Notes on the Existence of Self-Enforcing Insurance Schemes

Garance Genicot (UC Irvine) and Debraj Ray (NYU)

1. Introduction

These notes complement Genicot and Ray [2002]. Our purpose here is to outline a conjecture regarding the conditions under which a self-enforcing insurance scheme exists. We describe a sufficient condition for such schemes to be viable, which is weaker than known sufficient conditions in the literature. Our conjecture is that this condition is also necessary.

2. Model

A community of n agents is engaged in the production and consumption of a perishable good at each date. Each agent produces a random income which takes on two values: h and ℓ with $h > \ell > 0$. A *state* s is simply a listing of all output draws by agents. Denote by $y_i(s)$ the output draw of agent i.

Let $\pi(s)$ denote the probability that state s occurs. We will assume that $\pi(s) > 0$ for every s, and that $\pi(s)$ is symmetric in the sense that if s' is obtained from s by permuting the output draws in any arbitrary way across the agents, then $\pi(s') = \pi(s)$. State realizations are i.i.d over time.

Each agent has the same utility function, assumed to be increasing, smooth and strictly concave in consumption. We thus have an instance of a classical group insurance problem. The (symmetric) Pareto optimal allocation is reached by dividing equally — and among all members of the community — the aggregate available resources at each period. Of course, there are other first-best allocations, which are asymmetric.

An important reason why first-best allocations may not be achievable is the presence of an *enforcement constraint*. This refers to the possibility that at some date, an individual who is called upon to make transfers to others in the community (as part of some reciprocity arrangement) refuses to make those transfers. The constraint is then modeled by supposing that the individual is excluded from the insurance pool, so that he must bear stochastic fluctuations on his own.

To describe matters more formally, say that a nonnegative vector of consumptions $\mathbf{c} = (c_1, \ldots, c_n)$ is *feasible* at state s if $\sum_i c_i \leq \sum_i y_i(s)$. For any date t, an t-history ζ_t is a list of all past states and consumption activity at those states.[1] An *insurance scheme* is a list of functions $\sigma = \{\sigma_s\}_{s=0}^{\infty}$ such that for all $t \geq 0$, σ_t

[1] At $t = 0$, simply use any singleton to denote the 0-history. Note that, given our interest in stable insurance schemes, we won't concern ourselves with histories in which "defaults" have taken place. As far as the present model is concerned, the story is the over.

maps the product of t-histories and current state to feasible consumption vectors for that state.

An insurance scheme σ defines *continuation values* $v_i(\sigma, \zeta_s)$ for each person i following any t-history ζ_t. These are simply the discounted expected utility of all consumptions prescribed by σ following ζ_t. We may also define *lifetime standalone values* for each individual: $v^* \equiv (1-\delta)^{-1} \sum_s \pi(s) u\left(y_i(s)\right)$. By our symmetry assumptions v^* is obviously independent of the particular index i used to define it.

An insurance scheme is *nontrivial* if there is some t-history and some state for which the prescribed consumption vector fails to equal the vector of output draws at that state.

An insurance scheme σ is *stable* if for every current state s and every t-history ζ_t, the prescribed feasible consumption vector $\mathbf{c}$ satisfies the enforcement constraint:

$$u\left(c_i\right) + \delta v_i(\sigma, \zeta_{t+1}) \geq u\left(y_i(s)\right) + \delta v^*, \tag{1}$$

for every individual i, where ζ_{t+1} is simply the $(t+1)$-history obtained by concatenating ζ_t with $(s, \mathbf{c})$.

A recent literature (Kocherlakota [1996], Ligon, Thomas and Worrall [2002], Kletzer and Wright [2000]) describes the structure of nontrivial (and efficient) stable insurance schemes, assuming these exist. However, an open question remains: can one precisely characterize the set of parameters under which such schemes exist? These notes attempt to answer that question.

3. A Sufficient Condition

We begin with notation. Let $p(j)$ denote the probability of obtaining an outcome in which precisely j individuals draw high. Next, for some given set of individuals of size i, let $p(i,j)$ denote the probability of obtaining an outcome in which j individuals draw high, *conditional* on the event that *not all* individuals in the given set draw high. By symmetry, this definition is insensitive to the particular choice of set.

Next, define $\theta \equiv [u('(\ell) - u'(h)]/u'(h)$. By the strict concavity of u and the assumption that $h > \ell$, it is obvious that $\theta > 0$. An parametric increase in h may correspond to an increased spread between high and low draws, or it may correspond to increased curvature of the utility function. For these reasons, θ may be interpreted as the "need for insurance" (see Genicot and Ray [2001]).

To complete notation, recall by the Fröbenius theorem (see, e.g., Takayama [1974]) that every strictly positive square matrix A admits a unique strictly positive eigenvalue; call this eig(A).

Theorem **1.** *A nontrivial stable insurance scheme exists if*

$$\text{eig}(P) > \frac{1}{\delta}, \tag{2}$$

where P is an $n \times n$ matrix with (i,j) entry p_{ij} given by

$$p_{ij} = p(j) + \theta p(i,j) \tag{3}$$

for $1 \leq i \leq n$ and $1 \leq j < n$, and

$$p_{in} = p(0) + p(n) \tag{4}$$

for $1 \leq i \leq n$.

4. Proof of the Theorem

4.1. **Preliminaries.** It will be convenient on to concentrate on *excess equilibrium payoffs*, which we define to be the difference between various equilibrium payoffs and the baseline quantity v^*. Sometimes we shall drop the qualifier "excess" and use the term "equilibrium payoffs" or simply "payoffs". Denote by E the set of all (excess) equilibrium payoffs. To be sure, $E \subseteq \mathbb{R}^n_+$.

Well known arguments (see, e.g., Abreu, Pearce and Stacchetti [1986]) allow us to express E as the fixed point of a suitable mapping. This viewpoint is critical to our approach, so we develop it in some detail. Let B be any symmetric[2] compact set of nonnegative excess payoffs. Say that the payoff $\mathbf{e} \geq 0$ is *supported by B* if there are payoff vectors $\mathbf{e}(s) \in B$ and transfer vectors $\mathbf{t}(s) \in \mathbb{R}^n$ (for every conceivable state s) such that for every individual i,

$$e_i = \sum_s \pi(s) \left[u\left(y_i(s) + t_i(s)\right) + \delta e_i(s) \right] - (1-\delta) v^*, \tag{5}$$

while at the same time,

$$\sum_{j=1}^{n} t_j(s) = 0 \tag{6}$$

and

$$u\left(y_i(s) + t_i(s)\right) + \delta e_i(s) \geq u\left(y_i(s)\right) \tag{7}$$

for every state s.

Notice that there is always some payoff vector that can be supported in this way: simply consider any payoff vector generated by assigning zero transfers. Denote by $\Psi(B)$ the convex hull of the set of all (excess) payoff vectors which are supported by B. Notice that $\Psi(B)$ is symmetric, because B is. Moreover, observe that if B itself is convex, then the requirement of taking the convex hull is redundant: the set of all payoffs supported by B is automatically convex. This is certainly true of the set of *equilibrium* payoffs, which standard arguments inform us to be a fixed point of Ψ. Nevertheless, it will be technically convenient to define the mapping Ψ by requiring convexification, as this permits us to consider a wider domain of support sets.

[2] A set $B \subseteq \mathbb{R}^n$ is symmetric if $x \in B$ implies every permutation of x is also in B.

Lemma **1.** *A nontrivial stable insurance scheme exists if and only if* $\Psi(B) \supseteq B$ *for some nonempty symmetric compact set* B *of excess payoffs, with* $B \neq \{\mathbf{0}\}$.

Proof. Suppose that a nontrivial stable insurance scheme exists. Set $B = E$, the set of all excess stable payoffs. Then B is symmetric and compact, and by nontriviality, $B \neq \{\mathbf{0}\}$. Moreover, $\Psi(B) = B$, so it is trivially the case that $\Psi(B) \supseteq B$.

Conversely, suppose that $\Psi(B) \supseteq B$ for some nonempty symmetric compact set $B \neq \{\mathbf{0}\}$. Note that there is some $M < \infty$ such that the infinite-horizon payoff to an individual cannot exceed M. Let $\bar{B}$ be the set $\{\mathbf{e} \in \mathbb{R}^n | 0 \leq e_i \leq M \text{ for all } i\}$. It is easy to see that B must be a subset of $\bar{B}$.[3] Define $\mathcal{C}(B)$ to be the collection of all nonempty symmetric compact sets B' of payoff vectors such that $B \subseteq B' \subseteq \bar{B}$. It is trivial to check that $\mathcal{C}(A)$ is a Moore family of subsets of $\bar{B}$ (Birkhoff [1995, p. 111]), and is therefore a complete lattice under the set-inclusion ordering (Birkhoff [1995, Chapter V, Theorem 2]). Moreover, because Ψ is isotone and $\Psi(B) \supseteq B$, it follows that Ψ maps $\mathcal{C}(B)$ to itself.[4] By the Tarski fixed point theorem (see, e.g., Birkhoff [1995, Chapter V, Theorem 11]), there is some set B' such that $\Psi(B') = B'$. Because B' must be convex, no randomization is needed, so $\Psi(B')$ is *precisely* the set of points supported by B'. It is now easy to see every point in B' must represent an equilibrium payoff. Because $B' \supseteq B \neq \{\mathbf{0}\}$, it follows that a nontrivial stable insurance scheme exists. ∎

4.2. **Main Proof.** We begin the main proof by showing that if (2) is satisfied, then there exists a nontrivial stable insurance scheme. By Lemma 1, it suffices to exhibit a nonempty compact set B of excess payoffs, with $B \neq \{\mathbf{0}\}$, such that $\Psi(B) \supseteq B$.

To this end, pick strictly positive numbers $(e^1, \ldots, e^n)$, and $\lambda \in (0, 1)$. For each index k, consider an excess payoff vector of the form $(\lambda e^k, \ldots, \lambda e^k, 0, \ldots, 0)$, where $\lambda e^k > 0$ pertains to the first k entries, and the remaining entries are zero. Now consider all permutations of this vector. Do the same for every k. Consider the resulting collection of *all* such vectors, along with the collection $(0, \ldots, 0)$. This is a symmetric set of excess payoff vectors which depends on both $(e^1, \ldots, e^n)$ and λ. It will be convenient to explicitly track all dependence on λ so we denote this set of excess payoffs by B^λ.

Consider, for any index k, the problem of finding the largest symmetric excess payoff for some given set of k individuals that can be supported by B^λ. For ease in discussing the problem, we call the individuals in this given set *masters*, and the remainder the *slaves*.

[3] If this is not true, then there is some $e \in B$ with $e_i > M$ for some i. But then, the definition of Ψ will not permit $\Psi(B) \supseteq B$.

[4] If $B' \in \mathcal{C}(B)$, then $\bar{B} \supseteq \Psi(\bar{B}) \supseteq \Psi(B') \supseteq \Psi(B) \supseteq B$. Also, if B is compact and symmetric, so is the set of all payoff vectors supported by B, and consequently so is the convex hull of this set. Therefore $\Psi(B') \in \mathcal{C}(B)$.

This problem can be solved "state by state".[5] We shall employ a particular feasible solution to this subproblem. [Later, we show why this particular feasible solution gives us the exact characterization that we need.]

Consider any state s. Denote by $a(s)$ the number of masters with high draws, and by $b(s)$ the number of slaves with high draws. Let $j(s) \equiv a(s) + b(s)$ be the total number of high draws. Use the following rules:

[I] If $j(s) = 0$, then choose as continuation payoff λe^n for *every* individual. Define $r_1(\lambda)$ by

$$u\left(\ell - r_1(\lambda)\right) + \delta\lambda e^n \equiv u(\ell), \tag{8}$$

and require all slaves (who are all low) to make this transfer.[6] The entire transfer is divided equally among the k masters (who are all low too), so that each master receives

$$t_1(\lambda) \equiv \frac{(n-k)r_1(\lambda)}{k}. \tag{9}$$

[II] If $j(s) > 0$ and $a(s) < k$, then choose as continuation payoff $\lambda e^{j(s)}$ for all the high draws, and 0 for all the low draws. In addition, make no transfer to (nor demand any from) any low slaves. Define $r_2(\lambda)$ by

$$u\left(h - r_2(\lambda)\right) + \delta\lambda e^{j(s)} \equiv u(h), \tag{10}$$

and require all high drawers to make this transfer. The entire transfer is divided equally among the $k - a(s)$ low masters, so that each low master receives

$$t_2(\lambda) \equiv \frac{j(s)r_2(\lambda)}{k - a(s)}. \tag{11}$$

[III] If $j(s) > 0$ and $a(s) = k$, then choose as continuation payoff $\lambda e^{j(s)}$ for all the high draws, and 0 for all the low draws. In addition, make no transfer to (nor demand any from) any low slave. Recall the definition of $r_2(\lambda)$ from (10) and require all high slaves (if there are any) to make this transfer. Divide the entire transfer equally among the k (high) masters, so that each master receives

$$t_3(\lambda) \equiv \frac{b(s)r_2(\lambda)}{a(s)} = \frac{b(s)r_2(\lambda)}{k}. \tag{12}$$

With this description in mind, we can write down the expected payoff (call it $\hat{e}^k(\lambda)$) to a master. The easiest way to do this is to write the average payoff of all

[5]The reason is that when continuation payoffs for each player is only restricted to be nonnegative, the enforcement constraints imply the participation constraints. This means that there are no ex-ante constraints.

[6]Of course, if $k = n$, there may not be any low slaves, but this is automatically handled here (and in what follows later) by the notation.

the masters in every state, weight by the probability of that state, and then add over all states. Doing so, we see that

$$\begin{aligned}\hat{e}^k(\lambda) &= \pi(s_\ell)\left[u\left(\ell+t_1(\lambda)\right)+\delta\lambda e^n\right]+\sum_{\substack{s:j(s)>0\\a(s)<k}}\pi(s)\left[\frac{a(s)}{k}u(h)+\frac{k-a(s)}{k}u\left(\ell+t_2(\lambda)\right)\right]\\&+\sum_{\substack{s:j(s)>0\\a(s)=k}}\pi(s)\left[u\left(h+t_3(\lambda)\right)+\delta\lambda e^{k+b(s)}\right],\end{aligned}$$

where s_ℓ denotes the state in which all draws are low.

Now suppose that we can show that for some $\lambda > 0$ and some choice of $(e^1, \ldots, e^n) \gg 0$,

$$\hat{e}^k(\lambda) \geq \lambda e^k \tag{13}$$

for every $k = 1, \ldots, n$. Then it must be the case that $\Psi(B^\lambda) \supseteq B^\lambda$. To see this, observe that the zero vector $\mathbf{0}$ is supported by B^λ, and so — by construction — are all vectors of the form $\left(\hat{e}^k(\lambda), \ldots, \hat{e}^k(\lambda), 0, \ldots, 0\right)$ (in which $\hat{e}^k(\lambda)$ appears in the first k entries and 0 thereafter), and permutations thereof. Now take convex combinations of all these and recall (13) to complete the claim.

So it remains to establish (13). Observe that both sides of this inequality go to zero as $\lambda \to 0$, so a sufficient condition for the desired result is

$$\frac{d\hat{e}^k(\lambda)}{d\lambda}_{|\lambda=0} > e^k \tag{14}$$

for all k. Differentiation of the expression $\hat{e}^k(\lambda)$ with respect to λ involves evaluation of the derivatives of t_1, t_2 and t_3 with respect to λ. Carrying out these computations and making appropriate substitutions, we see that (14) can be rewritten as

$$\pi(s_\ell)M^n + \sum_{\substack{s:j(s)>0\\a(s)<k}}\pi(s)M^{j(s)}\frac{u'(\ell)}{u'(h)} + \sum_{\substack{s:j(s)>0\\a(s)=k}}\pi(s)M^{j(s)} > \frac{M^k}{\delta}, \tag{15}$$

where $M^k \equiv ke^k$ for each k.

Further simplification of this inequality is possible. Recall that $p(j)$ stands for the overall probability of j high draws, and that $p(k, j)$ denotes the probability of j high draws conditional on the k masters *not all* drawing high simultaneously. Then (15) may be rewritten as

$$\sum_{j=1}^{n-1}[p(j)+\theta p(k,j)]M^j + [p(0)+p(n)]M^n > \frac{M^k}{\delta}$$

for every k, for some vector $(M^1, \ldots, M^n)$. It is easy to see that this condition is equivalent to (2), and the proof is complete.

5. Conjecture

It appears that (2) is weaker than known sufficient conditions in the literature. For instance, it can be checked that (2) is implied by condition (5) in Proposition 1 of Genicot and Ray [2002].

Indeed, our conjecture is that (2) is not just sufficient, it is *necessary*. We have not been able to establish this result generally, though we have done so when $n = 2$.

6. References

(1) Genicot, G. and D. Ray (2002), "Group Formation in Risk-Sharing Arrangements," forthcoming, *Review of Economic Studies*.

(2) Kletzer, K. and B. Wright (2000), "Sovereign Debt as Intertemporal Barter," *American Economic Review* **90**, 621–639.

(3) Kocherlakota, N. (1996), "Implications of Efficient Risk Sharing without Commitment," *Review of Economic Studies* **63**, 595–609.

(4) Ligon, E., Thomas, J. and T. Worrall (2002), "Mutual Insurance and Limited Commitment: Theory and Evidence in Village Economies," *Review of Economic Studies* **69** 115–139.