Online Appendix for

ASPIRATIONS AND INEQUALITY

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This Appendix provides supplementary material to accompany the main text. Appendix A studies alternative specifications of milestone utility. Appendix B concerns stationary and steady states in the model with bounded income. It characterizes stationary states for the model with bounded income. It shows by example that stationary states may not always exist and proves the existence of the weaker notion of a steady state. It contains an extension to technological progress in the model with bounded income. Appendix C provides some comparative statics results in the constant elasticity growth model. Appendix D proves Observations 2 and 3 of Section 5.4.1 in the main text. Appendix E discusses minimal monotonicity and connects this idea to upward-looking aspirations. Throughout, we number equations, figures, propositions and lemmas using an "x" prefix, to avoid possible confusion with references to similar objects in the main text.

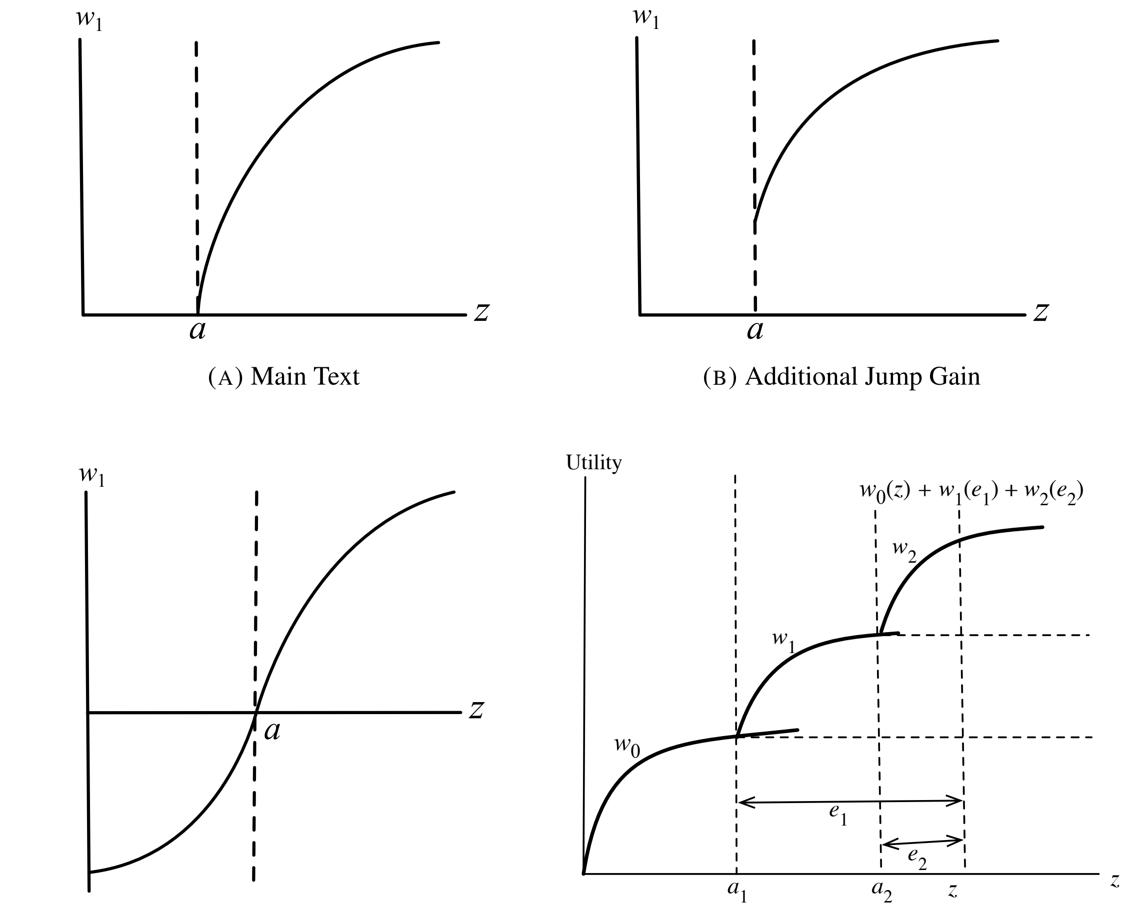
APPENDIX A. ALTERNATIVE MILESTONE UTILITIES

In the main text, the aspiration utility is given by $w_1(e)$, illustrated in Figure x1a, where $e = \max\{z - a, 0\}$ is the excess of their child's wealth z over the aspiration a of the parent and w_1 is increasing, smooth and strictly concave. Our results are robust to an alternative utility term, illustrated in Figure x1b, in which the crossing of the threshold also engenders a jump in utility; i.e., w_1 is strictly increasing with $w_1(0) > 0$.

We could also assume a more general w_1-function that incorporates additional disutility in departing downwards from a, as illustrated in Figure x1c (and used in Genicot and Ray 2009). We work with such a form below to deliver a richer version of Proposition 2 in the main text.

A final alternative, illustrated in Figure x1d, would be to define utilities around multiple "milestones" and interpret those thresholds as an *aspiration vector*. The crossing of each milestone is "celebrated" by an extra payoff. These "add-on" payoff functions are defined on the extent to which outcomes exceed the milestone in question, and are exogenous. But the social environment determines the milestones, and consequently individual incentives to invest and bequeath. Even if this aspiration vector were to be common to all in society, heterogeneity in wealth would play an important role as higher thresholds would become more relevant as an individual moves further up the income scale.

For the rest of this section, we return to the specification in Figure x1c. Consider an aspirational utility function of the form $w_1(z, a) = s(a)w(z/a)$ where w picks up a purely relative component and s picks up scale effects. Assume (a) w is smooth and increasing with $w' > 0$, (b)



(A) Main Text (B) Additional Jump Gain (C) S-Shaped (D) Multiple Thresholds

FIGURE X1. ASPIRATIONS AND PAYOFFS

$w''(x) > 0$ for $x < 1$ and $w''(x) < 0$ for $x > 1$, (c) $s(a)$ is smooth, with $s(a) > 0$ for all $a > 0$ and $s(a)/a$ nonincreasing.

Assumption (b) imparts a S-shaped form to aspirational utility, as shown in Figure x1c. Assumption (c) allows for scale effects but does not insist on them ($s(a)$ is permitted to be a constant or even decline in a). What is important is that there be a restriction on how quickly utility can increase in the scale term, which is captured by the requirement that $s(a)/a$ is weakly decreasing.

Notice that, in contrast to the specification used in our paper, the choice of continuation wealth z is no longer insensitive to a for the frustrated (unlike Proposition 2 in the main text). The first order condition that guides the choice of continuation wealth is given by

$$w_0'(z) + \frac{s(a)}{a} w(z/a) = u'(y - k(z)) / f'(k(z)). \tag{x1}$$

As in the main text, by the concavity of w to the right of a, there can be at most one solution to (x1) that exceeds a. However, owing to the convexity of w to the left of a there could be

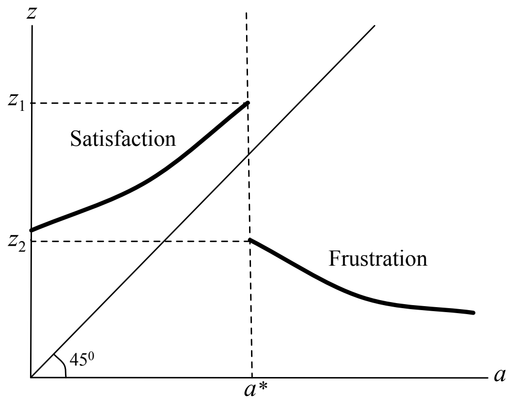


FIGURE X2. SATISFACTION AND FRUSTRATION AS ASPIRATIONS CHANGE.

several solutions to the first order condition that involve frustration. Finding an optimal solution involves comparing payoffs over all the continuation incomes for which (x1) holds.

Inequality (4) from the main text, which we continue to assume, guarantees a solution that strictly exceeds aspirations when aspirations close to zero, and so aspirations are satisfied when they are low. As aspirations continue to rise, there comes a threshold when the solution makes a switch from satisfaction to frustration; typically, this switch will arrive with a discontinuous fall in investment, as in the main text.

But higher aspirations can do more than switch individuals from satisfaction to frustration. Once in the "frustration zone," economic growth is actually *lowered* by an increase in a: higher aspirations encourage less investment. This is because the function becomes *flatter* to the left of a. It follows that every candidate for an optimal solution already below a must decrease still further. These findings are formalized in Proposition x1 (which is the counterpart of Proposition 2 in the main text) and illustrated in Figure x2.

Proposition x1. *For given y, there is a unique threshold value of aspirations below which aspirations are satisfied, and above which they are frustrated. Once aspirations are frustrated, chosen wealth declines as aspirations continue to grow.*

Proof. By (4), if aspirations are small enough, then aspirations must be satisfied. Because y is fixed, aspirations must be frustrated once a is high enough. So there is certainly a threshold at which a changeover occurs from satisfaction to frustration. Below, we shall prove that such a threshold must be unique. To go further, we employ the following lemma:

Lemma x1. *Consider any selection from the optimality correspondence that links a to an optimal choice z. Then that mapping cannot exhibit a discontinuous upward jump.*

Proof. Suppose that z_1 and z_2 are both optimal choices at a, with $z_2 > z_1$. We claim that

$$(z_2/a)w'(z_2/a) - w(z_2/a) > (z_1/a)w'(z_1/a) - w(z_1/a) \tag{x2}$$

To prove this, recall the first-order condition (x1) for z_1 and z_2:

$$w_0'(z_i) + \frac{s(a)}{a} w'(z_i/a) = u'(y - k(z_i))\, k'(z_i) = u'(c_i)/f'(y - c_i) \tag{x3}$$

for $i = 1, 2$, where c_i is consumption under z_i. At the same time, by the joint optimality of z_1 and z_2, $u(c_1) + w_0(z_1) + s(a)w(z_1/a) = u(c_2) + w_0(z_2) + s(a)w(z_2/a)$, or equivalently,

$$s(a)w(z_2/a) - s(a)w(z_1/a) = [u(c_1) + w_0(z_1)] - [u(c_2) + w_0(z_2)]. \tag{x4}$$

Multiplying both sides of (x3) by $z_i = f(y_i - c_i)$, combining the result with (x4), and defining $\Delta_i \equiv (z_i/a)w'(z_i/a) - w(z_i/a)$ for $i = 1, 2$, we see that

$$\begin{aligned} s(a)(\Delta_2 - \Delta_1) &= \left[\frac{u'(c_2)f(y_2 - c_2)}{f'(y - c_2)} + u(c_2)\right] - \left[\frac{u'(c_1)f(y_1 - c_1)}{f'(y - c_1)} + u(c_1)\right] \\ &- [z_2 w_0'(z_2) - w_0(z_2)] + [z_1 w_0'(z_1) - w_0(z_1)]. \end{aligned} \tag{x5}$$

Simple differentiation plus the strict concavity of u, w_0 (and the concavity of f) show that $zw_0'(z) - w_0(z)$ is decreasing in z while $[u'(c)f(y - c)/f'(y - c)] + u(c)$ is decreasing in c (for given y). Using this information in (x5) along with the fact that $z_2 > z_1$ and $c_2 < c_1$, we establish (x2), as desired.

Let $z(a)$ be any selection from the optimality correspondence. Suppose, contrary to the lemma, that it jumps up at a. By the upperhemicontinuity of optimal choices, that implies (i) there are z_1^* and z_2^* with $z_2^* > z_1^*$, both optimal at a, (ii) z_1^* is a limit point of optimal choices $z(a')$ for $a' < a$, and (iii) z_2^* is a limit point of optimal choices $z(a')$ for $a' > a$. Note that (x2) holds with $z_i = z_i^*$ for $i = 1, 2$, so that transposing terms,

$$(z_2^*/a)w'(z_2^*/a) - (z_1^*/a)w'(z_1^*/a) > w(z_2^*/a) - w(z_1^*/a).$$

Because $z^* > z_1^*$ and w is increasing, both terms in the inequality above are positive. So, because $s(a) \geq as'(a)$ by Assumption (c),[1] we can conclude that

$$\frac{s(a)}{a}\left[(z_2^*/a)w'(z_2^*/a) - (z_1^*/a)w'(z_1^*/a)\right] > s'(a)\left[w(z_2^*/a) - w(z_1^*/a)\right],$$

so that transposing terms again,

$$s(a)(z_2^*/a^2)w'(z_2^*/a) - s'(a)w(z_2^*/a) > s(a)(z_1^*/a^2)w'(z_1^*/a) - s'(a)w(z_1^*/a). \tag{x6}$$

With (x6) in mind, we can pick $a_1 < a$ and $a_2 > a$ (but close enough) along with z_i optimal for a_i and close enough to z_i^* for $i = 1, 2$, such that

$$s(\eta_2)\frac{z_2}{\eta_2^2}w'(z_2/\eta_2) - s'(\eta_2)w(z_2/\eta_2) > s(\eta_1)\frac{z_1}{\eta_1^2}w'(z_1/\eta_1) - s'(\eta_1)w(z_1/\eta_1) \tag{x7}$$

for every η_1 and η_2 in the interval $[a_1, a_2]$.

Viewing $s(a)w(z_i/a)$ as a function of a, and applying the mean-value theorem,

$$s(a_1)w(z_i/a_1) - s(a_2)w(z_i/a_2) = (a_2 - a_1)\left[s(\eta_i)z_i/\eta_i^2 w'(z_i/\eta_i) - s'(\eta_i)w(z_i/\eta_i)\right] \tag{x8}$$

[1] To see this, simply differentiate $s(a)/a$ with respect to a and use assumption (c) on s(a).

for $i = 1, 2$, where η_1 and η_2 are the points in $[a_1, a_2]$ where the relevant mean values are attained. Combining (x7) and (x8), it follows that

(x9) $$s(a_1)w(z_2/a_1) - s(a_2)w(z_2/a_2) > s(a_1)w(z_1/a_1) - s(a_2)w(z_1/a_2).$$

Now, z_2 is an optimal choice at a_2, so in particular we have

$$u(c_2) + w_0(z_2) + w(z_2/a_2) \geq u(c_1) + w_0(z_1) + w(z_1/a_2),$$

where c_1 and c_2 are the levels of consumption corresponding to the choices z_1 and z_2. Applying (x9) to this inequality, we must conclude that

$$u(c_2) + w_0(z_2) + w(z_2/a_1) > u(c_1) + w_0(z_1) + w(z_1/a_1),$$

but this contradicts the fact that z_1 is an optimal choice at a_1. ■

Let's now return to the main proof, and suppose that aspirations are frustrated at a_1: z_1 is an optimal choice with $z_1 < a_1$. Consider an increase from a_1 to a_2, with z_2 optimal at a_2. Then

$$u(c_1) + w_0(z_1) + w_1(z_1, a_1) \geq u(c_2) + w_0(z_2) + w_1(z_2, a_1),$$

where c_1 and c_2 are the levels of consumption corresponding to the choices z_1 and z_2, and likewise

$$u(c_2) + w_0(z_2) + w_1(z_2, a_2) \geq u(c_1) + w_0(z_1) + w_1(z_1, a_2).$$

Adding both these inequalities and transposing terms, we must conclude that

(x10) $$w_1(z_1, a_1) - w_1(z_2, a_1) \geq w_1(z_1, a_2) - w_1(z_2, a_2).$$

For a small increase in aspirations from a_1 to a_2, Lemma x1 implies that $\max\{z_1, z_2\} < a_1 < a_2$ for any optimal choice z_2 at a_2. But over this zone, the cross partial derivative

$$\frac{\partial^2 w_1(z, a)}{\partial z \partial a}$$

is strictly negative (for details, see this footnote).[2] It follows from (x10) that z_1 must be no smaller than z_2. Moreover, the first order condition

$$w_0'(z_1) + \frac{s(a_1)}{a_1} w'(z_1/a_1) = u'(y - k(z_1))\, k'(z_1)$$

can no longer hold when a_1 increases to a_2, so z_1 is actually *strictly* larger than z_2.

This argument can obviously be extended to any change in aspirations, small or not, as long as aspirations are frustrated to begin with.

The above argument, coupled with Lemma x1, also proves that the critical threshold of movement from satisfaction to frustration is unique. For once aspirations are frustrated, they can never be satisfied at higher levels of aspirations. ■

[2] We have $w_1(z, a) = s(a)w(z/a)$. Differentiating with respect to z, we see that $\partial w_1(z, a)/\partial z = \phi(a)w'(z/a)$, where $\phi(a) \equiv s(a)/a$. Differentiating the result with respect to a, we see that $\partial^2 w_1(z, a)/\partial z \partial a = \phi'(a)w'(z/a) - \phi(a)w''(z/a)z/a^2$. We have $\phi'(a) \leq 0$ by assumption c) on w_1, while $w''(z/a) > 0$ by $z < a$ and assumption b) on w_1. Therefore $\partial^2 w_1(z, a)/\partial z \partial a < 0$, as claimed.

APPENDIX B. STEADY STATES AND STATIONARY STATES WITH BOUNDED INCOMES

B.1. **Characterization of Stationary States.** Recall the definitions from the main text: a *stationary state* is a distribution on positive wealths such that each dynasty replicates its starting wealth generation after generation. A *steady state* is a distribution that replicates itself period after period. Under the latter definition, dynasties might "cross paths," generating persistent mobility but with an unchanging distribution.

A natural setting for steady or stationary states is one in which wealths are bounded, as in the Solow model. It is implied by the end-point restriction: $f(x) < x$ for all x large enough. Proposition 6 in the main text tells us that if aspirations are range-bound, scale-invariant and socially monotone, then a stationary state is concentrated on just two positive values of incomes.

Let us examine what such a stationary state — call it F^* — would look like. Let's say that there are two incomes, y_ℓ and a higher level y_h, and p is the proportion of the population located at y_ℓ. For each group $i = \ell, h$, a_i is then given by

$$a_i = \Psi(y_i, F^*). \tag{x11}$$

The proof of Proposition 6 tells us that both these aspirations cannot be satisfied. Both can't be frustrated either, because the corresponding income for failed aspirations is determined *uniquely*, given condition (4) in the main text. So a_ℓ must be a failed aspiration and a_h a satisfied aspiration. In particular, y_ℓ is fully pinned down by

$$d(y_\ell) = 0, \tag{x12}$$

for which the solution is unique, as just discussed. On the other hand, because a_h is satisfied, y_h is determined by

$$d(y_h) + w_1'(y_h - \Psi(y_h, F^*)) = 0. \tag{x13}$$

This generates four equations for five unknowns that need to be satisfied in a stationary state $(y_\ell, y_h, p, a_\ell, a_h)$, but in part the extra degree of freedom will be used up in guaranteeing that we can find configurations that are compatible with the failure of aspirations at y_ℓ and the satisfaction of aspirations at y_h. Sometimes this will work, as in the case of Example 1 in the main text, where there is a continuum of stationary states. But sometimes a stationary state may fail to exist.

B.2. **Possible Nonexistence of a Stationary State.** To prove this last assertion, fix any utility function (u, w_0, w_1) satisfying our assumptions. Observe first that y_ℓ is uniquely given by (x12). Next, note that a_ℓ must be bounded below by $\underline{a}$, defined as the lowest aspiration for which an individual at y_ℓ is just indifferent between her lower choice $z = y_\ell$ and some upper choice $z' > \underline{a}$. In turn, that places a lower bound on the size of y_h. At that value, it is possible that $f(y_h) - y_h$ is low enough so that y_h is not worth maintaining for any aspiration a_h. That proves that a stationary state may not always exist.

The problem is that as a parent, I obtain my payoff from the value of the child's wealth, whereas the child does not value her own wealth directly; only the consumption she obtains from it. If there is enough curvature in the production function, the implied level of consumption that maintains wealth may be too low. This isn't to assert that non-existence is the rule rather than the exception, but only to caution that non-existence is a possibility.

B.3. **Existence of Steady States.** Faced with this possible nonexistence of a stationary state, one must then retreat to the weaker definition of a steady state, which is an invariant equilibrium distribution F^*. Despite the lack of stochastic shocks, there could be mobility within the distribution, so that the stationarity of the outcome only happens at the aggregate and not at the individual level.

In what follows we will assume that Ψ, the aspirations formation function, is additionally continuous in F, in the topology of weak convergence on distributions. This assumption is not entirely innocuous: for instance, in the case of upper mean aspirations, it is not automatically satisfied.[3] However, a little smoothing restores that continuity. For example, suppose that the edges of the cognitive window are not sharply defined but are delineated by a continuous weighting function that drops to zero. Then the resulting aspirations function will indeed be continuous in the weak topology on distributions.

Proposition x2. *If the aspirations formation function is continuous in* (y, F)*, then there exists a steady state.*

Proposition x2 is a corollary of the more general Proposition x3, which we now state and prove.

Fix some compact interval $[0, Y]$ of real numbers, with $Y > 0$. Let $\mathcal{M}$ be the space of all probability measures μ defined on $[0, Y]$, equipped with the weak convergence topology.

For each μ and each $y \in [0, Y]$, let $\Phi(y, \mu)$ be the set of "choices" made by y. We assume:

(i) $\Phi(y, \mu)$ is nonempty and takes values in $[0, Y]$.

(ii) Φ is uhc in (y, μ).

A transition probability p defined on $[0, Y]$ *agrees with* Φ and μ if for every B, and for μ-a.e. y, $p(y, B) > 0$ only if $B \cap \Phi(y, \mu) \neq \emptyset$.

Proposition x3. *Under Assumptions* (i) *and* (ii) *on* Φ*, there exists a probability measure* μ^* *on* $[0, Y]$ *and a transition probability* $p^*(y, .)$ *that agrees with* Φ *and* μ^* *such that*

$$\mu^*(B) = \int_{[0,Y]} p^*(y, B)\mu^*(dy), \tag{x14}$$

for every Borel set B.

Before proving the proposition, we make some remarks. First, Proposition x2 is a near-immediate consequence. If the aspirations function $a = \Psi(y, F)$ is continuous in (y, F), then the policy correspondence

$$\Phi(y, F) \equiv \arg\max_z \left[u\left(y_t - k(z)\right) + w_0\left(z\right) + w_1\left(\max\{z - a, 0\}\right)\right]$$

is nonempty-valued and uhc on $[0, Y]$, so that Assumptions (i) and (ii) are satisfied. Therefore an invariant measure μ^* and a transition probability p^* satisfying (x14) exist. Because p^* agrees

[3] Fix y, and think of a sequence of distributions in which a mass point approaches the boundary of the cognitive window from below. In the limit, the upper conditional expectation will move discontinuously.

with Φ and μ^*, it only picks out optimal continuation choices. The cdf F^* corresponding to μ^* is therefore a steady state.

Second, Proposition x3 is conceivably of independent interest as it applies to a variety of situations in which the ambient distribution of types influences individual choice, not just aspirations. There is also ample scope to expand the proposition to a larger domain, not just some compact subset $[0, Y]$ of the reals.[4]

Proof of Proposition x3. Let $\mathcal{R}$ be the space of all probability measures ρ on the product space $[0, Y]^2$. For any $\rho \in \mathcal{R}$, let $m_1(\rho)$ and $m_2(\rho)$ stand for the induced marginals on each dimension of $[0, Y]^2$.

Let $G(\mu)$ stand for the graph of Φ in $[0, Y]^2$; that is,

$$G(\mu) = \{(y, z) \in [0, Y]^2 | z \in \Phi(y, \mu)\}.$$

For any $\mu \in \mathcal{M}$, define

$$\Gamma(\mu) = \{\rho \in \mathcal{R} | \rho(G(\mu)) = 1 \text{ and } m_1(\rho) = \mu\},$$

and

$$H(\mu) = m_2(\Gamma(\mu)) = \{\nu \in \mathcal{M} | \nu = m_2(\rho) \text{ for some } \rho \in \Gamma(\mu)\}.$$

Lemma x2. Γ *is nonempty- and convex-valued, and is upper-hemicontinuous (uhc).*

Proof. To show that Γ is nonempty-valued, consider the upper-semicontinuous selection $\phi(y, \mu) \equiv \max \Phi(y, \mu)$. This is easily seen to be a deterministic transition probability,[5] so that $\bar{\rho}$ defined by

$$\bar{\rho}(A) \equiv \mu\{y : (y, \phi(y, \mu)) \in A\}$$

for all Borel $A \subseteq [0, Y]^2$ belongs to $\Gamma(\mu)$. Convex-valuedness is trivial: the convex combination of two joint probabilities sharing the same marginal must have that marginal, and if each places full weight on some common set, their convex combination must do so as well.

For uhc, let μ^n be a sequence in $\mathcal{M}$ with $\mu^n \Rightarrow \mu$, and suppose that for some sequence $\rho^n \in \Gamma(\mu^n)$, $\rho^n \Rightarrow \rho \in \mathcal{R}$. Because marginals are continuous in weak convergence (see, e.g., Billingsley 1999, p. 23), it must be that $m_1(\rho) = \mu$.

For any $\epsilon > 0$, define G_ϵ to be the set of all points $(y, z) \in [0, Y]^2$ such that $d((y, z), G(\mu)) \leq \epsilon$, where d is the infimum of Euclidean distance between (y, z) and points $(y', z') \in G(\mu)$. Because Φ is uhc and $[0, Y]^2$ is compact, it is easy to see that there exists $N(\epsilon)$ such that for all $n \geq N(\epsilon)$, $G(\mu^n) \subseteq G_\epsilon$. Because G_ϵ is closed, we know from the Portmanteau theorem for weak convergence (see. e.g., Billingsley 1999, Theorem 2.1, part iii) that

$$1 = \limsup_n \rho^n(G(\mu^n)) \leq \limsup_n \rho^n(G_\epsilon) \leq \rho(G_\epsilon);$$

or in other words, $\rho(G_\epsilon) = 1$. Because $\cap_\epsilon G_\epsilon = G(\mu)$ and all these sets are closed, it follows that $\rho(G(\mu)) = 1$. Therefore $\rho \in \Gamma(\mu)$, so Γ is uhc. ■

[4]We are very grateful to Andy McLennan who suggested the line of proof of this proposition. A similar approach — though for a result that does not apply to the case at hand — is also to be found in Duffie et al (1994).

[5]An upper-semicontinous function is certainly measurable, and therefore satisfies all the conditions for it to be a transition probability; see, e.g., Remark 2 in Neveu (1965, p. 74).

Again, by the continuity of marginals, H inherits all the properties of Lemma x2 from Γ, so it is also nonempty-valued, convex-valued, uhc. By the Fan-Glicksberg fixed point theorem, there exists (μ^*, ρ^*) such that $\rho^* \in \Gamma(\mu^*)$ and $\mu^* = m_2(\rho^*)$.

By the so-called disintegration theorem which can be proved at varying levels of generality (Parthasarathy 1967, p. 147, Theorem 8.1, can be easily adapted for our purposes), there exists a transition probability $p^*(y, .)$ such that for every product set $A \times B \subseteq [0, Y]^2$,

$$\rho^*(A \times B) = \int_A p^*(y, B)\mu^*(dy). \tag{x15}$$

Notice that $\rho^*(G(\mu^*)) = 1$, so that p^* must agree with μ^*.[6] Moreover, because $m_2(\rho^*) = \mu^*$, it follows that for every set B, $\mu^*(B) = \rho^*([0, Y] \times B)$. Applying this to (x15), we see that

$$\mu^*(B) = \int_{[0,Y]} p^*(y, B)\mu^*(dy),$$

which establishes (x14) and completes the proof. ■

B.4. **Bounded Incomes: Exogenous Technological Progress.** In this section, we introduce exogenous technological progress in the model with bounded income. Assume that a unit measure of individuals supplies one unit of labor inelastically and at zero disutility cost. Society has access to an aggregate production function $Y = G(K, L)$ where K is capital and L is effective labor. Effective labor is the product of the labor supply 1 and a labour enhancing technology that grows exogenously at a factor $\eta > 1$. We can therefore write

$$L_t = \eta^t \text{ for all } t \geq 0.$$

Assume that G is strictly increasing, strictly concave in each input and exhibits constant returns to scale. Define capital per effective unit of labor by $k_t \equiv K_t/L_t$ and income per effective unit of labor by $y_t = Y_t/E_t$. By constant returns to scale, these are connected by a "per-capita production function" $y_t = f(k_t) = G(k_t, 1)$ that is strictly increasing and strictly concave. It is well known that this device of exogenous technological change is often used to allow for growth in the Solow model. As in that model, we impose the assumption that $f(k) < k$ for all k large enough, and recall that while this bounds wealth per unit of effective labor, wealth per-capita is fully capable of growth without bound.

Given the possibility of unbounded exponential growth, it will be useful to assume (as with linear production) that utilities are constant-elasticity, with the same elasticity for each utility indicator:

$$u(c) = c^{1-\sigma}, w_0(z) = \delta z^{1-\sigma}, \text{ and } w_1(e) = \delta\pi e^{1-\sigma}, \tag{x16}$$

where $\sigma \in (0, 1)$, $\delta > 0$ is a measure of discounting and $\pi > 0$ is a measure of the importance of milestone utility relative to intrinsic utility.

An individual with starting wealth Y_t and aspirations A_t chooses K_{t+1} to maximize

$$(Y_t - K_{t+1})^{1-\sigma} + \delta\left[G(K_{t+1}, L_{t+1})^{1-\sigma} + \pi\left(\max\{g(K_{t+1}, L_{t+1}) - A_t, 0\}\right)^{1-\sigma}\right]. \tag{x17}$$

[6] Details: Suppose on the contrary that for some set B, $p(y, B) > 0$ for y in some set A with $\mu^*(A) > 0$, but at the same time, $B \cap \Phi(y, \mu^*) = 0$ for each such y. Then it should be obvious that $(A \times B) \cap G(\mu^*)$ is empty, but (x15) informs us that $\rho^*(A \times B) > 0$. That contradicts $\rho^*(G(\mu^*)) = 1$.

Dividing throughout by L_{t+1}, we see that the maximization in (x17) is equivalent to choosing y_{t+1} that maximizes

(x18) $$\left[\frac{y_t}{\eta} - f^{-1}(y_{t+1})\right]^{1-\sigma} + \delta\left[y_{t+1}^{1-\sigma} + \pi\left(\max\{y_{t+1} - a_t, 0\}\right)^{1-\sigma}\right]$$

where $a_t \equiv A_t / L_{t+1}$.

The central question here is: with *systematic* technical change, how might A_t be determined? If the rate of technical progress is used as a normalization factor for aspirations, we could have (for instance):

$$A_t = \eta\Psi(Y_t, F_t),$$

which is related to (though not the same as) the "future-based aspirations" variation in Genicot and Ray (2009), where aspirations are adjusted to take account of the future distribution of income.

With aspirations normalized in this way to account for trends in technical progress, Proposition 5 in the main text survives unscathed, and *equality is impossible in steady state*.

On the other hand, if aspirations are entirely unnormalized — that is, if they fail to take account of the fact that the next generation will be, on average, richer, then perfect equality is possible, just as it is in Proposition 7. By the assumption that Ψ is range bound, it must be that under perfect equality, $A_t = Y_t$, so that $a_t = Y_t / L_{t+1} = y_t/\eta$ for every t. In this case, it can be shown that if η is sufficiently large, it is possible to find $y^* > 0$ such that faced with the maximization problem

$$\max_z \left(\frac{y^*}{\eta} - f^{-1}(z)\right)^{1-\sigma} + \delta\left[z^{1-\sigma} + \pi\left(z - \frac{y^*}{\eta}, 0\}\right)^{1-\sigma}\right],$$

the individual at income y^* will choose the continuation $z = y^*$. But then y^* must be *a steady state with perfect equality*.

APPENDIX C. COMPARATIVE STATICS IN THE CONSTANT ELASTICITY GROWTH MODEL

In addition to the effect of a more equal initial distribution discussed in the main text, we may be interested in other factors that "affect the chances" of convergence to equality. One way to formalize this is to say that some parametric change makes convergence to equality *more likely* if for any initial distribution F_0 with convergence to perfect equality before the change, that convergence is unaffected, and for some distributions F_0 with bipolar divergence before the change, convergence to perfect equality occurs after the change. One such parameter is the extent of social monotonicity, and to vary it, we consider aspirations to be a weighted average of one's own income y and a common term $\psi(F)$ that lies in the range of the distribution F:

(x19) $$\Psi(y, F) = \gamma y + (1-\gamma)\psi(F) \text{ for } \gamma \in [0, 1],$$

so that a higher γ means we have aspirations that are less sensitive to social changes.

Proposition x4. *Consider the constant-elasticity growth model. Assume that aspirations are range-bound, scale-invariant and socially monotone. Everything else remaining the same, the set of initial distributions on a compact support for which convergence to perfect equality ensues* *expands with (i) a higher rate of return (ρ), (ii) a larger weight on aspirational utility (π), (iii) lower aspirations (that is, a decrease in $\Psi(y, F)$ for all (y, F)), and (iv) aspirations that are less sensitive to social outcomes (higher γ in* (x19)*).*

Proof. Proposition 3 showed that whether an individual with income y is frustrated or satisfied at time 0 depends on whether that individual's aspirations ratio $r_0(y)$ is above or below the threshold $r^* > 1$ (identified in the statement of that proposition). It is obvious from the maximization problem in (10) that a higher rate of return ρ or a higher π increases the threshold r^* that induces frustration, thereby increasing the set of initial distributions that will result in perfect equality. (Indeed, for any given distribution on a compact support there is a ρ high enough such that convergence to equality will ensue.) That establishes part (i) and (ii).

To establish parts (iii) and (iv), notice that the threshold r^* is unaffected by the aspirations formation process. Because lower aspirations (in the sense of a decrease in $\Psi(y, F)$ for every (y, F)) decrease $r_0(y)$ for all y, convergence to perfect equality becomes more likely, which proves part (iii). To prove part (iv), we note:

Lemma x3. *$r_t(y)$ is strictly increasing (decreasing) in γ if $r_t(y) < (>)1$, and is bounded above (below) by 1.*

Proof. Using (x19), aspirations ratios at time t are

$$r_t(y) \equiv \frac{1}{\gamma + (1-\gamma)\psi(F_t)/y} \text{ for every } y \in \text{Supp } F_t.$$

The effect of γ is given by $dr_t(y)/d\gamma = -r_t(y)^2(1 - \psi(F_t)/y)$. Hence, an increase in γ raises (lowers) $r_t(y)$ if $\psi(F_t) > (<)y$. Since the latter inequality depends on how r_t compares to 1,

$$\frac{dr_t(y)}{d\gamma} > (=, <)0 \text{ for } r_t(y) < (=, >)1,$$

while $r_t(y)$ is correspondingly bounded above (below) by 1. ■

It follows from this lemma that all aspirations ratios converge to 1 as γ increases (i.e., as aspirations become less socially sensitive). Because $r^* > 1$, lowering the social sensitivity of aspirations therefore bunches more individual aspirations ratios in an interval below r^*. That reduces the proportion of frustrated individuals, increasing the likelihood of convergence to perfect equality. (Observe that without condition (4) in the main text, $r^* < 1$ and $\underline{g} < 1$ would be possible. In this case, more — and not less — social sensitivity would reduce the proportion of frustrated individuals.) ■

APPENDIX D. PROOFS OF OBSERVATIONS 2 AND 3 OF SECTION 5.4.1

Observation 2. *Assume upper mean aspirations. Then balanced growth with growth factor $g > \underline{g}$ is possible if (and only if) the distribution of normalized incomes $y/(1+g)^t$ is Pareto*:

$$F\left(\frac{y}{[1+g]^t}\right) \equiv F(w) = 1 - (A/w)^{r/(r-1)}$$

for all $w \geq A$ and (A, r) such that $r \in (1, r^*]$.

Proof. Balanced growth at rate g among the satisfied requires the aspirations ratio to be constant in y, say at value r. Define normalized income w at time t by $w = y/(1+g)^t$, and let $F(w)$ be the distribution of w; it will be time-invariant. Aspirations ratios (expressed as a function of w in the support of F) are easily seen to be

$$r = r(w) = \frac{\mathbb{E}(w'|w' \geq w)}{w} = \frac{1}{w}\int_0^\infty \left[1 - \max\left\{\frac{F(x) - F(w)}{1 - F(w)}, 0\right\}\right] dx$$
$$= 1 + \frac{1}{w[1 - F(w)]}\int_w^\infty [1 - F(x)]\, dx$$

where the penultimate inequality is a standard rewriting of the expectation formula for a nonnegative random variable.[7] Defining $H(w) \equiv \int_w^\infty [1 - F(x)]dx$ for all $w \geq 0$, this means that

$$\frac{H(w)}{wH'(w)} = -(r-1),$$

for all w in the support of F, from which it follows that for all such w,

$$hw^{-1/(r-1)} = H(w) = \int_w^\infty [1 - F(x)]dx \quad \text{(x20)}$$

for some nonzero constant h, to be suitably chosen soon. Differentiating both sides of (x20) and transposing terms, we have

$$F(w) = 1 - Aw^{-r/(r-1)},$$

for all w in its support, where $A = h/(r-1)$ is a suitably chosen constant so that F is a bonafide cdf. This is a Pareto distribution, and all such distributions with $r \in (1, r^*]$ are compatible with perennially satisfied individuals growing at the rate of $g(r)$. ■

Observation 3. *Consider the local income neighborhood model. The balanced growth is possible from a distribution with compact support, and no individual remains frustrated forever.*

Proof. It is easy to construct a distribution with balanced growth, even under initial distributions with bounded support. Create a set of clusters of individuals, all with the same income within the cluster but with incomes that differ enough across clusters that they are not in each other aspirations windows. Each cluster then grows at a rate $g(1)$.

Now we prove that in no equilibrium can an individual be frustrated forever.

Recall that $\Psi(y, F)$ is completely insensitive to the distribution outside the local range $[y(1-\beta), y(1+\beta)]$, for every y. We first show that this implies that $\min\{y, \min F_{y,\beta}\} \leq \Psi(y, F) \leq \max\{y, \max F_{y,\beta}\}$, where $F_{y,\beta}$ is the restriction of the distribution over $[y(1-\beta), y(1+\beta)]$.

To do so, take a person with income y. Let $y_1 = \min\{y, \min F_{y,\beta}\}$ and $y_2 = \max\{y, \max F_{y,\beta}\}$. Create a new distribution F_1 by taking all the mass of the distribution F that is not in $[y(1-$

[7] For any nonnegative random variable with cdf G, its expectation is given by $\int_0^\infty [1 - G(x)]dx$. In the case of a distribution conditional on values no less than w, $G(x) = \max\left\{\frac{F(x)-F(w)}{1-F(w)}, 0\right\}$ for all $x \geq 0$.

$\beta), y(1+\beta)]$ and place it at $y(1-\beta)-\epsilon$, just below our individual's window. By insensitivity outside the window, $\Psi(y, F) = \Psi(y, F_1)$, and by range-boundedness $\Psi(y, F_1) \in [y(1-\beta) - \epsilon, y_2]$. Combining these two observations and taking an appropriate sequence with $\epsilon \to 0$, we must conclude that $\Psi(y, F) \in [y(1-\beta), y_2]$. By the same token, we can create a new distribution F_2 by taking all the mass of the distribution F that is not in $[y(1-\beta), y(1+\beta)]$ and place it at $y(1+\beta)+\epsilon$, just above our individual's window. By insensitivity outside the window $\Psi(y, F) = \Psi(y, F_2)$, and by range-boundedness, $\Psi(y, F_2) \in [y_1, y(1+\beta)+\epsilon]$, so that passing to the limit as $\epsilon \to 0$, $\Psi(y, F) \in [y_1, y(1+\beta)]$. Intersecting these two findings, we must conclude that $y_1 \leq \Psi(y, F) \leq y_2$.

Now suppose, on the contrary, that there is some largest income level y at some time t that remains frustrated forever. Since $r^* > 1$, it must be that $\Psi(y, F_t) > y$ and therefore — by the above claim — that $\max F_{t,y,\beta} > y$. Take any $y' \in (y, \max F_{t,y,\beta}]$. Because those individuals initially at y' are not forever frustrated, their income must grow at a rate strictly exceeding $\underline{g}$n(at least along a subsequence of dates) and in time leave the aspirations window of individuals who started at y. In time, there cannot be any higher income in that person's aspirations window. But that contradicts the assumption that that person remains frustrated forever. Hence no individual remains frustrated forever. ■

APPENDIX E. MINIMAL MONOTONICITY

Recall the definition of minimal monotonicity. Take a distribution F. Let $\bar{y}$ be the supremum income in it. For any y, let $F_-(y)$ denote the left-hand limit of the distribution at y. Now take another distribution F' that weakly dominates F. Aspirations are *minimally monotone* if

1. $\Psi(\bar{y}, F') \geq \Psi(\bar{y}, F)$;
2. $\Psi(y, F') \geq \Psi(y, F)$ for $y < \bar{y}$ if $F'_-(y) = F_-(y)$; and
3. $\Psi(y, F') > \Psi(y, F)$ for $y < \bar{y}$ if $F'_-(y) = F_-(y)$ and $F'(x) < F(x)$ for all $x > y$ with $F'(x) < 1$.

Let us pause to understand minimal monotonicity. F' weakly dominates F. Minimal monotonicity requires that the aspirations at the highest income level under F do not decrease. For other levels of income $y < \bar{y}$, matters can go either way, as we showed by example in the main text. That is, aspirations can increase or decrease if $F'_-(y) \neq F_-(y)$ (there are more or fewer people who are poorer than y). Minimal monotonicity asks that if $F'_-(y) = F_-(y)$, then aspirations for $y < \bar{y}$ cannot decrease, and indeed, they must strictly increase if there is strict dominance above y.

For any distribution function F and any income y, define F_y to be the conditional distribution function of F on the domain $[y, \infty)$. Consider any aspirations formation function Ψ that respects the following conditions:

[Upward Looking] $\Psi(y, F) = \Psi(F')$ whenever $F_y = F'_y$.

[Upper Monotone] $\Psi(y, F') \geq \Psi(F)$ whenever F'_y dominates F'_y, with strict inequality when the domination is strict.

Recall that "upper mean aspirations," in which aspirations are given by the conditional mean of income above one's own income:

$$\Psi(y, F) = \mathbb{E}_F(x|x \geq y),$$

are "upward looking " and "upper monotone." But one can think of other examples as well, including those that truncate the cognitive window at some upper bound. For instance, suppose that an individual were to look at some average of conditional incomes belonging only to the nearest p percentiles above her. Then, too, the conditions above are satisfied.

Proposition x5. *Suppose that aspirations are upward looking and upper monotone. Then they are minimally monotone.*

Proof. Consider two distributions F and F', with supremum incomes $\bar{y}$ and $\bar{y}'$ respectively, such that F' weakly dominates F. Clearly $\bar{y}' \geq \bar{y}$, so that it is trivially the case that $F'_{\bar{y}}$ dominates $F_{\bar{y}}$. It is also easy to check the same relationship for any other income y provided that $F'_-(y) = F_-(y)$. Similarly, it readily follows that F'_y strictly dominates F_y under these circumstances when F' strictly dominates F. Therefore, we have

$$\Psi(y, F') \geq \Psi(y, F),$$

with strict inequality whenever F' strictly dominates F.

It follows that aspirations that are upward looking and upper monotone, are minimally monotone. ■

References

P. Billingsley (1999), *Convergence of Probability Measures*, second edition, John Wiley and Sons.

D. Duffie, J. Geanakoplos, A. Mas-Colell and A. McLennan (1994), "Stationary Markov Equilibria," *Econometrica* **62**, 745–781.

G. Genicot and D. Ray (2009), "Aspirations and Mobility," Background paper for the LAC-Regional Report on Human Development 2009, UNDP.

J. Neveu (1965), *Mathematical Foundations of the Calculus of Probability*, San Francisco, CA: Holden Day.

K. R. Parthasarathy (1967), *Probability Measures on Metric Spaces*, New York, NY: Academic Press.