Reciprocity in Groups and the Limits to Social Capital.

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Robert Putnam defines social capital as "features of social organization, such as networks, norms and social trust that facilitate coordination and cooperation" (Putnam 1995: 67). Such networks are typically associated with norms that promote coordination, cooperation and reciprocity for the mutual benefit of network members. These norms, coupled with the appropriate use of sanctions in case of noncompliance, are often thought to enable these groups to deal smoothly and effectively with multiple social and economic issues. At the same time, some authors have noted that strongly bonded groups may have adverse consequences for others (see, for instance Alejandro Portes and Patricia Landolt, 1996) or even for themselves (see for instance George Akerlof, 1976, or Kaushik Basu, 1986). It is this latter aspect that we wish to emphasize in these notes.

Based on our earlier work on risk sharing in groups and networks (Garance Genicot and Debraj Ray, 2003, 2005 and Francis Bloch, Garance Genicot and Debraj Ray, 2006), this paper proposes a simple model of mutual help in groups and networks. We argue that, if social capital can promote cooperation among groups of individuals, it can also hurt it. When groups of individuals can jointly deviate from a social norm, the fact that they have built strong ties among themselves may in fact make deviations easier, and weaken cooperation in society as a whole.

In the process of making this argument, we develop a model with two distinctive features. First, we construct a simple framework for studying mutual help and cooperation within groups. While admittedly abstract, this tractable model retains some of the features of risk sharing arrangements, but allows for more flexibility and richer structure. Specifically, we assume that at every date, an individual either *needs* help or is in a position to *offer* it (at some cost to herself). The state of

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needing help occurs with independent and identical probability, and is observable. Individuals needing help, and those capable of providing it, are matched according to a fixed matching technology. The cost of providing assistance is always positive, and depends on two components: the proportion of people in the community who need help, and an idiosyncratic cost shock which is privately observed by the individual who is being asked to help. At any period and for any state, agents who are asked to provide help may default on their commitment, and mutual help arrangements need to be self-enforcing.

The second feature of our model is a new concept to study self-enforcing arrangements in groups and networks, that we label *fragility.* It attempts to capture the idea that a group may well survive relatively small shocks but may break down under more stressful circumstances. The probability that a group breaks up is an endogenous outcome, and this is the focus of our analysis. In contrast, in the existing literature (for instance Steven Coate and Martin Ravallion, 2003, Narayana Kocherlakota, 1996 and our own previous work), self-enforcement is considered to be an all-or-nothing concept. Either a group norm is *fully* sustainable in *every* state, or the norm is deemed unstable. When all information is perfectly available, this is often without any loss of generality: norms that achieve results in some states may be suitably slackened in other states to avoid deviations. When there is private information — as there is here regarding the costs of rendering assistance— matters are different. A norm may work well for some states but not for others. We define the *degree of fragility* of the norm as the overall probability of states in which some self-enforcement constraint fails.[1]

I. Mutual Help in Groups

We consider a society with n identical individuals. Let p denote the probability that a person needs help. Call such a person a potential *receiver* and the remaining individuals potential *donors.* Receivers and donors are matched according to a fixed matching technology. If a proportion q of the community is in need of help, then a given fraction $d(q)$ of randomly chosen potential donors is enlisted to provide that

[1]Only a non-stationary scheme where potential donors reveal their cost would make it possible to suitably adjust the norm, see Mobius, 2001, and Hugo Hopenhayn and Christine Hauser, 2004.

assistance to another given fraction $r(q)$ of randomly chosen receivers. Let $c(q,\alpha)$ be an individual donor's cost of helping, where c is continuous in both q and α, a private cost shock. Without loss of generality suppose that c is nondecreasing in α. Assume that α is independently drawn from some continuous distribution and that for every $q > 0$, $\min_\alpha c(q,\alpha) > 0$. We assume that *if* help is given, it implies an indivisible benefit of b. Providing help is always socially optimal, that is $r(q)b - d(q)\max_\alpha c(q,\alpha) > 0$ for all q.

This model accommodates two important special cases. If a single donor provides help to a single recipient, then $r(q) = d(q) = \min\{1, (1-q)/q\}$, and c is independent of q and may be set equal to α. If donors can potentially help several recipients, then $r = d = 1$, and $c = \frac{q}{1-q}\alpha$ is increasing in q.

If everyone abides by the social norm at every date, it is very easy to calculate the expected value of membership in this community as

$$(1) \qquad \widetilde{v}(n) \equiv \sum_{k=0}^{n-1} p(k, n-1)\left[p\, r\left(\frac{k+1}{n}\right) b - (1-p)\, d\left(\frac{k}{n}\right) Ec\left(\frac{k}{n}, \alpha\right)\right]$$

where $p(k,n)$ is the probability that k persons are in need, within n people.[2]

We consider social norms such that $\widetilde{v}(n)$ is increasing in n. There is nothing particularly remarkable about this assumption. It says that larger communities can pool their resources better (ex ante) when it comes to acts of reciprocity. It is easy to verify that this is indeed the case for our two examples.

II. Stability and Fragility

Whether a social norm can be enforced or not depends on the assumptions made on the behavior of the agents following a deviation. One may distinguish between situations where *individuals* deviate and situations where *groups of agents deviate.* In the first case, after refusing to provide help, an agent is ostracized and obtains his autarky payoff, denoted $\widetilde{v}(1)$. In the second case, a deviant group of agents can form a smaller subgroup, and obtain a payoff $\widetilde{v}(s)$, where s denotes the size of the stable subgroup formed.

[2] That is, $p(k,n) = \frac{n!}{k!(n-k)!}p^k(1-p)^{n-k}$.

The classical 0-1 definition of *stability* is as follows. A social norm for a group of size n is called *stable* (under individual or group deviations) if and only if, for all states of nature, no individual (or group) has an incentive to deviate at any state. Our concept of *fragility* corresponds to a weaker requirement. We allow for deviations to occur in some states, and define fragility as the overall probability that a deviation occurs. Because agents do not abide by the social norm at all states, the expected values are not given by $\widetilde{v}(n)$ as provided in equation (1). Instead, agents anticipate that deviations will occur at some states, and compute accordingly new values denoted by $\hat{v}(n)$ (in the case of individual deviations) and by $v^*(n)$ (in the case of group deviations). However, an agent's incentive to deviate also depends on the expected values, $\hat{v}(n)$ or $v^*(n)$. Hence, incentives to deviate and values must be computed simultaneously, as a fixed point of a circular process.

Say that a group S of donors of size s has a *strictly (weakly) profitable deviation* in a given state if

$$(2) \qquad (1-\delta)c(q,\alpha_i) \; > \; (\geq) \; \delta[v^*(n) - v^*(s)] \quad \text{[group deviation]}$$

for every $i \in S$, and a *weakly profitable deviation* if a weak inequality holds for every $i \in S$ in (2). Similarly, if only individual deviations are possible, we say that individual i has a strictly profitable deviation in a state if

$$(3) \qquad (1-\delta)c(q,\alpha_i) \; > \; \delta[\hat{v}(n) - \widetilde{v}(1)] \quad \text{[individual deviation]},$$

and a weakly profitable deviation if (3) holds with a weak inequality. holds with a weak inequality. Our view is that, when a strictly profitable deviation is possible in some state, the group is *fragile*: it will indeed suffer a deviation and subsequently break up.

Implicit in our definitions of profitable deviation are two restrictions that keep the exposition simple without changing the final results in a significant way. First, the right hand sides of (2) and (3) presume that if our group of donors were *not* to break away from the group, no other group or individual would. This is a restriction on two counts. There may well be coordinated breakaways (much in the same way as a currency crisis occurs): lots of groups rush for the door because the fear that others will, thereby reducing the continuation value of compliance. Moreover, other groups may deviate, not because of coordination failure, but simply because a condition akin to (2) or (3) is met for *them* as well. In both these cases the right hand side of (2) and (3) may be misspecified, because the values $v^*(n)$ or $\hat{v}(n)$ may not be available to the group under consideration in the event that they do comply.

One way to justify our specification is that among the potential deviant groups in a given state, one and only one group — perhaps selected at random — obtains the opportunity to break away. Then our specification is the correct way to describe fragility. We take this route for a straightforward reason: a more comprehensive description would involve the use of additional fixed-point arguments to define continuation values, without adding anything of substance to the results. (As it is a fixed-point argument must be employed to define the values; see below.) We return to this point after the statement of our main proposition below.

Second, we only permit actual donors to deviate. It is entirely conceivable that others deviate as well. But this would happen only in the dismal scenario where the right-hand side of (2) is actually negative. But in that case a group of size n wouldn't even be *ex ante* stable, and it adds nothing to investigate its fragility *ex post*. Therefore we may ignore this case without any loss of generality.

We now develop the notion of fragility and deviations more carefully.

Group Deviations. Suppose that recursively, all values $v^*(s)$ have been defined for $s \leq n$. To define an equilibrium at n, we must associate a value $v^*(n)$ and a probability system $\{p(\omega, S), q(\omega)\}$, where $p(\omega, S)$ is to be interpreted as the probability that group S will deviate at state ω, and $q(\omega)$ as the probability that no deviation will occur at state ω. To form an equilibrium, these objects must satisfy the following conditions:

[1] (Anonymity) Let ω' be a state created from ω by permuting all identities by a permutation σ. Then, if $S' = \sigma[S]$, we must have $p(\omega', S') = p(\omega, S)$.

[2] (Single deviation) In keeping with our simplifying assumption that only one group can deviate at a time, we have

$$q(\omega) + \sum_S p(\omega, S) = 1. \tag{4}$$

[3] (Consistent Values) The value $v^*(n)$ must be consistent with the probability system (p, q). Denote by $u_i(\omega)$ the *one-period* payoff to person i if there is no breakup of the group in that state, and by $v_i(\omega, S)$ the *overall* expected payoff (including current payoff) to i if the subgroup S deviates in that state.[3] Then we must have

$$(5) \qquad v^*(n) = E_\omega \left[q(\omega)\{(1-\delta)u_i(\omega) + \delta v^*(n)\} + \sum_S p(\omega, S) v_i(\omega, S) \right].$$

Note that because p and q are symmetric, the i-subscript on the right hand side of (5) drops out after integrating over states, so that $v^*(n)$ is well-defined and independent of i.

[4] (Consistent Probabilities) If a state is fragile, then q *must* equal zero; and if $p(\omega, S) > 0$, then S must have at least a weakly profitable deviation in that state.

The associated *fragility* of a group of a size n, denoted by Frag(n) is the *equilibrium* probability that a state is fragile.

Naturally, this expected utility needs to satisfy a *participation constraint.* If the value generated in this fixed point is less than what a group $s = \arg\max_{s'=1,..n-1} v^*(s')$ can expect $v^*(s)$, we would expect the group of n to split directly into smaller groups. A group of size s would form and, using the symmetry in the probability of belonging to this group,[4]

$$(6) \qquad v^*(n) = \frac{s}{n} v^*(s) + \frac{n-s}{n} v^*(n-s).$$

The fragility of n is then 1.

Notice that fragility and group value are intimately connected and are determined simultaneously. Not only does fragility determine group worth (see (5)), but group worth in turn feeds back on fragility (see (2)).

The following proposition guarantees that the existence of an equilibrium.[5]

Proposition 1. *An equilibrium exists.*

[3]The former is well-defined simply because the norm prescribes what to do in each state. The latter is well-defined by recursion: if a group S of size s leaves, $v_i(\omega, S)$ equals $(1-\delta)*$[current payoff]$+\delta v^*(s)$ for all $i \in S$, and equals $(1-\delta)*$ [current payoff] $+\delta v^*(n-s)$ for all i not in S.
[4]$v^*(n-s)$ accounts for the possible creation of further subgroups.
[5]Proofs of all propositions are in the appendix on the AER Website.

Individual Deviations. We can easily define the *equilibrium* values $\hat{v}(n)$, when only individuals can deviate, in a very similar way. Only individual deviations are allowed,[6] and the probability system $\{p(\omega, S), q(\omega)\}$ must be consistent with the individual fragility (3).[7]

III. The Fragility of Mutual Help Groups

When only individuals can deviate, the strength of subgroups can only enhance the value of a group and decrease its fragility. If someone in the group deviates, the very fact that the remaining subgroup is still valuable provides additional incentives to nondeviants to abide by the norm. In contrast, when group deviations are possible, the value of subgroups has conflicting effects. On the one hand, it increases the value of staying within the group but on the other hand in raises the value of deviating as a group. The following example illustrates these effects.

Example 1: Consider the model of indivisible help. Individuals have a probability $p = 1/2$ to be in need and satisfying a need generates a utility $b = 150$. Assume that the cost of helping can take only two values $\underline{c} = 42.5$ or $\overline{c} = 80$. The cost is $\underline{c}$ with probability $q = 0.8$, so that the average cost of helping is 50. Finally, the discount factor δ is set at 0.75.

In this case, it is easy to check that groups of 2 and 3 have a fragility of 0.1 and a worth of $v = 16.5$ – short of the utility of 25 that they could achieve in the absence of commitment problems; whether individual or group deviations are allowed.

Groups of 4 or more have a zero fragility for individual deviations. In contrast, when group deviations are possible, groups of 4 are fully fragile, breaking up in smaller groups. Even at the highest possible values, groups of 2 with high costs would deviate when asked to help. This reduces the value of the group by enough that individuals with high cost and groups of 2 with lower costs would also renege on their transfer. This, in turn, makes it not worth for anyone to ever help. Hence, a group or 4 would break up immediately.

[6]This implies that, if a state is fragile, the expected utility of agents who have not deviated moves to $\hat{v}(n-1)$ while the utility of the deviator becomes $\widetilde{v}(1)$

[7]$p(\omega, S) = 0$ for all group S larger than a singleton.

The preceding example illustrates a general phenomenon. When the size of the society becomes large enough, fragility tends to one when group deviations are allowed.

Proposition 2. *For any* $\epsilon > 0$*, there exists* $\overline{n}$ *so that for all* $n \geq \overline{n}$*,* $\mathrm{Frag}(n) > 1-\epsilon$*.*

In contrast the fragility of groups tend to zero when only individuals can deviate.

Proposition 3. *For any* $\mu > 0$*, there exists* $\overline{\delta} < 1$ *and* $\overline{n}$ *so that, for all* $n \geq \overline{n}$ *and all* $\delta \geq \overline{\delta}$*,* $\mathrm{Frag}(n) < \mu$*.*

IV. Social Networks

So far, we have assumed that agents were organized into groups. We now consider a setting where agents form bilateral agreements rather than groups, and mutual help is provided along bilateral links. We suppose that , instead of belonging to a group, agents are organized in a fixed social network g. While the existence of a social network may affect mutual help in a number of ways (for example, by conveying information about defections, about the location of donors and recipients), we emphasize here the fact that the matching between donors and recipients depends on the network of social contacts. Agents will be able to provide or receive help only if they are linked directly or at relatively short distances in the social network. More precisely, we consider a model of indivisible mutual help, and let the probabilities of being matched for recipients and donors (now denoted $\rho_i(g,\omega)$ and $\eta_i(g,\omega)$ respectively) depend on the network as well as on the state. Hence, when all agents abide by the social norm, the value of agent i in social network g is given by:

$$\widetilde{v}_i(g) = \sum_{\omega|i=R} \Pr(\omega)\rho_i(g,\omega)b - \sum_{\omega|i=D} \Pr(\omega)\eta_i(g,\omega)Ec(\alpha).$$

We first analyze the relation between an agent's location and her value. In the next example, we consider a society of three agents organized along a line. The central agent can provide and receive help from the two peripheral agents, but peripheral agents can only provide or receive help from the central agent.

Example 2 Consider a line among three agents, labeled 1, 2 and 3. Suppose that the matching technology prescribes that agents can only provide or receive help from their direct neighbor, and that when agents 1 and 3 both need help, agent 2 helps any of them with equal probability, and when agent 2 needs help that both 1 and 3 can provide, any of the two extreme agents provides help with equal probability. With this specification of the matching technology,

$$\begin{aligned} \widetilde{v}_1(g) = \widetilde{v}_3(g) &= \tfrac{1}{2}p(1-p)\left[b(2-p) - c(1+p)\right], \\ \widetilde{v}_2(g) &= p(1-p)(b(1+p) - c(2-p), \\ \widetilde{v}_i(g^c) &= p(1-p)(b-c) \;\; i \in \{1,2,3\}. \end{aligned}$$

This example shows that there is no clear monotonic relation between an agent's position in the network (measured for example by his number of connections) and his value: the middle agent obtains a higher value than the peripheral agents when p is large, but a lower value when p is low. This phenomenon is actually quite general, as shown by the next observation.

Let ω_{ri} and ω_{di} denote the states at which i is the only recipient (respectively donor) of help. Furthermore, suppose that the matching is efficient at these states, i.e. $\rho_i(g, \omega_{ri}) = \eta_i(g, \omega_{di}) = 1$. Consider two agents i and j and say that agent i is *better connected* than agent j if and only if $\sum_{k,ik \in g} \eta_i(g, \omega_{rk}) > \sum_{k,jk \in g} \eta_j(g, \omega_{rk})$ and $\sum_{k,ik \in g} \rho_i(g, \omega_{dk}) > \sum_{k,jk \in g} \rho_j(g, \omega_{dk})$.

Observation 1: Suppose that agent i is better connected than agent j as defined above. Then there exists $\underline{p}$ and $\overline{p}$ such that, for all $p \in [0, \underline{p}]$, $\widetilde{v}_i(g) < \widetilde{v}_j(g)$ and for all $p \in [\overline{p}, 1]$, $\widetilde{v}_i(g) > \widetilde{v}_j(g)$.

Next, we analyze the relation between the fragility of a network and its connectivity. Because the addition of new links increases the number of matches between donors and recipients, one may expect that social networks with higher density of links are less fragile (with respect to individual deviations). The next Proposition formalizes this intuition.

Assume that when the graph is complete, $g = g^c$, the matching technology is both *efficient* and *anonymous*, so that, for any state ω:

$$\sum_{i|i=R} \rho_i(g^c,\omega) = \sum_{i|i=D} \eta_i(g^c,\omega) \quad = \quad \min\{k(\omega), n-k(\omega)\},$$
$$\forall i,j, i=j=R, \rho_i(g^c,\omega) = \rho_j(g^c,\omega) \quad \text{and} \quad \forall i,j, i=j=D, \eta_i(g^c,\omega) = \eta_j(g^c,\omega),$$

where $k(\omega)$ denotes the number of recipients of help at stat ω. Furthermore, let $\delta(g) = \min \delta|\text{Frag}(g) = 0$, where fragility is measured with respect to individual deviations. Hence, $\delta(g)$ denotes the minimal discount factor for which network g is stable.

Proposition 4. *For any graph $g \subset g^c$, $\delta(g^c) \leq \delta(g)$ and the inequality is strict for some g.*

This proposition shows that the complete network is more likely to be stable (with respect to individual deviations) than any other network. Furthermore, there exist values of the discount factor for which the fragility of the complete network is zero, while other networks have a positive fragility.

V. Conclusion

This paper presents a model of *self-enforcing* reciprocity in groups and networks that stresses the importance of subsets of individuals and introduces the concept of fragility. More valuable and less fragile subgroups are shown to have conflictual effects. On the one hand, they raise the value of abiding to the mutual aid norm even if some individuals might leave the group. On the other hand, they increase the value of coordinated deviation, thereby making the original group more fragile. The latter effect dominates for large groups. This is the limit to social capital.

VI. References

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Appendix

Existence of Equilibrium Values.

Proposition 5. *An equilibrium exists.*

Proof . Fix any value v in some large compact interval V containing all feasible payoffs. For any state ω and group S of size s, define

$$\Delta(\omega, v, S) = \max\{\min_i\{(1-\delta)c(q,\alpha_i) - \delta[v - v^*(s)]\}, 0\}.$$

Obviously, $\Delta(\omega, v, S) > 0$ if and only if S has a strictly profitable deviation under v.

Define a *collection* $\phi^1(\omega, v)$ of symmetric probability distributions on (ω, v) by:
$[i]$ the *singleton* set containing

$$q(\omega) = 0 \text{ and } p(\omega, v, S) = \frac{\Delta(\omega, v, S)}{\sum_T \Delta(\omega, v, T)} \;\forall S \tag{7}$$

if some *strictly* profitable deviation exists at (ω, v), and
$[ii]$ the collection of *all* symmetric probability distributions of the form (p, q) otherwise, under the restriction that $p(\omega, v, S)$ can have positive value only if S has a weakly profitable deviation.

Lemma 1. *For each ω, $\phi^1(\omega, v)$ is nonempty, convex-valued, and upper-hemi-continuous (uhc) in v.*

Proof. Obviously, $\phi^1(\omega, v)$ is nonempty and convex-valued for each (ω, v). Now we claim that it is uhc in v. To this end, let v^k be some sequence in V converging to v. Consider a corresponding sequence $(p^k, q^k) \in \phi^1(\omega, v^k)$ (we omit the explicit dependence on ω and S for notational ease) and extract a convergent subsequence converging to some (p, q) (but retain the original sequence notation). We claim that $(p, q) \in \phi^1(\omega, v)$.

This claim is obviously true if no strictly profitable deviation exists at (ω, v).[8] So suppose that a strictly profitable deviation does exist at (ω, v). But then a strictly profitable deviation must exist for k large enough, so that far enough out in the

[8] All we need to observe is that if S has a strictly profitable deviation for the sequence (ω, v^k), then it must have a weakly profitable deviation at (ω, v).

sequence, (p^k, q^k) must be uniquely pinned down by the condition (7). Because $\Delta(\omega, v, S)$ is obviously continuous in v, we must have that $(p^k, q^k) \to (p, q)$ in this case as well. ■

We now construct a second map — this time, a function — that links symmetric probability systems to values. In line with condition [3] of an equilibrium (Consistent Values), simply define it by

$$(8) \qquad \phi^2(p, q) = \frac{1}{1 - \delta E_\omega q(\omega)} E_\omega \left[q(\omega)(1-\delta)u_i(\omega) + \sum_S p(\omega, S) v_i(\omega, S) \right].$$

The reason ϕ^2 is well-defined is precisely because p is symmetric, so that the subscript i on the right hand side of (8) no longer appears on the left hand side after integrating.

It is trivial to see that ϕ^2 is continuous in (p, q). Now compose the two correspondences, by defining a third correspondence $\phi : V \mapsto V$:

$$(9) \qquad \phi(v) = \{v' \in V | v' = \phi^2(p, q) \text{ for } (p, q) \text{ with } (p(\omega), q(\omega)) \in \phi^1(\omega, v)\ \forall \omega\}.$$

Since ϕ^2 is a continuous function on a non-empty, covex and upper-hemi continuous correspondence (by Lemma 1), $\phi(v)$ is nonempty, convex-valued, and upper-hemi-continuous (uhc) in v. Hence, ϕ must have a fixed point, call it $v^*(n)$. Define an associated probability system (p, q) by the particular value of (p, q) in (9) that permits the fixed point to be attained. One can now check that all the five conditions for an equilibrium are satisfied. ■

Fragility of Mutual Help Groups

Proposition 6. *For any $\epsilon > 0$, there exists $\overline{n}$ so that for all $n \geq \overline{n}$,* $\text{Frag}(n) > 1 - \epsilon$.

Proof. Assume that the proposition is false. Then there exists $\epsilon > 0$ and an infinite set N of group sizes such that for all $n \in N$, $\text{Frag}(n) \leq 1 - \epsilon$.

By our assumptions on the cost function, there exists a closed interval $I(p)$ containing p in its interior and a value $\underline{c} > 0$ such that $c(q, \alpha) \geq \underline{c}$ for all cost shocks α and $q \in I(p)$. Fix this interval and the lower bound on costs in what follows.

Note that per-capita group payoffs are obviously bounded. It follows that for every $\mu > 0$, there exists $n(\mu) \in N$ so that $v^*(n) - v^*(n(\mu)) < \mu$ for all $n > n(\mu)$, $n \in N$. Pick any such $\mu < \frac{1-\delta}{\delta}\underline{c}$, where $\underline{c}$ is described in the previous paragraph.

An application of the weak law of large numbers yields the following implication: For every $\epsilon > 0$ there exists a group size $\hat{n}$ such that for every $n \in N$ with $n > \hat{n}$, the joint event that

(a) the number of actual donors exceeds $n(\mu)$, and

(b) the proportion of those in need lies within $I(p)$

has probability exceeding $1 - \epsilon$. But in this event, we may use the definition of μ to conclude that for all individuals i in some subgroup of size $n(\mu)$,

$$c(q, \alpha_i) \geq \underline{c} > \frac{\delta}{1-\delta}\mu > \frac{\delta}{1-\delta}[v^*(n) - v^*(n(\mu))],$$

so that (2) holds for $s = n(\mu)$. We may therefore conclude that the fragility of all such $n \in N$ exceeds $1 - \epsilon$, which is a contradiction.

Proposition 7. *For any $\mu > 0$, there exists $\overline{\delta} < 1$ and $\overline{n}$ so that, for all $n \geq \overline{n}$ and all $\delta \geq \overline{\delta}$,* $\mathrm{Frag}(n) < \mu$.

Proof. Fix $\mu > 0$. Now choose a size $m > 1$ and a value ϵ with $\widetilde{v}(m) - \widetilde{v}(1) > \epsilon > 0$, and define

$$\overline{c}(\delta) \equiv \frac{\delta}{1-\delta}[\widetilde{v}(m) - \epsilon - \widetilde{v}(1)].$$

Denote as $I(c)$ the set of q so that $c(q, \alpha) \geq c$ for any cost shock α. It is easy to select $\overline{\delta}$ so that $max_\alpha c(p, \alpha) < \overline{c} \equiv \overline{c}(\overline{\delta})$.

The probability that $q \in I(\overline{c})$ is an upper bound on the i-fragility of a group with a value v of at least $\underline{v} = \widetilde{v}(m) - \epsilon$.

It follows that for any group of size n, if $\hat{v}(n) \geq \underline{v}$, a lower bound on the utility of the agents is given by

$$v(n) = \frac{1}{[1 - \delta Pr(q \notin I(\overline{c}))]} \sum_{k=0}^{n-1} p(k, n-1) \left[p\, r\left(\frac{k+1}{n}\right) b - (1-p)\, d\left(\frac{k}{n}\right) Ec\left(\frac{k}{n}, \alpha\right)\right].$$

An application of the weak law of large numbers yields the following implication: for every $\eta > 0$ there exists a group size $\hat{n}(\eta)$ such that for every $n > \hat{n}(\eta)$, the probability that the proportion of those in need lies within $I(\overline{c})$ has probability less than η. It follows that we can choose η sufficiently small that (i) $v(n) \geq \underline{v}$ for all $n > \hat{n}(\eta)$, and (ii) $\eta < \mu$.

Since $\overline{c}(\delta)$ is increasing, it follows that, for $\delta > \overline{\delta}$, the fragility of all groups of size $n > \hat{n}(\eta)$ will be less than μ.

Social Networks.

Proposition 8. *For any graph $g \subset g^c$, $\delta(g^c) \leq \delta(g)$ and the inequality is strict for some g.*

Proof. Let $w_i(g,\omega)$ denote the net payoff of i at state ω. For any ϵ, let $\underline{p}$ be such that for all i, $\sum_{\omega|\omega\neq\omega_{ri}\text{for some i}} w_i(g,\omega) < \epsilon$. Then,

$$\begin{aligned}
\widetilde{v}_j(g) - \widetilde{v}_i(g) &> pb(\rho_j(g,\omega_{rj}) - \rho_i(\omega_{ri}) + pc(\sum_{k,ik\in g} \eta_i(g,\omega_{rk}) - \sum_{k,jk\in g} \eta_j(g,\omega_{rk})) - \epsilon \\
&> pc(\sum_{k,ik\in g} \eta_i(g,\omega_{rk}) - \sum_{k,jk\in g} \eta_j(g,\omega_{rk})) - \epsilon.
\end{aligned}$$

Hence, one can choose ϵ and hence $\underline{p}$ so that $\widetilde{v}_j(g) - \widetilde{v}_i(g) > 0$.

By a similar argument, for any ϵ, define $\overline{p}$ so that for all i, i, $\sum_{\omega|\omega\neq\omega_{di}\text{for some i}} w_i(g,\omega) < \epsilon$. Then,

$$\begin{aligned}
\widetilde{v}_i(g) - \widetilde{v}_j(g) &> (1-p)b(\sum_{k,ik\in g} \rho_i(g,\omega_{dk}) - \sum_{k,jk\in g} \rho_j(g,\omega_{dk})) + (1-p)c(\eta_j(g,\omega_{dj}) - \eta_i(g,\omega_{di}) - \epsilon \\
&> (1-p)b(\sum_{k,ik\in g} \rho_i(g,\omega_{dk}) - \sum_{k,jk\in g} \rho_j(g,\omega_{dk})) - \epsilon.
\end{aligned}$$

establishing the result. ∎

Proof. We show that $\widetilde{v}_i(g^c) \geq \min_i \widetilde{v}_i(g)$ for any graph g, with strict inequality for some graphs g.

Notice that for any state ω,

$$\sum_{i|i=R} \rho_i(g,\omega) = \sum_{i|i=D} \eta_i(g,\omega) \leq \min\{k(\omega), n-k(\omega)\} = \sum_{i|i=R} \rho_i(g^c,\omega) = \sum_{i|i=D} \eta_i(g^c,\omega),$$

Hence,

$$\sum_{i|i=R} \rho_i(g,\omega)b - \sum_{i|i=D} \eta_i(g,\omega)Ec(\alpha) \leq \sum_{i|i=R} \rho_i(g^c,\omega)b - \sum_{i|i=D} \eta_i(g^c,\omega)Ec(\alpha).$$

Taking expectations over ω,

$$\sum_i \widetilde{v}_i(g) \leq \sum_i \widetilde{v}_i(g^c).$$

By anonymity, $\widetilde{v}_i(g^c) = \frac{\sum_i \widetilde{v}_i(g^c)}{n}$.

This shows that $\widetilde{v}_i(g^c) \geq \min_i \widetilde{v}_i(g)$ for any graph g. Furthermore, if the graph g is not symmetric or disconnected, $\widetilde{v}_i(g^c) > \min_i \widetilde{v}_i(g)$.

Now consider $\overline{\delta}$ such that

$$\frac{\overline{\delta}}{1-\overline{\delta}}(\widetilde{v}_i(g^c) - \widetilde{v}(1)) = \max_\alpha c(\alpha).$$

For any $\delta \geq \overline{\delta}$, Frag($g^c$)=0 and as $\widetilde{v}_i(g^c) \geq \min_i \widetilde{v}_i(g)$ for all g with strict inequality for some g, the conclusion follows. ∎