm{Cov}(\theta, s_n))$ between the fundamental θ and each signal s_i. So

$$\Sigma = \begin{pmatrix} \mathrm{Var}(\theta) & \mathrm{Cov}(\theta, \mathrm{s}) \\ \mathrm{Cov}(\mathrm{s}, \theta) & \Sigma_{\mathbf{ss}} \end{pmatrix},$$

where a typical covariance term in $\Sigma_{\mathbf{ss}}$ is just $\mathrm{Cov}(i, j)$.

We refer to $\mathrm{s} = (s_1, \ldots, s_n)$ as the *aggregate signal structure*. We impose minimal restrictions on this structure. We take it that each aggregate signal s_i is positively correlated with the state; that is, $\mathrm{Cov}(\theta, i) > 0$. Apart from presuming the existence of some correlation, this is without loss of generality (it is a matter of arranging the signals so that they "point" the right way).

We also presume that no component of s can be expressed as a linear combination of the others. In particular, this means that the aggregate structure is "imperfect": no one can pin down the fundamental even if she observes all the aggregate signals.[2] Formally, we assume that $\Sigma_{\boldsymbol{\theta},\boldsymbol{\theta}} - \Sigma_{\boldsymbol{\theta},\mathbf{s}} \Sigma_{\mathbf{s},\mathbf{s}}^{-1} \Sigma_{\boldsymbol{\theta},\mathbf{s}}'$ is positive definite.

Apart from these restrictions, we allow the aggregate signals to exhibit any degree of asymmetry or heterogeneous correlation structure.

3.2. **Traders and Trader Types.**

There is a unit measure of market participants or traders. Each trader has a quadratic utility function. That is, if they hold k units of the risky asset with a perceived

[2]For instance, if $e(1) = -e(2)$, the aggregate signal structure permits us to completely pin down the value of θ.

mean value μ and variance v, then their payoff is given by

$$k[\mu - p] - \frac{k^2 v\gamma}{2}, \tag{1}$$

where p is the going market price of the asset and γ is a parameter that measures the degree of that participant's risk aversion. Notice that the risk is borne on fluctuating value whether or not the position k is positive or negative.

A trader is described by her type t, which lies in some finite set T. Let $\tau(t)$ be the measure of individuals of type t. An individual type t has two components: $\{i(t), \gamma(t)\}$, where $i(t)$ denotes her informational group membership, and $\gamma(t)$ denotes her risk-aversion. Now for a more detailed description.

First, a trader belongs to one of n information groups, each of positive mass. Each group i reads a distinct set of newspapers that effectively allows one of the aggregate signals s_i to be "directed" towards them. If an individual trader α belongs to informational group i, she sees a (conditionally) independent idiosyncratic signal

$$z_i(\alpha) = s_i + \epsilon_i \tag{2}$$

where ϵ_i is normal with mean 0 and variance Var(ϵ_i). Of course, Var$(z_i(\alpha))$ = Var(s_i) + Var(ϵ_i).

Second, we wish to allow for an general distribution of risk attitudes, but to avoid arbitrarily large short or long term positions we assume that everyone is risk-averse ($\gamma > 0$). We make no assumption about the interaction between information and risk (nor the size of any of these groups), but only ask that each information group have strictly positive size.

3.3. **Demand.** A typical agent α of type t maximizes the payoff function in (1) with μ and v set equal to her conditional expectation and variance of her fundamental value, given the information available to her. That is, if her private information is $z_{i(t)}(\alpha)$, then $\mu = \mathbb{E}(\theta | z_{i(t)}(\alpha), p)$ and $v =$ Var$(\theta | z_{i(t)}(\alpha), p)$. The consequent demand for the asset by α is given by

$$k(z_{i(t)}(\alpha), p, \gamma(t)) = \frac{\mathbb{E}(\theta | z_{i(t)}(\alpha), p) - p}{\text{Var}(\theta | z_{i(t)}(\alpha), p)\gamma(t)}. \tag{3}$$

3.4. **Noise Trade.** There is additionally some noise trade u, distributed independently of everything else, with mean 0 and variance Var(u). In the sequel we will take Var(u) to zero.

3.5. **Price Functions.** It is customary in Gaussian models to focus on *linear equilibria*, and we follow that lead here. Accordingly, we concentrate on the class of price functions given by

$$p = \boldsymbol{\lambda}\mathbf{s} + \omega u + c, \tag{4}$$

where $\boldsymbol{\lambda} \equiv (\lambda_1, \ldots, \lambda_n)$ represents weights on signals for each information group, and ω is the weight on the noise u.

We make three remarks on this conjectured price function. First, because there is a continuum of independent idiosyncratic signals of each type, all drawn from the same group, linearity will guarantee that only the aggregate signals will matter; hence just the appearance of s's in the price function and the disappearance of all idiosyncratic signals. Second, the prior mean of θ is taken to be zero and so does not explicitly enter the price function, but of course the prior will cast its influence so that $\sum_i \lambda_i$ won't generally equal 1. Finally, the λ_i's will depend in an intimate way on the overall stochastic structure of signals, asset supply, group sizes, and the distribution of risk attitudes and payoff functions.[3]

3.6. **Expectations and Demand.** Given any price function as in (4), we can describe the conditional expectation $\mathbb{E}(\theta|z_{i(t)}(\alpha), p)$ for a trader of type t. The relevant predictor is the variable $q \equiv p - c$, where the intercept term has been netted out. It is described by a system of weights $\{a(t), b(t)\}$ such that for any such individual α and any signals $(z_{i(t)}(\alpha), p)$ received by her,

$$\mathbb{E}(\theta|z_{i(t)}(\alpha), p) = a(t)z_{i(t)}(\alpha) + b(t)q. \tag{5}$$

We will provide explicit formulae for these weights below.

By Bayes' Rule, the values of $a(t)$ and $b(t)$ will be independent of an agent's risk type, though of course, the actions she takes in response will depend on her risk-aversion. Similarly, by a familiar property of normal updating, the conditional variance $\text{Var}(\theta|z_{i(t)}(\alpha), p)$ *only* depends on informational group identity $i = i(t)$ but not on the particular signals received nor on the risk type. Call it $\text{Var}(\theta|i)$. To be sure, this conditional variance depends on the price *function* π as well, but we do not record that dependence explicitly here, for ease of notation. Using this definition along with (5) in equation (3), we see that the aggregate demand for the asset by all individuals of type t depends on p and the aggregate signal $s_{i(t)}$, and is given by

$$\tau(t)\frac{a(t)s_{i(t)} + b(t))q - p}{\text{Var}(\theta|i(t))\gamma(t)} = \delta(t)\frac{a(t)s_{i(t)} - (1 - b(t))p - b(t)c}{\text{Var}(\theta|i(t))}.$$

where $\delta(t) \equiv \tau(t)/\gamma(t)$ is the ratio of population to risk-aversion coefficient for each type t.

We now aggregate over risk groups. Define, for each i,

$$\Delta_i \equiv \sum_{i(t)=i} \delta(t) = \sum_{i(t)=i} \frac{\tau(t)}{\gamma(t)} > 0.$$

Also remember that $a(t)$ and $b(t)$ are independent of risk types, and therefore can be written as a_i and b_i respectively. Then the aggregate demand by each (informational) group i is

$$\Delta_i \frac{a_i s_i - (1 - b_i)p - b_i c}{\text{Var}(\theta|i)}.$$

3.7. **Equilibrium.** Summing noise trade and demand over all i and noise trade, equilibrium price is given by

$$\sum_{i=1}^{n} \Delta_i \frac{a_i s_i - (1 - b_i)p - b_i c}{\text{Var}(\theta|i)} + u = X. \tag{6}$$

Notice that in any equilibrium, we must have $\sum_{i=1}^{n} \Delta_i(1 - b_i)/\text{Var}(\theta|i) \neq 0$, otherwise (6) cannot hold for all u. We therefore have

$$p = \sum_{i=1}^{n} \lambda_i s_i + \omega u + c, \tag{7}$$

[3] In particular, λ_i *cannot* be guaranteed to be nonnegative for all i.

where

$$\omega = \frac{1}{\sum_{j=1}^{n} \Delta_j (1-b_j)/\text{Var}(\theta|j)}. \tag{8}$$

$$c = \frac{X}{\sum_{j=1}^{n} \Delta_j (1-2b_j)/\text{Var}(\theta|j)}. \tag{9}$$

and for each i,

$$\lambda_i = \frac{\Delta_i a_i/\text{Var}(\theta|i)}{\sum_{j=1}^{n} \Delta_j (1-b_j)/\text{Var}(\theta|j)}, \tag{10}$$

Equations (7)–(10) are crucially different from the solution concept described in Grossman (1976). In particular, (10) explicitly demands that the price vary with signals only to the extent that individuals actually use those signals. To complete our description, it will be useful to record closed forms for the vectors of weights $\mathbf{a}$ and $\mathbf{b}$, as well as for the posterior variances $\text{Var}(\theta|i)$. Standard arguments from normal updating show that

$$a_i = \frac{[\text{Var}(\boldsymbol{\lambda}) + \omega^2 \text{Var}(u)]\text{Cov}(\theta, i) - \text{Cov}\,(\boldsymbol{\lambda}, i)\,\text{Cov}(\theta, \boldsymbol{\lambda})}{[\text{Var}(\boldsymbol{\lambda}) + \omega^2 \text{Var}(u)]\text{Var}(z_i) - \text{Cov}\,(\boldsymbol{\lambda}, i)^2} \tag{11}$$

and

$$b_i = \frac{\text{Var}(z_i)\text{Cov}(\theta, \boldsymbol{\lambda}) - \text{Cov}(\theta, i)\text{Cov}\,(\boldsymbol{\lambda}, i)}{[\text{Var}(\boldsymbol{\lambda}) + \omega^2 \text{Var}(u)]\text{Var}(z_i) - \text{Cov}\,(\boldsymbol{\lambda}, i)^2}. \tag{12}$$

Lemma 2 makes the elementary observation that the common denominator of these expressions is strictly positive, so that a_i and b_i are well-defined for every i.

For information to seep into prices, it is intuitively clear that $\mathbf{a}$ cannot be zero. Formally, suppose that $\mathbf{a} = 0$ in some equilibrium, then by (10), $\boldsymbol{\lambda} = 0$ as well. But using this information in (11), we must conclude that $a_i = \text{Cov}(\theta, i)/\text{Var}(z_i) \neq 0$ for every i, which is a contradiction. So $\mathbf{a} \neq 0$, and the same is true of $\boldsymbol{\lambda}$.

The posterior variance $\text{Var}(\theta|i)$, i.e. the variance of θ conditional on the idiosyncratic signal and the price function $\boldsymbol{\pi}$, is given by

$$\text{Var}(\theta|i) = \text{Var}(\theta) - [a_i \text{Cov}(\theta, i) + b_i \text{Cov}(\theta, \boldsymbol{\lambda})]. \tag{13}$$

Equations (7)–(13), along with the parameters already defined, fully describe an equilibrium price function. We are interested in the behavior of this function as the noise trade vanishes; i.e., as $\text{Var}(u) \to 0$. We observe:

THEOREM **1.** *There exists an equilibrium.*

Existence is a technical matter, though in the light of the peculiar problems associated with informational aggregation, it is perhaps of separate interest in this context. We use a fixed point argument with some modifications suggested by the work of Halpern (1968) and Halpern and Bergman (1968); see Lemma 6 which generates a suitable "inward mapping." We do not know if more standard methods would suffice for existence. In a different model of oligopolistic trading with risk-neutral agents, Lambert, Ostrovsky and Panov (2016) obtain an elementary and impressive proof of existence; to the best of our understanding, their methods do not apply here.

4. THE FULL-INFORMATION PREDICTOR

Suppose that an observer were to see the entire set of aggregate signals $\mathrm{s} = (s_1, \ldots, s_n)$, and asked to infer θ. By a standard property of normal updating, the weights $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_n^*)$ of the *full-information predictor* on the aggregate signals are given by

$$\boldsymbol{\lambda}^* = \text{Cov}(\theta, \mathrm{s})\Sigma_{\mathbf{ss}}^{-1} \tag{14}$$

so that $\mathbb{E}(\theta|\mathrm{s}) = \boldsymbol{\lambda}^*\mathrm{s} = \text{Cov}(\theta, \mathrm{s})\Sigma_{\mathbf{ss}}^{-1}\mathbf{s}$, and the variance of the composite signal thus constructed is given by

$$v(\boldsymbol{\lambda}^*) = \boldsymbol{\lambda}^*\Sigma_{ss}\boldsymbol{\lambda}_{\mathbf{w}}^{*'} = \text{Cov}(\theta, \mathrm{s})\Sigma_{\mathbf{ss}}^{-1}\text{Cov}(\mathbf{s}, \theta), \tag{15}$$

where the second equality uses (14).

For each $i = 1, \ldots, n$, define

$$\Gamma_i(\boldsymbol{\lambda}) \equiv \text{Var}(\boldsymbol{\lambda})\text{Cov}(\theta, i) - \text{Cov}\,(\boldsymbol{\lambda}, i)\,\text{Cov}(\theta, \boldsymbol{\lambda}). \tag{16}$$

The zeros of this function and an additional covariance restriction characterize the full-information predictor:

OBSERVATION **1.** *For each $i = 1, \ldots, n$,*

$$\Gamma_i(\boldsymbol{\lambda}^*) = 0, \tag{17}$$

and

$$\text{Cov}(\theta, \boldsymbol{\lambda}^*) = \text{Var}(\boldsymbol{\lambda}^*). \tag{18}$$

and the converse is also true: any non-null vector $\boldsymbol{\lambda}$ solving (17) and (18) must be the full-information predictor that solves (14).

The proof follows immediately from rewriting (14), and is omitted.

The full information predictor is such that all the information in s is combined optimally in $\boldsymbol{\lambda}^*$ to predict θ. Adding any signal to it with any weight will be redundant. In particular, for each i, $\mathbb{E}(\theta|\boldsymbol{\lambda}^*, s_i) = \mathbb{E}(\theta|\boldsymbol{\lambda}^*)$. To see this more formally in a market context, imagine that the price function $\boldsymbol{\pi} = (\boldsymbol{\lambda}, \omega)$ has $\boldsymbol{\lambda} = \boldsymbol{\lambda}^*$, and suppose that there is no noise at all. Then, using (11) with $\text{Var}(u) = 0$,

$$a_i = \frac{\Gamma_i(\boldsymbol{\lambda}^*)}{v(\boldsymbol{\lambda}^*)\text{Var}(z_i) - \text{Cov}\,(\boldsymbol{\lambda}^*, i)^2},$$

and because the denominator of this expression is strictly positive (Lemma 2 below), the weight a_i on the signal s_i will be zero, by Observation 1. In our setting, the full information predictor can never be an equilibrium in a world with no noise, because the resulting zero individual weights on signal cannot be built back into a non-trivial price function. This is at the heart of our departure from Grossman (1976), as already discussed in the Introduction.

As a final remark on the full-information predictor: there is no guarantee that $\boldsymbol{\lambda}^*$ will place positive (or even nonnegative) weight on all the signals, *even if* each signal is separately informative and positively pairwise correlated. The generality of our signal setting precludes this. As an example, suppose there are two iid mean zero normal random variables given by e_1 and e_2. Suppose that the aggregate signal structure is given by $s_1 = \theta + e_1$, and $s_2 = \theta + \zeta e_1 + (1-\zeta)e_2$, where $\zeta \in (1/2, 1)$. Then it can be checked that $\lambda^*(1) < 0$.

4.1. **Information Aggregation.** In what follows, we will take the variance of the noise trade to zero and examine the characteristics of limit equilibria. Therefore, when we speak of equilibrium sequences (indexed by s), we will refer to any sequence of equilibria as $\text{Var}_s(u) \to 0$. The following is our main result:

THEOREM **2.** *Along any sequence* $\text{Var}_s(u)$ *such that* $\text{Var}_s(u) \to 0$*, any corresponding sequence of equilibrium price functions* $(\boldsymbol{\lambda}_s, \omega_s, c_s)$ *must:*

(i) *involve* $\boldsymbol{\lambda}_s$ *converging to* $\boldsymbol{\lambda}^*$*, the full-information predictor, while*

(ii) $\omega_s^2 \text{Var}_s(u) \to 0$ *and* $c_s \to -\frac{X[\text{Var}(\theta) - \text{Cov}(\theta, \boldsymbol{\lambda}^*)]}{\sum_{j=1}^n \Delta_j}$.

The theorem has two parts. The central assertion is Part (i). In general, an equilibrium price function will not aggregate information efficiently. After all, different signals are observed by groups that may vary both in their overall numbers and in their within-group distribution of risk attitudes. Because the volume of group-specific trade also goes into determining the equilibrium price function, and because numbers and risk attitudes affect those volumes, the equilibrium price function will incorporate not just pure information but also group sizes and the full distribution of attitudes to risk. From this perspective, it is of interest that as the impact of noise trade vanishes, *all* these additional effects on the price function endogenously vanish, leaving only the efficient aggregation of information.

Part (ii) states two ancillary observations. First, as the variance of the noise goes to zero, its overall impact on prices goes to zero as well. This is intuitive.[4] Second, when the supply of the asset X is non-zero, the intercept term c of the price function does retain the influence of group sizes and attitudes to risk, as captured by the Δ_i's. But it does so in a trivial way, in the intercept term, which does not impede efficient information aggregation. The main point is that all group-level heterogeneity must complete vanish from the coefficients on s, as already described in Part (i).

We illustrate this result with an example. Suppose that the aggregate structure is made up of just two signals s_1 and s_2. Suppose, moreover, that for $i = 1, 2$, $s_i = \theta + e_i$, where e_1 and e_2 are iid. For additional simplicity, suppose that the prior on the fundamental is improper, or close to it, if you want to exactly embed this example in our model.[5] Then the full information predictor is obviously given by equal weighting: $(\lambda_1^*, \lambda_2^*) = (1/2, 1/2)$. That said, if the groups observing these signals are heterogeneous, the equilibrium price function $(\boldsymbol{\lambda}, \omega)$ will *not* place equal weight on the two signals; that is, in general, $\lambda_1 \neq \lambda_2$.

But this discrepancy must vanish as the variance of noise trades goes to zero. To see this, take a sequence of noise variances going to zero, and let the corresponding sequence of regular equilibrium weights be given by $(\boldsymbol{\lambda}_s, \omega_s)$. Suppose for this discussion that $\boldsymbol{\lambda}_s$ lie on the unit simplex (it is easy to verify that this will indeed be the case for a diffuse prior.) We want to show that $\boldsymbol{\lambda}_s \to (1/2, 1/2)$. Suppose that along some subsequence, $(\boldsymbol{\lambda}_s, \omega_s) \to (\boldsymbol{\lambda}, \omega)$, where $\boldsymbol{\lambda} \neq (1/2, 1/2)$. Without loss of generality, let $\lambda_1 > 1/2$. In this exposition, we shall also suppose that $\lambda_1 < 1$.

Consider an agent from group 1 who sees her own private signal z_1 along with p, conjecturing the weights for p to be given by $\boldsymbol{\lambda}$. What weight a_1 should she place on z_1 so as to predict θ? The answer is that if noise trade is negligible, the weight *must be strictly negative*. Intuitively, a lower

[4]However, it is also the case that ω_s^2 will blow up to infinity.

[5]However, an exact analogue of Theorem 2 can be established for improper priors.

value of the private signal z_1, controlling for price, means two things. It certainly means a lower value of s_1, and therefore of θ. But it *also* means a higher conjectured value for s_2 (holding p constant), and indeed, because we are controlling for the price and because $\lambda_1 \in (1/2, 1)$, it must mean that s_2 goes up by more than s_1 comes down. Consequently, the individual's prediction of the fundamental, which is given by $\mathbb{E}((s_1 + s_2)/2|z_1, p)$, *must go up* as z_1 comes down.

But now we have a contradiction: we started off with $\lambda_1 > 0$ (indeed, $\lambda_1 > 1/2$), but in such a situation every individual in group 1 reacts by placing a negative weight on their own signal. When these reactions are aggregated to form the equilibrium price function, the overall weight placed on s_1 will also be negative. But that weight, in equilibrium, is just λ_1, which we've taken to be positive to begin with!

A similar argument can be applied to $\lambda_1 > 1$ as well, though now the contradiction is not as stark as a comparison of positives and negatives. In fact, the formal analysis is far more general. As long as prices fail to converge to the full-information predictor (with noise trades vanishing), their equilibrium status can be shown to be compromised. Trade volumes along the equilibrium price sequence must delicately adjust to asymptotically negate all the "interferences" of group heterogeneity regarding size and risk attitudes.

4.2. **Proofs.** An *environment* is any set of parameters with a non-zero variance for noise trades. For any price function $(\boldsymbol{\lambda}, \omega, c)$, observe that c plays only a role in determining demand but not in the inference of the fundamental — it is just the intercept term in the price function. For this reason, we will be referring to just (λ, ω) in most of the arguments below.

LEMMA **1.** *There is $v > 0$ such that for any environment, any $(\boldsymbol{\lambda}, \omega)$ and any i, if* $\text{Var}(\theta|i)$ *is given by (13), then* $v \leq \text{Var}(\theta|i) \leq \text{Var}(\theta)$.

Proof. Obvious, as $\text{Var}(\theta|i)$ must be bounded above by the unconditional variance $\text{Var}(\theta)$ and below by the variance of θ conditional on having access to all the aggregate signals $\mathbf{s}$. This latter bound is strictly positive by the assumed positive definiteness of Σ. □

LEMMA **2.** *For every i and every $(\boldsymbol{\lambda}, \omega)$,*

$$[\text{Var}(\boldsymbol{\lambda}) + \omega^2\text{Var}(u)]\text{Var}(z_i) - \text{Cov}\,(\boldsymbol{\lambda}, i)^2 \geq [\text{Var}(\boldsymbol{\lambda}) + \omega^2\text{Var}(u)]\text{Var}(\epsilon_i).$$

Proof. By the Cauchy-Schwarz inequality, $\text{Cov}\,(\boldsymbol{\lambda}, i)^2 \leq v(\boldsymbol{\lambda})\text{Var}(s_i) \leq [\text{Var}(\boldsymbol{\lambda}) + \omega^2\text{Var}(u)]\text{Var}(s_i)$, and $\text{Var}(z_i) = \text{Var}(s_i) + \text{Var}(\epsilon_i)$, so that $[\text{Var}(\boldsymbol{\lambda}) + \omega^2\text{Var}(u)]\text{Var}(z_i) - \text{Cov}\,(\boldsymbol{\lambda}, i)^2 \geq [\text{Var}(\boldsymbol{\lambda}) + \omega^2\text{Var}(u)]\text{Var}(\epsilon_i)$. □

For any set of weights $\mathbf{a} = (a_1, \ldots, a_n)$ as given by (11), define

$$r_i \equiv \frac{\Delta_i a_i}{\text{Var}(\theta|i)} \tag{19}$$

and observe from (10) and (8) that any equilibrium price function $(\boldsymbol{\lambda}, \omega, c)$ is then fully characterized by (11)–(13), (19) and

$$\boldsymbol{\lambda} = \omega\mathbf{r} \text{ and } \omega = \left[\sum_j \frac{\Delta_j(1 - b_j)}{\text{Var}(\theta|j)}\right]^{-1}. \tag{20}$$

along with (9) to determine the intercept term c.

For any vector $\mathbf{x}$, denote by $||\mathbf{x}||$ its norm $\sum_i |x_i|$.

LEMMA **3.** *If* $\mathbf{a}$ *is given by (11) and* $\mathbf{r}$ *by (19), then* $||\mathbf{a}||$ *and* $||\mathbf{r}||$ *are uniformly bounded over all environments and all* $(\boldsymbol{\lambda}, \omega)$ *with* $\omega \neq 0$.

Proof. Using (11) and Lemma 2, we see that for all $\boldsymbol{\pi} = (\boldsymbol{\lambda}, \omega)$ with $\omega \neq 0$, a_i is well-defined and

$$\begin{aligned} |a_i| &\leq \frac{[\text{Var}(\boldsymbol{\lambda}) + \omega^2 \text{Var}(u)]|\text{Cov}(\theta, i)| + |\text{Cov}\,(\boldsymbol{\lambda}, i)\,||\text{Cov}(\theta, \boldsymbol{\lambda})|}{[\text{Var}(\boldsymbol{\lambda}) + \omega^2 \text{Var}(u)]\text{Var}(\epsilon_i)} \\ &= \frac{|\text{Cov}(\theta, i)| + [|\text{Cov}\,(\boldsymbol{\lambda}, i)\,||\text{Cov}(\theta, \boldsymbol{\lambda})|/[\text{Var}(\boldsymbol{\lambda}) + \omega^2 \text{Var}(u)]]}{\text{Var}(\epsilon_i)} \\ &\leq \frac{|\text{Cov}(\theta, i)| + \left[\sqrt{\text{Var}(s_i)\text{Var}(\theta)}\text{Var}(\boldsymbol{\lambda})/[\text{Var}(\boldsymbol{\lambda}) + \omega^2 \text{Var}(u)]\right]}{\text{Var}(\epsilon_i)} \\ &\leq \frac{|\text{Cov}(\theta, i)| + \sqrt{\text{Var}(s_i)\text{Var}(\theta)}}{\text{Var}(\epsilon_i)} < \infty, \end{aligned}$$

where the penultimate inequality applies Cauchy-Schwartz to $|\text{Cov}\,(\boldsymbol{\lambda}, i)\,|$ and $|\text{Cov}(\theta, \boldsymbol{\lambda})|$. This gives us a bound on $|a_i|$ which is uniform over all environments, and of course can be taken to be independent of i. By Lemma 1 and (19), the same must be true of $||\mathbf{r}||$. $\square$

Consider the domain of all vectors $(\mathbf{r}, \omega)$ with $\omega > 0$ and $||\mathbf{r}|| \leq R < \infty$, where R is some upper bound given by Lemma 3. We will construct a mapping Ψ on this domain. To this end, define $\boldsymbol{\lambda} \equiv \omega \mathbf{r}$. Now define a_i by (11), b_i by (12), $\text{Var}(\theta|i)$ by (13), and r'_i by (19). Next, define an "auxiliary vector" $\mathbf{d}$ the meaning of which will become clearer below: for each i, let

$$d_i = \frac{\text{Cov}(\theta, \mathbf{r}) - \text{Cov}(\mathbf{r}, i)\frac{\text{Var}(\theta|i)}{\Delta_i} r_i}{\text{Var}(\mathbf{r}) + \text{Var}(u)}. \tag{21}$$

Complete the mapping by setting

$$\omega' = \left[\sum_j \frac{\Delta_j \left(1 - \frac{1}{\omega} d_j\right)}{\text{Var}(\theta|j)}\right]^{-1}. \tag{22}$$

We now have $(\mathbf{r}', \omega') = \Psi(\mathbf{r}, \omega)$. Notice that while $||\mathbf{r}'|| \leq R$, ω' may not be positive, so that this function does not necessarily map back to the domain from where we started.

LEMMA **4.** $||\mathbf{d}||$ *is bounded uniformly over all* $\mathbf{r}$ *with* $||\mathbf{r}|| \leq R < \infty$. *Moreover, there is* $\zeta > -1$ *such that*

$$\sum_j \frac{\Delta_j}{\text{Var}(\theta|j)} d_j \geq \zeta > -1 \tag{23}$$

uniformly on the subdomain of $\mathbf{r}$ *with* $\text{Cov}(\theta, \mathbf{r}) \geq 0$.

Proof. Because $\mathbf{r}$ is bounded, the absolute value of the numerator in (21) is bounded above, while the denominator is bounded below by $\text{Var}(u)$. The boundedness of $||\mathbf{d}||$ follows immediately. To establish (23), observe that for each j, using (21) and $\text{Cov}(\theta, \mathbf{r}) \geq 0$ (on the subdomain),

$$\frac{\Delta_j}{\text{Var}(\theta|j)} d_j = \frac{\Delta_j}{\text{Var}(\theta|j)} \left[\frac{\text{Cov}(\theta, \mathbf{r}) - \text{Cov}(\mathbf{r}, j) \frac{\text{Var}(\theta|i)}{\Delta_i} r_j}{\text{Var}(\mathbf{r}) + \text{Var}(u)} \right] \geq -\frac{\text{Cov}(\mathbf{r}, j) r_j}{\text{Var}(\mathbf{r}) + \text{Var}(u)},$$

so that summing over all j, we have

$$\sum_j \frac{\Delta_j}{\text{Var}(\theta|j)} d_j \geq -\sum_j \frac{\text{Cov}(\mathbf{r}, j) r_j}{\text{Var}(\mathbf{r}) + \text{Var}(u)} = -\frac{\text{Var}(\mathbf{r})}{\text{Var}(\mathbf{r}) + \text{Var}(u)}.$$

Because $\text{Var}(u) > 0$ and $||\mathbf{r}||$ is bounded, the existence of $\zeta > -1$ is immediate. □

To proceed further, let K (resp. k) be a finite and strictly positive upper (resp. lower) bound on $\sum_j \Delta_j / \text{Var}(\theta|j)$ and D be some finite positive upper bound on $\sum_j \Delta_j d_j / \text{Var}(\theta|j)$, which is well-defined by Lemma 4. Let $m < M$ be any strictly positive and finite numbers that satisfy

$$m \leq (1 + \zeta)/K \text{ and } M \geq (D + 1)/k. \tag{24}$$

Because $\zeta > -1$ (Lemma 4), M and m satisfy all the properties required of them.

LEMMA **5.** Ψ *has the following properties:*

(i) *If* $\text{Cov}(\theta, \mathbf{r}) = 0$*, then* $\text{Cov}(\theta, \mathbf{r}') \geq 0$.

(ii) *If* $\omega \geq M$*, then* $\omega' \leq M$.

(iii) *If* $\omega = m$*, then* $\omega \geq m$.

Proof. (i) In constructing the mapping Ψ, we define $\boldsymbol{\lambda} = \omega \mathbf{r}$ and a_i by (11). Combining the two, we see that

$$\begin{aligned} a_i &= \frac{[\text{Var}(\boldsymbol{\lambda}) + \omega^2 \text{Var}(u)]\text{Cov}(\theta, i) - \text{Cov}(\boldsymbol{\lambda}, i) \text{Cov}(\theta, \boldsymbol{\lambda})}{[\text{Var}(\boldsymbol{\lambda}) + \omega^2 \text{Var}(u)]\text{Var}(z_i) - \text{Cov}(\boldsymbol{\lambda}, i)^2} \\ &= \frac{[\text{Var}(\mathbf{r}) + \text{Var}(u)]\text{Cov}(\theta, i) - \text{Cov}(\mathbf{r}, i) \text{Cov}(\theta, \mathbf{r})}{[\text{Var}(\mathbf{r}) + \text{Var}(u)]\text{Var}(z_i) - \text{Cov}(\mathbf{r}, i)^2}, \end{aligned}$$

where we've divided through above and below by the common term ω^2. Because the denominator of this last expression is strictly positive (Lemma 2) and because $\text{Cov}(\theta, i) \geq 0$ by assumption, it follows that if $\text{Cov}(\theta, \mathbf{r}) = 0$, then $a_i \geq 0$ for all i. Therefore r'_i given by (19) is nonnegative for all i. It follows that

$$\text{Cov}(\theta, \mathbf{r}') = \sum_j r'_j \text{Cov}(\theta, j) \geq 0.$$

(ii) Suppose that $\omega \geq M$. Then, using (22), we have that

$$\omega' = \left[\sum_j \frac{\Delta_j \left(1 - \frac{1}{\omega} d_j\right)}{\text{Var}(\theta|j)} \right]^{-1} = \left[\sum_j \frac{\Delta_j}{\text{Var}(\theta|j)} - \frac{1}{\omega} \sum_j \frac{\Delta_j}{\text{Var}(\theta|j)} d_j \right]^{-1} \leq \left[k - \frac{1}{\omega} D \right]^{-1} \leq M,$$

where the last inequality uses the fact that $\omega = M \geq (D + 1)/k$; see (24).

(iii) Finally, suppose that $\omega = m$. Again using (22), we see that

$$\omega' = \left[\sum_j \frac{\Delta_j \left(1 - \frac{1}{\omega} d_j\right)}{\text{Var}(\theta|j)}\right]^{-1} = \left[\sum_j \frac{\Delta_j}{\text{Var}(\theta|j)} - \frac{1}{\omega}\sum_j \frac{\Delta_j}{\text{Var}(\theta|j)} d_j\right]^{-1} \geq \left[K - \frac{1}{\omega}\zeta\right]^{-1} \geq m,$$

where the last inequality uses $\omega = m \leq (1+\zeta)/K$; see (24). □

Now consider the subdomain $\mathcal{F}$ of all $(\mathbf{r}, \omega)$ such that $||\mathbf{r}|| < R < \infty$ (as before), and satisfying the additional conditions: (i) $\text{Cov}(\theta, \mathbf{r}) \geq 0$ and (ii) $m \leq \omega \leq M$.

LEMMA **6.** *Ψ has a fixed point $(\mathbf{r}^*, \omega^*) \in \mathcal{F}$.*

Proof. Clearly, $\mathcal{F}$ is a nonempty, compact, convex subset of Euclidean space, and Ψ is continuous on $\mathcal{F}$. In general, however, Ψ will fail to map from from $\mathcal{F}$ to $\mathcal{F}$. However, the map is *inward* in the sense of Halpern (1968) and Halpern and Bergman (1968); for an exposition, see Aliprantis and Border (2006, Definition 17.53). That is, for every $\mathbf{r} \in \mathcal{F}$, there exists $\alpha > 0$ such that

$$(\mathbf{r}, \omega) + \alpha[\Psi(\mathbf{r}, \omega) - (\mathbf{r}, \omega)] \in \mathcal{F}. \tag{25}$$

To verify this property, consider any point $(\mathbf{r}, \omega) \in \mathcal{F}$. If, in addition $\text{Cov}(\theta, \mathbf{r}) > 0$ and $\omega \in (m, M)$, then (25) is trivially true for some $\alpha > 0$.[6] Otherwise, at least one of the following is true: $\text{Cov}(\theta, \mathbf{r}) = 0$, $\omega = m$, or $\omega = M$. But then it follows from Lemma 5 (i)–(iii) that $\text{Cov}(\theta, \mathbf{r}) \geq 0$, *and* either $\omega \geq m$ or $\omega \leq M$. Because $\text{Cov}(\theta, \mathbf{r})$ is linear in $\mathbf{r}$ and $m < M$, it follows that (25) must hold for some $\alpha > 0$.

By the Halpern-Bergman fixed point theorem (see Aliprantis and Border 2006, Theorem 17.54), there exists $(\mathbf{r}^*, \omega^*) \in \mathcal{F}$ such that $\Psi(\mathbf{r}^*, \omega^*) = (\mathbf{r}^*, \omega^*) \in \mathcal{F}$. □

Proof of Theorem 1. Suppose that $(\mathbf{r}^*, \omega^*)$ is a fixed point of the mapping Ψ. Let $\boldsymbol{\lambda}^* \equiv \omega^* \mathbf{r}^*$. Define for each i, a_i^* by (11), b_i^* by (12), $\text{Var}^*(\theta|i)$ by (13) and c^* by (9). We claim that (8) and (10) hold for all i.[7] We reproduce these equations here for convenience:

$$\lambda_i^* = \frac{\Delta_i a_i^* / \text{Var}^*(\theta|i)}{\sum_{j=1}^n \Delta_j (1 - b_j^*)/\text{Var}^*(\theta|j)} \tag{26}$$

and

$$\omega^* = \frac{1}{\sum_{j=1}^n \Delta_j (1 - b_j^*)/\text{Var}^*(\theta|j)}. \tag{27}$$

Note that at the fixed point, the auxiliary vector $\mathbf{d}^*$ is given by (21). Also, (19) holds at the fixed point. Combining these two equations, we see that for every i

$$d_i^* = \frac{\text{Cov}(\theta, \mathbf{r}^*) - \text{Cov}(\mathbf{r}^*, i)\frac{\text{Var}^*(\theta|i)}{\Delta_i} r_i^*}{\text{Var}(\mathbf{r}^*) + \text{Var}(u)} = \frac{\text{Cov}(\theta, \mathbf{r}^*) - \text{Cov}(\mathbf{r}^*, i) a_i^*}{\text{Var}(\mathbf{r}^*) + \text{Var}(u)}. \tag{28}$$

[6] In this case, some convex combination of $(\boldsymbol{\lambda}, \omega)$ and $\Psi(\boldsymbol{\lambda}, \omega)$ must trivially lie in $\mathcal{F}$ as well.

[7] Because of the circuitous definition of ω' in (22) via the auxiliary variable $\mathbf{d}$ defined in (21), the proof that a fixed point of Ψ is an equilibrium is non-trivial. We could not find a way to directly define ω' via the equilibrium condition and proceed from there. The reason is that discontinuities appear in the mapping; in particular, we could not find a way to keeping the mapping away from $(\mathbf{r}, \omega) = (\mathbf{0}, 0)$, where singularities occur. Our use of the auxiliary variable $\mathbf{d}$ circumvents these discontinuities.

On the other hand, invoking (11) and the fact that $\boldsymbol{\lambda}^* \equiv \omega^* \mathbf{r}^*$,

$$
\begin{aligned}
a_i^* &= \frac{[\text{Var}(\boldsymbol{\lambda}^*) + \omega^{*2}\text{Var}(u)]\text{Cov}(\theta, i) - \text{Cov}\left(\boldsymbol{\lambda}^*, i\right)\text{Cov}(\theta, \boldsymbol{\lambda}^*)}{[\text{Var}(\boldsymbol{\lambda}^*) + \omega^{*2}\text{Var}(u)]\text{Var}(z_i) - \text{Cov}\left(\boldsymbol{\lambda}^*, i\right)^2} \\
&= \frac{[\text{Var}(\mathbf{r}^*) + \text{Var}(u)]\text{Cov}(\theta, i) - \text{Cov}\left(\mathbf{r}^*, i\right)\text{Cov}(\theta, \mathbf{r}^*)}{[\text{Var}(\mathbf{r}^*) + \text{Var}(u)]\text{Var}(z_i) - \text{Cov}\left(\mathbf{r}^*, i\right)^2}, \qquad (29)
\end{aligned}
$$

where we've divided through above and below by the common term ω^{*2}. Combining (28) and (29),

$$
\begin{aligned}
d_i^* &= \frac{\text{Cov}(\theta, \mathbf{r}^*)}{\text{Var}(\mathbf{r}^*) + \text{Var}(u)} - \frac{\text{Cov}(\mathbf{r}^*, i)a_i^*}{\text{Var}(\mathbf{r}^*) + \text{Var}(u)} \\
&= \frac{\text{Cov}(\theta, \mathbf{r}^*)}{\text{Var}(\mathbf{r}^*) + \text{Var}(u)} - \frac{\text{Cov}(\mathbf{r}^*, i)}{\text{Var}(\mathbf{r}^*) + \text{Var}(u)} \left\{ \frac{[\text{Var}(\mathbf{r}^*) + \text{Var}(u)]\text{Cov}(\theta, i) - \text{Cov}\left(\mathbf{r}^*, i\right)\text{Cov}(\theta, \mathbf{r}^*)}{[\text{Var}(\mathbf{r}^*) + \text{Var}(u)]\text{Var}(z_i) - \text{Cov}\left(\mathbf{r}^*, i\right)^2} \right\} \\
&= \frac{[\text{Var}(\mathbf{r}^*) + \text{Var}(u)]\text{Var}(z_i)\text{Cov}(\theta, \mathbf{r}^*) - [\text{Var}(\mathbf{r}^*) + \text{Var}(u)]\text{Cov}(\theta, i)\text{Cov}(\mathbf{r}^*, i)}{[\text{Var}(\mathbf{r}^*) + \text{Var}(u)]\left\{[\text{Var}(\mathbf{r}^*) + \text{Var}(u)]\text{Var}(z_i) - \text{Cov}\left(\mathbf{r}^*, i\right)^2\right\}} \\
&= \frac{\text{Var}(z_i)\text{Cov}(\theta, \mathbf{r}^*) - \text{Cov}(\theta, i)\text{Cov}(\mathbf{r}^*, i)}{[\text{Var}(\mathbf{r}^*) + \text{Var}(u)]\text{Var}(z_i) - \text{Cov}\left(\mathbf{r}^*, i\right)^2} \\
&= \omega^* \frac{\text{Cov}(\theta, \boldsymbol{\lambda}^*)\text{Var}(z_i) - \text{Cov}(\theta, i)\text{Cov}(\boldsymbol{\lambda}^*, i)}{[\text{Var}(\boldsymbol{\lambda}^*) + \omega^{*2}\text{Var}(u)]\text{Var}(z_i) - \text{Cov}\left(\boldsymbol{\lambda}^*, i\right)^2} = \omega^* b_i^*.
\end{aligned}
$$

Using this equality in (22) at the fixed point, so that $\omega = \omega' = \omega^*$, we obtain precisely (27). Now combine (19) (again at the fixed point) along with (27) and $\boldsymbol{\lambda}^* \equiv \omega^* \mathbf{r}^*$ to obtain (26). □

Recall that for any i and $\boldsymbol{\lambda}$, $\Gamma_i(\boldsymbol{\lambda}) = \text{Var}(\boldsymbol{\lambda})\text{Cov}(\theta, i) - \text{Cov}\left(\boldsymbol{\lambda}, i\right)\text{Cov}(\theta, \boldsymbol{\lambda})$.

LEMMA **7.** *For any vector $\boldsymbol{\lambda}$, $\sum_i \lambda_i \Gamma_i(\boldsymbol{\lambda}) = 0$.*

Proof. Note that $\sum_i \lambda_i \text{Cov}(\theta, i) = \text{Cov}(\theta, \boldsymbol{\lambda})$ and $\sum_i \lambda_i \text{Cov}\left(\boldsymbol{\lambda}, i\right) = \text{Cov}\left(\boldsymbol{\lambda}, \boldsymbol{\lambda}\right) = \text{Var}(\boldsymbol{\lambda})$. Expanding $\Gamma_i(\boldsymbol{\lambda})$, we see that $\sum_i \lambda_i \Gamma_i(\boldsymbol{\lambda}) = \text{Var}(\boldsymbol{\lambda}) \sum_i \lambda_i \text{Cov}(\theta, i) - \text{Cov}(\theta, \boldsymbol{\lambda}) \sum_i \lambda_i \text{Cov}\left(\boldsymbol{\lambda}, i\right) = \text{Var}(\boldsymbol{\lambda})\text{Cov}(\theta, \boldsymbol{\lambda}) - \text{Cov}(\theta, \boldsymbol{\lambda})\text{Var}(\boldsymbol{\lambda}) = 0$. □

LEMMA **8.** *In any equilibrium, each $\lambda_i a_i$ (when nonzero) must have the same sign, which is also the same as the signs of ω and* $\text{Cov}(\theta, \lambda)$.

Proof. In equilibrium, for each i, $\lambda_i = \omega \Delta_i a_i / \text{Var}(\theta|i)$ by (10) and (8). Multiplying through by λ_i,

$$
\lambda_i^2 = \omega \frac{\Delta_i \lambda_i a_i}{\text{Var}(\theta|i)},
$$

which proves that all $\lambda_i a_i$ have the same sign when non-zero, and which in turn is equal to the sign of ω. Note from (11) and the formula for Γ_i that the numerator of $\lambda_i a_i$, call it $\text{Num}(\lambda_i a_i)$, is just $\Gamma_i(\boldsymbol{\lambda}) + \omega^2 \text{Var}(u)\text{Cov}(\theta, i)$, and that the denominator is strictly positive (Lemma 2). By Lemma 7,

$$
\sum_i \text{Num}(\lambda_i a_i) = \sum_i \lambda_i \left[\Gamma_i(\boldsymbol{\lambda}) + \omega^2 \text{Var}(u)\text{Cov}(\theta, i)\right] = \sum_i \lambda_i \Gamma_i(\boldsymbol{\lambda}) + \omega^2 \text{Cov}(\theta, \boldsymbol{\lambda}) = \omega^2 \text{Cov}(\theta, \boldsymbol{\lambda}),
$$

so that each $\lambda_i a_i$ (and ω) must also have the same sign as $\text{Cov}(\theta, \boldsymbol{\lambda})$. □

We will consider sequences of equilibria $(\boldsymbol{\lambda}_s, \omega_s, c_s)$ in environments (indexed by s) such that $\text{Var}_s(u) \to 0$ as $s \to \infty$. For any $\boldsymbol{\lambda} \neq 0$, let $\ell \equiv ||\boldsymbol{\lambda}|| > 0$ and $\boldsymbol{\mu} \equiv \boldsymbol{\lambda}/\ell$. Note that $||\boldsymbol{\mu}|| = 1$. By (11) and (12),

$$\begin{aligned} a_{is} &= \frac{[\text{Var}(\boldsymbol{\lambda}_s) + \omega_s^2 \text{Var}_s(u)]\text{Cov}(\theta, i) - \text{Cov}\,(\boldsymbol{\lambda}_s, i)\,\text{Cov}(\theta, \boldsymbol{\lambda}_s)}{[\text{Var}(\boldsymbol{\lambda}_s) + \omega_s^2 \text{Var}_s(u)]\text{Var}(z_i) - \text{Cov}\,(\boldsymbol{\lambda}_s, i)^2} \\ &= \frac{[\text{Var}(\boldsymbol{\mu}_s) + \gamma_s^2 \text{Var}_s(u)]\text{Cov}(\theta, i) - \text{Cov}\,(\boldsymbol{\mu}_s, i)\,\text{Cov}(\theta, \boldsymbol{\mu}_s)}{[\text{Var}(\boldsymbol{\mu}_s) + \gamma_s^2 \text{Var}_s(u)]\text{Var}(z_i) - \text{Cov}\,(\boldsymbol{\mu}_s, i)^2} \end{aligned} \tag{30}$$

and

$$\begin{aligned} b_{is} &= \frac{\text{Var}(z_i)\text{Cov}(\theta, \boldsymbol{\lambda}_s) - \text{Cov}(\theta, i)\text{Cov}\,(\boldsymbol{\lambda}_s, i)}{[\text{Var}(\boldsymbol{\lambda}_s) + \omega_s^2 \text{Var}_s(u)]\text{Var}(z_i) - \text{Cov}\,(\boldsymbol{\lambda}_s, i)^2}. \\ &= \frac{1}{\ell_s} \frac{\text{Var}(z_i)\text{Cov}(\theta, \boldsymbol{\mu}_s) - \text{Cov}(\theta, i)\text{Cov}\,(\boldsymbol{\mu}_s, i)}{[\text{Var}(\boldsymbol{\mu}_s) + \gamma_s^2 \text{Var}_s(u)]\text{Var}(z_i) - \text{Cov}\,(\boldsymbol{\mu}_s, i)^2}, \end{aligned} \tag{31}$$

where $\gamma_s \equiv \omega_s/\ell_s$. Because $\boldsymbol{\mu}_s$ is bounded in s, all its limit points $\boldsymbol{\mu}$ are finite. We also know that $\mathbf{a}_s$ is uniformly bounded; Lemma 3. By taking an appropriate subsequence if needed, we can presume that $\boldsymbol{\mu}_s \to \boldsymbol{\mu}$ and $\mathbf{a}_s \to \mathbf{a}$.

LEMMA **9.** *At the limit,* $\text{Cov}(\theta, \boldsymbol{\mu}) \neq 0$.

Proof. Suppose on the contrary that $\text{Cov}(\theta, \boldsymbol{\mu}) = 0$. Noting that $\boldsymbol{\mu} \neq 0$ so that $\text{Var}(\boldsymbol{\mu}) > 0$ by the positive definiteness of Σ, and recalling that $\text{Cov}(\theta, i)$ for all i, we can pass to the limit in (30) to conclude that the limit value a_i is strictly positive for every i. Because

$$\mu_{is} = \frac{1}{\ell_s} \lambda_{is} = \gamma_s \frac{\Delta_i}{\text{Var}_s(\theta|i)} a_{is}$$

for every i, we see that in the limit, every μ_i must have the same sign when non-zero (and some are indeed nonzero, because $\boldsymbol{\mu} \neq 0$). Because

$$\text{Cov}(\theta, \boldsymbol{\mu}) = \sum_i \mu_i \text{Cov}(\theta, i)$$

and $\text{Cov}(\theta, i) > 0$ for all i, it follows that $\text{Cov}(\theta, \boldsymbol{\mu}) \neq 0$, a contradiction. □

LEMMA **10.** *Along any sequence of equilibria,* $\{\ell_s\}$ *is bounded above.*

Proof. Suppose on the contrary that for some subsequence (retain original index s) $\ell_s \to \infty$. Then from (31) and Lemma 2 applied to the denominator of (31), it is easy to see that for each i, $b_{is} \to 0$. On the other hand, from (30) and Lemma 2 it is clear that a_{is} is bounded. It follows that for each i, invoking (10) and Lemma 1,

$$\lambda_{is} = \frac{\Delta_i a_{is}/\text{Var}_s(\theta|i)}{\sum_{j=1}^n \Delta_j (1 - b_{js})/\text{Var}_s(\theta|j)}$$

is bounded in s. That contradicts the presumption that $\ell_s \to \infty$. □

LEMMA **11.** *Along any sequence of equilibria,* $\gamma_s^2 \text{Var}_s(u) \to 0$.

Proof. Suppose that along some subsequence of s, $\gamma_s^2\text{Var}_s(u)$ is bounded away from zero. Extract such a subsequence (but retain the index s), ensuring in addition that $\boldsymbol{\mu}_s \to \boldsymbol{\mu}$ and $\gamma_s^2\text{Var}_s(u)$ converges to a strictly positive (possibly infinite) limit in the extended reals; call it G.

For each i, $\lambda_{is} = \omega_s\Delta_i a_i/\text{Var}_s(\theta|i)$ by (10) and (8). Dividing through by ℓ_s, we see that

$$\mu_{is} = \gamma_s\Delta_i a_i/\text{Var}_s(\theta|i). \tag{32}$$

Because $\text{Var}_s(u) \to 0$ while $\gamma_s^2\text{Var}_s(u)$ is bounded away from zero, $\gamma_s^2 \to \infty$. Therefore, because the sequence $\{\boldsymbol{\mu}_{is}\}$ is bounded, $\Delta_i a_{is}/\text{Var}_s(\theta|i) \to 0$ as $m \to \infty$. Because $\text{Var}_s(\theta|i) \geq v$ for all i and s (Lemma 1), and $\Delta_i > 0$,

$$a_{is} \to 0 \text{ as } m \to \infty. \tag{33}$$

At the same time, (30) and the definition of Γ_i (see (16)) together tell us that for every i and s,

$$a_{is} = \frac{\gamma_s^2\text{Var}_s(u)\text{Cov}(\theta, i) + \Gamma_i(\boldsymbol{\mu}_s)}{[\text{Var}(\boldsymbol{\mu}_s) + \gamma_s^2\text{Var}_s(u)]\text{Var}(z_i) - \text{Cov}\,(\boldsymbol{\mu}_s, i)^2}, \tag{34}$$

Because $\boldsymbol{\mu}_s \to \boldsymbol{\mu}$, all terms involving $\boldsymbol{\mu}$ in (34) converge to some finite limit. Therefore, $G = \lim_s \gamma_s^2\text{Var}_s(u)$ must be finite as well, for if not, (34) implies that $a_{is} \to \text{Cov}(\theta, i)/\text{Var}(z_i) > 0$, which contradicts (33). Therefore $G < \infty$ and so, combining (33) and (34), we see that for each i,

$$\Gamma_i(\boldsymbol{\mu}) = -G\text{Cov}(\theta, i). \tag{35}$$

Multiplying both sides of (35) by μ_i, summing over all i, and invoking Lemma 7, we have

$$0 = \sum_i \mu_i\Gamma_i(\boldsymbol{\mu}) = -G\sum_i \mu_i\text{Cov}(\theta, i) = -G\text{Cov}(\theta, \boldsymbol{\mu}),$$

Because G is non-zero, this contradicts Lemma 7. □

LEMMA **12.** *At the limit as $s \to \infty$, $a_i = 0$ and $\Gamma_i(\boldsymbol{\mu}) = 0$ for all i, and $|\gamma_s| \to \infty$.*

Proof. Passing to the limit as $s \to \infty$ in Equation (30) and invoking Lemma 11, we see that

$$a_i = \frac{\text{Var}(\boldsymbol{\mu})\text{Cov}(\theta, i) - \text{Cov}\,(\boldsymbol{\mu}, i)\,\text{Cov}(\theta, \boldsymbol{\mu})}{\text{Var}(\boldsymbol{\mu})\text{Var}(z_i) - \text{Cov}\,(\boldsymbol{\mu}, i)^2} = \frac{\Gamma_i(\boldsymbol{\mu})}{\text{Var}(\boldsymbol{\mu})\text{Var}(z_i) - \text{Cov}\,(\boldsymbol{\mu}, i)^2}. \tag{36}$$

By Lemma 8, we know that for every s, $a_{is}\lambda_{is}$ and therefore $a_{is}\mu_{is}$ have the same sign (when non-zero) over all i, so this is also true of the limit vectors $\mathbf{a}$ and $\boldsymbol{\mu}$. But using (36),

$$\sum_i \text{Num}(\mu_i a_i) = \sum_i \mu_i\Gamma_i(\boldsymbol{\mu}) = 0,$$

so *every* individual term must be zero as well. That is, $\Gamma_i(\mu_i) = 0$ — and therefore $a_i = 0$ — when $\mu_i \neq 0$.

Now observe that along any equilibrium sequence, $\lim_s |\gamma_s| = \infty$ as claimed in the Lemma. To see this, note that $\boldsymbol{\mu} \neq 0$, so $\lambda_i \neq 0$ for some i. For that i, $\Gamma_i(\lambda_i) = 0$ as we've just shown, which means that $a_i = \lim_s a_{is} = 0$. Recalling (32),

$$\mu_{is} = \lim_s \gamma_s \sum_i \frac{\Delta_i}{\text{Var}_s(\theta|i)} a_{is} \to \mu_i \neq 0,$$

so (keeping Lemma 1 in mind) it must be that $\lim_s |\gamma_s| = \infty$.

We now complete the proof by showing that $\Gamma_i(\boldsymbol{\lambda})$ must *also* be zero when $\lambda_i = 0$. For if not, then by (36), $a_i \neq 0$. But then, using $\lim_s |\gamma_s| = \infty$, $\lim_s |\mu_{is}| = \lim_s |\gamma_s||a_{is}| = \infty$, which contradicts $||\boldsymbol{\mu}_s|| = 1$. So $\Gamma_i(\boldsymbol{\lambda})$ (and therefore, by (36), a_i) is zero for all i. □

LEMMA **13.** *As $s \to \infty$, ℓ_s has a strictly positive limit ℓ, with*

$$\ell = \frac{\text{Cov}(\theta, \boldsymbol{\mu})}{\text{Var}(\boldsymbol{\mu})}. \tag{37}$$

Proof. Combine (30) and (31) to obtain (after some elementary manipulation):

$$b_{is} = \frac{\text{Cov}(\theta, \boldsymbol{\lambda}_s) - \text{Cov}(\boldsymbol{\lambda}_s, i)a_{is}}{\text{Var}(\boldsymbol{\lambda}_s) + \omega_s^2 \text{Var}_s(u)} = \frac{1}{\ell_s}\frac{\text{Cov}(\theta, \boldsymbol{\mu}_s) - \text{Cov}(\boldsymbol{\mu}_s, i)a_{is}}{\text{Var}(\boldsymbol{\mu}_s) + \gamma_s^2 \text{Var}_s(u)},$$

which proves that

$$\ell_s b_{is} = \frac{\text{Cov}(\theta, \boldsymbol{\mu}_s) - \text{Cov}(\boldsymbol{\mu}_s, i)a_{is}}{\text{Var}(\boldsymbol{\mu}_s) + \gamma_s^2 \text{Var}_s(u)}, \tag{38}$$

By Lemma 11, $\gamma_s^2 \text{Var}_s(u) \to 0$, and by Lemma 12, $a_{is} \to 0$. Therefore, passing to the limit in (38), we see that

$$\lim_s \ell_s b_{is} = \frac{\text{Cov}(\theta, \boldsymbol{\mu}_s)}{\text{Var}(\boldsymbol{\mu}_s)}, \tag{39}$$

for all i and s. By Lemma 10, all limit points of ℓ_s are finite. Let ℓ be any such point. We claim that $\ell > 0$.

Suppose not; then $\ell_s \to 0$. Notice that $\text{Cov}(\theta, \mu) \neq 0$ by Lemma 9). If it is positive, then from (38) we see that $b_{is} \to +\infty$. By (8), $\omega_s \to -\infty$. Also, because $\text{Cov}(\theta, \mu) > 0$, so is $\text{Cov}(\theta, \mu_s)$ for s large. These last two implications contradict Lemma 8. A parallel argument holds if $\text{Cov}(\theta, \mu) < 0$. Therefore the claim is true: $\ell > 0$.

Now, by Lemma 12, $|\gamma_s| \to \infty$. Because $\ell > 0$, it follows that $|\omega_s| = |\gamma_s|/\ell_s \to \infty$ as well. By (8), it follows that

$$\sum_j \frac{\Delta_j}{\text{Var}_s(\theta|j)}[1 - b_{is}] \to 0 \tag{40}$$

as $s \to \infty$. But (39) tells us that b_{is} has a limit *independent* of i. Therefore $b_{is} \to 1$ for every i and s. Using this information in (39), we must conclude that $\ell = \text{Cov}(\theta, \boldsymbol{\mu})/\text{Var}(\boldsymbol{\mu})$, and the proof is complete. □

Proof of Theorem 2. Part (i). Lemma 12 informs us that $\Gamma_i(\boldsymbol{\mu}) = 0$ for all i. Because the limit value $\boldsymbol{\lambda}$ is just a scaling of $\boldsymbol{\mu}$, the same is true of $\boldsymbol{\lambda}$: $\Gamma_i(\boldsymbol{\lambda}) = 0$ for all i. Also,

$$\frac{\text{Cov}(\theta, \boldsymbol{\lambda})}{\text{Var}(\boldsymbol{\lambda})} = \frac{1}{\ell}\frac{\text{Cov}(\theta, \boldsymbol{\mu})}{\text{Var}(\boldsymbol{\mu})} = 1$$

by Lemma 13. By Observation 1, $\boldsymbol{\lambda}$ must equal the full-information predictor $\boldsymbol{\lambda}^*$.

Part (ii). Lemma 11 and $\inf_s \ell_s > 0$ proves that $\omega_s^2 \text{Var}_s(u) \to 0$. Passing to the limit in (13), we see that $\text{Var}_s(\theta|i) \to \text{Var}(\theta) - \text{Cov}(\theta, \boldsymbol{\lambda}^*)$ for all i, Moreover, by (40) and the discussion following it, $b_{is} \to 1$ for all i. Applying these observations to (9), we must conclude that $c_s \to -\frac{X[\text{Var}(\theta)-\text{Cov}(\theta,\boldsymbol{\lambda}^*)]}{\sum_{j=1}^{n}\Delta_j}$. The proof of the Theorem is now complete. $\square$

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