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Nonpaternalistic Intergenerational Altruism*

DEBRAJ RAY

Stanford University, Stanford, California, and the Indian Statistical Institute, New Delhi - 110016, India

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The paper develops a concept of equilibrium behaviour and establishes its existence in a model of nonpaternalistic intergenerational altruism. Each generation derives utility from its own consumption and the utilities of its successors. Equilibrium capital stocks are time-monotone and therefore converge to a steady state. Finally. when each generation's utility depends on that of at least two successors, equilibria may be inefficient. This is shown by an example, where feasible programs exist providing greater consumption (compared to the equilibrium path) for every generation. *Journal of Economic Literature* Classification Numbers: 021, 111. © 1987 Academic Press, Inc.

I. INTRODUCTION

Many economic issues need to be posed within a framework of intergenerational altruism, if they are to be analyzed in a proper way. Examples of such issues are numerous and varied. Rawls [28] postulated intergenerational altruism in order to set his theory of justice in an intertemporal context.[1] Several models of the evolution of inequality over time rest critically on altruism (Becker and Tomes [4, 5], Bevan and Stiglitz [10], and Loury [21]). The Ricardian equivalence theorem, as formulated by Barro [3], leans heavily on an assumption of intergenerational altruism, and has well-known macroeconomic implications.

Studies in the conceptual foundations of the framework are also numerous. There is the familiar literature on consistent plans, initiated by Strotz [33], with subsequent contributions by Phelps and Pollak [27],

[1] The implications of Rawlsian theory for a "just savings" principle were studied by Arrow [2] and Dasgupta [13], using a model of intergenerational altruism.

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Peleg and Yaari [26], Goldman [16], and Bernheim and Ray [8], among others. These bear a formal identity to models of intergenerational altruism. The latter models have been directly studied by Kohlberg [18], Lane and Mitra [19], Bernheim and Ray [6, 7, 9], Leininger [20], Streufert [32], and others.

Intergenerational altruism has been modelled in two ways. First, there is what one might call the *paternalistic model*. Here, the utility of each generation depends on its own consumption and the *consumptions* of other generations. The term "paternalism" is used to emphasize the fact that generations care about what others actually *consume*, and not the *utilities* they derive from the act of consumption.[2] Secondly, there is the *nonpaternalistic model*, in which each generation derives utility from its own consumption and the *utilities* of other generations.[3]

There is now a substantial body of work on the paternalistic model. The entire literature on consistent plans falls into this category. Many applications of the altruism model, such as Arrow [2] and Dasgupta [13], belong here. Equilibria in this framework (which form the basis for an analysis of the properties of the model) have been recently studied in an extensive way (see Bernheim and Ray [6, 8, 9], Leininger [20], Goldman [16], and Harris [17]). And finally, the positive and normative properties of these equilibria have been explored (Kohlberg [18], Lane and Mitra [19], and Bernheim and Ray [7]).

In contrast, there have been very few rigorous studies of nonpaternalistic altruism, though this framework appears to be of somewhat greater interest in the context of applications (see, e.g., Barro [3] and Loury [21]). This is understandable. After all, the position that generations care about others' utilities, and not in the manner these utilities are achieved, has a certain persuasiveness.[4]

A few studies do explore one aspect of nonpaternalism, and this concerns its connections with the paternalistic model. A natural consistency property (sometimes also called "nonpaternalism" (Archibald and Donaldson [1])) allows the representation of some paternalistic utility indicators in a simple nonpaternalistic form, where the utility of each generation depends on its own consumption and the utility of its immediate successor (this is the form explored in Barro [3], for example). Streufert [32] contains a comprehensive study of this consistency property. The representation of nonpaternalistic functions in paternalistic form has also been the object of

[2] Some paternalistic functions may indeed express intergenerational harmony of interests. But then they will have nonpaternalistic representations (more on this below).

[3] There is also, of course, a mixed framework with both paternalistic and nonpaternalistic components. But I know of no work addressing such a model.

[4] The persuasiveness is not complete though, as a little reflection on the "generation gap" will reveal!

limited attention; a fairly general treatment of a finite horizon model appears in Pearce [25]. But a systematic analysis of the relationship between these two frameworks is yet to be written, and appears to be quite a challenge, especially for models with an infinite horizon.

This paper conducts a *direct* study of the behavioural foundations of the nonpaternalistic model. The first line of business is the definition of an equilibrium outcome, and a demonstration of its existence in a general setting. This is one of the main tasks of the present paper. The notion of an equilibrium outcome is adapted in an obvious way from the concept of subgame perfect Nash equilibria in extensive-form games (the nonpaternalistic structure is not itself a game, though conflicts of interest may well be present).[5]

In this paper, I state and prove an existence theorem for a class of models that exhibits an extremely general form of nonpaternalistic altruism. In particular, each generation's utility may depend on those of any number of future generations.

The existence result asserts the following, in the context of a stationary model.[6] There exist an *indirect utility function* and a *savings policy*, both depending on current endowment, such that each generation finds it optimal to adopt that savings policy provided its descendants use the same policy and exhibit the given indirect utility. In addition, the indirect utility function generated by the generation's maximization problem is also the same as that "announced" by its descendants.

Such equilibria correspond to Markov subgame-perfect equilibria in paternalistic models (Bernheim and Ray [6, 9]).[7] It is of interest to note that the paternalistic models do not turn out to be equally well-behaved with respect to existence. While the existence of Markov equilibrium is guaranteed in simple paternalistic structures, where each generation's concern extends to only the consumption of its immediate successor (Bernheim and Ray [6], Leininger [20]), there are counterexamples to the existence of Markov equilibria when the utility dependence extends to more than one successor (Peleg and Yaari [26]).[8] It turns out that the introduction of

[5] Nonpaternalistic models do not represent games, because payoff functions are not necessarily well defined on the *actions* (consumptions) of all agents. The concept of subgame-perfect equilibrium (Selten [29]) must therefore be modified.

[6] Stationary of the model can be dropped at the expense of sacrificing the stationarity of equilibria.

[7] More general history-dependent policies may be allowed, but Markov equilibria continue to be perfect equilibria, only provided that each generation's preferences are "ancestor-insensitive."

[8] More general history-dependent equilibria continue to exist (see Goldman [16] and Harris [17]). But the absence of Markov equilibria is troubling (for a discussion, see Bernheim and Ray [8]).

uncertainty removes these counterexamples and guarantees existence (Bernheim and Ray [8, 9]).

It is therefore surprising that nonpaternalistic models yield the existence of an equilibrium even in the context of perfect certainty. In fact, the introduction of uncertainty (an extension not pursued here) only makes the existence proof *simpler*, and can be easily accommodated with the techniques developed here.

Some existence results do appear in the literature, dealing with nonpaternalistic structures where altruism is confined to one's *immediate* successor. Streufert [32] proves a very general result for a multisectoral model in this context, and Loury [21] has an earlier proof for the aggregative model. These are certainly of interest.

The problem, though, with the simple nonpaternalistic structure is that it leaves no room for conflicts of interest. Indeed, as I mentioned before, there must be perfect harmony of interests between each generation and its descendant for this specification of altruism to be operative. Predictably, the *equilibrium* solution to this model is often an *optimal* one, viewed from the vantage point of each generation. This is easily seen in the case where the utility function of each generation is additively separable in its own consumption and the utility of the next generation (discounted). Here, one can use the notion of one-step unimprovability, familiar in dynamic programming, to yield the optimality of every equilibrium path.[9] There also appears to be a general connection between unimprovability and the non-additive simple nonpaternalistic structure (Streufert [32]).

However, conflicts of interest play a central role in more general forms of altruism. This is expected to manifest itself in the lack of some optimality properties of the equilibrium solution. A natural candidate is the examination of Pareto-optimality. But there are problems with this concept, unless the nonpaternalistic structure can be shown to have a paternalistic representation and the examination is carried out in the transformed model. Such an exercise is not attempted in the present paper.

However, it can be shown that equilibrium programs may even fail to achieve a far weaker efficiency property. Below, I describe a class of cases (Sect. IIc) where equilibrium programs are inefficient in the sense that more consumption can be feasibly provided at *every* date (Malinvaud [22]). The equilibrium path can therefore display efficiency failures of a drastic sort. In the light of the discussion above, such failures are only possible if altruism extends to the utilities of descendants other than one's immediate successor.

The paper also contains results on the behaviour of equilibrium paths (Sect. IIb) in the case where the current generation's utility is additively

[9] This is because in the particular context at hand, an unimprovable strategy is easily interpretable as an equilibrium for the nonpaternalistic structure.

separable in its own consumption and some function of the utilities of its descendants. Here, equilibrium paths are shown to converge monotonically to a steady state from any initial stock (Theorem C). Moreover, equilibrium programs can be conveniently "ordered" in the sense that equilibrium stocks from a higher initial endowment must be higher at *each* date.

Throughout, I restrict myself to an aggregative framework of intertemporal accumulation, with stationary technology and no uncertainty. An initial endowment is given, which is divided by the current generation into consumption and investment. Investment yields output via a production function, and this forms the endowment of the next generation. The process then repeats itself. This model has been widely used in the literature on efficient and optimal growth,[10] and in the particular context of intergenerational altruism.[11] Analysis of the multidimensional case is an open question.[12] It should be reiterated, though, that uncertainty only makes the analysis simpler, so that the assumption of perfect certainty is not restrictive. And finally, identical techniques can be used to establish the existence of nonstationary equilibria in a nonstationary model.

In Section 2, I spell out the model and its assumptions. The main results are stated and discussed here. Proofs are relegated to Section 3.

2. The Model

The *technology* is given by a production function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$. I shall assume

(F.1) f is increasing and continuous, and

(F.2) there is $\hat{y}$ such that $f(y) < y$ for all $y > \hat{y}$.

Initial endowments are taken to lie in the interval $[0, \hat{y}]$. So given an initial stock $\mathbf{y}$ in this interval, a feasible program $\langle x, y, c \rangle$ is given by

$$
\begin{aligned}
y_0 &= \mathbf{y} \\
y_t &= c_t + x_t, \qquad t \geqslant 0 \\
y_{t+1} &= f(x_t), \qquad t \geqslant 0.
\end{aligned}
\tag{1}
$$

[10] See, for example, Solow [31], von Weizsacker [34], Gale and Sutherland [15], Cass [12], Mitra [23], and Mitra and Ray [24].

[11] See, for example, Strotz [33], Arrow [2], Dasgupta [13], Kohlberg [18], Leininger [20], and Bernheim and Ray [6].

[12] One possible route is proving the existence of an equilibrium involving history-dependent policy and indirect utility functions along the lines developed by Goldman [16] and Harris [17] for the paternalistic case. Streufert [32] does obtain a Markovian equilibrium in a multidimensional model, but restricts himself to simple nonpaternalistic altruism. The question of existence of such equilibria in the case of general altruism *and* a multidimensional model is an important, though unresolved issue.

Define $Y \equiv \max(\hat{y}, \tilde{y})$. Note that $y_t \leqslant Y$ for all feasible programs.

Remarks. (1) Technology (and preferences, below) are taken to be time-stationary. But as I have mentioned above, this assumption can be dropped.

(2) Assumption (F.2) can be dispensed with at the expense of some work, using truncation arguments described in Bernheim and Ray [6].

Let v_s denote the utility of generation s, and let $v^t = (v_{t+1}, v_{t+2},...)$. The *preferences* of each generation are described by a nonpaternatistic, ancestor-insensitive utility function $u: \mathbb{R}_+^\infty \to \mathbb{R}_+$,[13] which maps, for each t, (c_t, v^t) into utilities of generation t. I shall assume that for each t,

(u.1) u is increasing and strictly concave in c_t, and non-decreasing in v_s, $s > t$.

(u.2) u is continuous in the product topology on nonnegative real-valued sequences.

(u.3) c_t and v^t are weak complements in the following sense; for all c_t, $\tilde{c}_t$, v^t, $\tilde{v}^t$, with $c_t \geqslant \tilde{c}_t \geqslant 0$, $v^t \geqslant \tilde{v}^t \geqslant 0$,

$$u(c_t, v^t) - u(\tilde{c}_t, v^t) \geqslant u(c_t, \tilde{v}^t) - u(\tilde{c}_t, \tilde{v}^t).^{14}$$

(u.4) For each $c > 0$, there exists $M(c) < \infty$ such that $u(c, M(c), M(c),...) \leqslant M(c)$.

Remarks. (1) It is assumed that the concern of each generation extends to only its descendants. Concern for one's ancestors can be easily incorporated if this is separable with respect to u. If not, the concept of equilibrium that I present below must be expanded to include history-dependence. However, the method used here is readily applicable to that case.

(2) The assumption of weak complementarity (u.3) is essential for the way I establish the existence result. Whether the theorem is true without this assumption is an interesting open question.[15]

(3) The boundedness assumption (u.4) is really a generalization of some notion of time-preference or discounting. Consider for a moment an additively separable utility function of the simple form

$$u(c_t, v^t) = w(c_t) + \delta v_{t+1}, \qquad \delta > 0.$$

[13] The assumption that u maps into $\mathbb{R}_+$ is really not a restriction, but only makes the notation easier. What *is* important is assumption (u.2), which rules out unboundedly negative utilities at the origin.

[14] If u is differentiable, this amounts to the nonnegativity of the cross-partial derivatives $\partial^2 u / \partial c_t \, \partial v_s$, $s > t$.

[15] Below (proof of Theorem A) it will be seen that the assumption allows me to restrict attention to spaces of nondecreasing functions, on which I can place a tractable topology that works in the present context.

A necessary and sufficient condition for u to satisfy (u.4) is clearly $\delta < 1$. For a more general (yet additive) form of altruism,

$$u(c_t, v^t) = w(c_t) + \sum_{s=1}^{n} \delta^s v_{t+s}, \qquad \delta > 0;$$

the necessary and sufficient condition is $\sum_{s=1}^{n} \delta^s < 1$.

Now, some definitions. A *(Markov) policy* is a function $s: [0, Y] \to [0, Y]$ with $0 \leqslant s(y) \leqslant y$, $y \in [0, Y]$. An *indirect utility* is a function $v: [0, Y] \to \mathbb{R}$.

We are now ready to define our main concept. Call a pair (v^*, s^*), where v^* is an indirect utility and s^* a policy, an *equilibrium* if for each $y \in [0, Y]$, $s^*(y)$ is the value of x_t that solves the problem

$$\max_{0 \leqslant x_t \leqslant y} u(c_t, v^t) \tag{2}$$

subject to

$$\begin{aligned} y_t &= y && \\ c_s + x_s &= y_s, && s \geqslant t \\ y_{s+1} &= f(x_s), && s \geqslant t \\ v_s &= v^*(y_s), && s \geqslant t+1 \\ x_s &= s^*(y_s), && s \geqslant t+1, \end{aligned} \tag{3}$$

and if

$$v^*(y) = \max_{0 \leqslant x_t \leqslant y} \{u(c_t, v^t), \text{subject to (3)}\}. \tag{4}$$

Call the feasible program obtained from some $y \in [0, Y]$ by applying the equilibrium an *equilibrium path*.

Remarks. (1) Clearly, the definition implicitly states that problem (2) is well defined for all $y \in [0, Y]$. This requirement is handled below in the proof.

(2) Observe that for special forms of altruism, such as the one-step recursion $u(c_t, v_{t+1})$, the last equalities in (3) involving the function s^* are redundant in the constraints of problem (2).

(3) Equilibria may be analogously defined for general history dependent policies and indirect utilities. It can be checked that the present version continues to be a solution in this more general class of functions.

It follows from the definition of an equilibrium that if generations $t+1$, $t+2$,... all "announce" a policy s^* and indirect utility v^*, then it is optimal for generation t to do so. It is in this sense that an equilibrium can be said to depict a form of consistent behaviour. Note again the close analogy between this solution concept and that of (Markov) subgame perfect equilibrium developed for extensive form games.

2a. *Existence*

THEOREM A. *Under* (F.1), (F.2), (u.1)–(u.4), *an equilibrium* (v^*, s^*) *exists. The equilibrium indirect utility* v^* *must be an increasing function. There is an equilibrium policy* s^* *that is nondecreasing.*

Theorem A establishes that the equilibrium concept developed here is not vacuous in a wide class of situations.

The equilibrium indirect utility is increasing in endowment, and this is natural, given (u.1). The fact that there is always a non-decreasing equilibrium *policy* is of greater interest. The result hinges on (u.1) and (u.3), and parallels those obtained in paternalistic models (Kohlberg [18], Bernheim and Ray [6, 9], Leininger [20]). For a discussion of the underlying intuition, see Bernheim and Ray [6].

2b. *Steady State Properties*

The results here depend upon strengthening the complementarity assumption (u.3) to separability:

(u.5) There is $w: \mathbb{R}_+ \to \mathbb{R}_+$ and $W: \mathbb{R}_+^\infty \to \mathbb{R}_+$ such that

$$u(c_t, v^t) = w(c_t) + W(v^t).$$

We can now establish

THEOREM B. *Under* (u.1) *and* (u.5), *if* (v, s) *is an equilibrium, then* s *must be nondecreasing.*

Theorem B parallels, as I said, results for paternalistic models. Similar results have also been useful in the analysis of aggregative optimal growth models with nonconvexities in production (see, e.g., Dechert and Nishimura [14] and Mitra and Ray [24]). Specifically, it yields in a straightforward way the time-monotonicity of equilibrium paths, and in particular, their convergence to some steady state.

To state this, we need some notation and definitions. Call x a *fixed point of* s (*under* f) *if* $x = s \circ f(x)$. For a correspondence Φ: $[0, Y] \Rightarrow [0, Y]$ call x a *fixed point* of Φ (*under* f) if $x \in \Phi \circ f(x)$. And for a nondecreasing policy s, define $\Phi(s)$: $[0, Y] \Rightarrow [0, Y]$ by

$$\Phi(s)(x) = \{\lim_{x' \uparrow x} s(x), \lim_{x' \downarrow x} s(x')\}.$$

We then have

THEOREM C. *Under* (F.1), (u.1), *and* (u.5), *every equilibrium path of* (v, s) *is monotone in stocks, which converge to a fixed point of* $\Phi(s)$ *under* f. *Every fixed point of* s *has the stocks of at least one equilibrium path converging to it. Finally, if* $\langle x, y, c \rangle$ (*resp.* $\langle x', y', c' \rangle$) *are equilibrium paths from* $\mathbf{y}$ (*resp.* $\mathbf{y}'$), *then* $\mathbf{y} \geqslant \mathbf{y}'$ *implies* $x_t \geqslant x'_t$, $y_t \geqslant y'_t$ *for all* t.

Theorem C shows that equilibrium programs have a great deal of structure and in particular, display steady state properties in a fairly general class of situations. Specifically, the result holds for very broad versions of nonpaternalistic altruism, extending to an arbitrary number of descendants.

2c. *Normative Properties*

The most obvious normative question that one can ask of these equilibria is: Are they Pareto-optimal? But we are on slippery ground here, for the lack of Pareto-optimality implies the existence of a feasible program with utilities higher for all concerned. However nothing in the model so far allows us to *define* utility levels for each generation, given *any* feasible program. Utilities are defined on equilibrium programs,[16] but this is not a rich enough set to test for Pareto-optimality with respect to *all* feasible programs, equilibrium or not. One is therefore inexorably led to the following question: What kinds of nonpaternalistic functions have utility representations on the set of all consumption programs, or in short, paternalistic representations? (The reverse link is *also* important for different reasons.)[17]

This question (and its converse) is the subject of future work. Here, I shall use a weaker normative notion, that of consumption-efficiency (see, e.g., Malinvaud [22]). This is a relatively mild efficiency concept that requires that there be no other feasible program with *consumptions* higher at every date. In this model, Pareto-optimality (given that the representation problem is solved) would *imply* consumption efficiency, but not vice versa. It follows, therefore, that a failure of consumption efficiency is indeed a failure of a rather drastic kind.

Here, I provide an example (actually sufficient conditions in a specific model) where equilibrium programs are inefficient. Note at the outset that altruism extends to *more* than one descendant in the example.

[16] Simply associate the indirect utility of a stock that generates a given equilibrium program with the equilibrium program.

[17] One reason is that a class of paternalistic utility functions would be exhibited for which Markov perfect equilibria are guaranteed to exist. One would have to translate the equilibria of the nonpaternalistic model into those of the paternalistic model which it represents.

EXAMPLE (Inefficiency of equilibrium programs). Let

$$u(c_t, v^t) = c_t^\delta v_{t+1}^\beta v_{t+2}^\gamma, \qquad \delta, \beta, \gamma > 0,$$

and

$$f(x) = x^\alpha, \qquad 0 < \alpha < 1.$$

I now claim the following[18]: *There exists an equilibrium* (v^*, s^*), *provided that* $\beta + \gamma \neq 1$ *and* $1 - \alpha[\beta + \alpha\gamma] > 0$, *of the following form*:

$$v^*(y) = by^a,$$

$$s^*(y) = dy,$$

where b, a, d *are positive constants determined by the parameters of the model.*

If $\beta + \alpha\gamma \leqslant 1$, *all equilibrium programs of* (v^*, s^*) *are efficient. But if* $\beta + \alpha\gamma > 1$, *all of them are inefficient.*

Consider this for a moment in the context of a paternalistic model. There, limited altruism is actually a form of myopia, relative to, say, a planner who would consider the *entire* stream of consumptions. However, in a large class of models, this myopia, while it does cause Pareto-inoptimality, does *not* do away with consumption-efficiency (the equilibrium program *under*accumulates stocks, if anything (see Bernheim and Ray [7])). In the nonpaternalistic case, only the simple one-step altruistic formulation $(u(c_t, v^t) = \hat{u}(c_t, v_{t+1}))$ can guarantee optimality properties (see, e.g., Streufert [32]). But caring for any further descendants through their *utilities* results in a tendency to *overaccumulate* stocks.[19] This is the source of the inefficiency.

3. PROOFS

I start with some notation. First take some $\hat{Y} > Y$, and define $M \equiv M(\hat{Y})$, by assumption (u.4). We will deal with functions on $[0, \hat{Y}]$ rather than on $[0, Y]$ to avoid technical difficulties at the upper value of the domain.[20]

[18] One point regarding the example: my assumptions as stated do not cover this case. By (u.1), u must be increasing in c, but this is not satisfied in the example when $v_{t+1} = v_{t+2} = 0$. So a fixed-point argument (the one used in the proof of Theorem A) cannot rule out trivial equilibria of the form $v^*(y) = 0$ in these cases. These equilibria *are* inefficient but I do not push them as serious cases of efficiency failure. The claim below holds for nontrivial equilibria which are explicitly computed.

[19] Specifically, in this example, when the conditions for inefficiency are met, the resulting equilibrium program results in limit stocks that exceed the "golden rule."

[20] These concern the convergence arguments made in the crucial continuity lemma (Lemma 6). The present construction sidesteps these difficulties.

These functions will be given preassigned values of $\hat{Y}$ at $\hat{Y}$ (see below); in Lemma 1, particular restrictions of these to $[0, Y]$ are shown to yield an equilibrium.

Let S^o be the space of all functions s: $[0, \hat{Y}] \to [0, \hat{Y}]$ with $0 \leqslant s(y) \leqslant y$ and $s(\hat{Y}) = \hat{Y}$. Let $S = \{s \in S^o / s$ is uppersemicontinuous (usc) and nondecreasing$\}$. Let K^o be the space of all functions v: $[0, \hat{Y}] \to \mathbb{R}$ with $v(\hat{Y}) = M$, and $K = \{v \in K^o / v$ is usc, nondecreasing with $0 \leqslant v(y) \leqslant M\}$. Let Z^o (resp. Z) be the product $K^o \times S^o$ (resp. $K \times S$); a typical element z is thus of the form (v, s).

For any $z = (v, s) \in Z^o$ and $x \in [0, \hat{Y}]$, define

$$\hat{V}_k(x) \equiv v \circ [f \circ s]^{k-1} \circ f(x), \qquad k \geqslant 1 \tag{5}$$

and

$$\hat{V}(x) \equiv \langle \hat{V}_k(x) \rangle_{k=1}^{\infty}. \tag{6}$$

Say that $z^* = (v^*, s^*) \in Z^o$ *solves* $z = (v, s) \in Z^o$ if, constructing $\hat{V}(x)$ as in (6), for each $y \in [0, \hat{Y})$ the problem

$$\max_{0 \leqslant x \leqslant y} u(y - x, \hat{V}(x)) \tag{7}$$

can be solved to yield

$$v^*(y) = \max_{0 \leqslant x \leqslant y} u(y - x, \hat{V}(x)) \tag{8}$$

and

$$s^*(y) \in \arg\max_{0 \leqslant x \leqslant y} u(y - x, \hat{V}(x)). \tag{9}$$

LEMMA 1. *Consider* $z \in Z^o$ *such that* z *solves* z. *Let* $\tilde{z} = (\tilde{v}, \tilde{s})$ *be the restriction of* z *to* $[0, Y] \times [0, Y]$. *Then* $(\tilde{v}, \tilde{s})$ *is an equilibrium.*

Proof. Observe that the maximization of (7) for $y \in [0, Y]$ is equivalent to the problem in (2). Also, as $y \in [0, Y]$ implies $f(y) \leqslant f(Y) \leqslant Y$, the behavior of v and s outside $[0, Y]$ is irrelevant when (7) is solved for $y \in [0, Y]$. But this yields an equilibrium. Q.E.D.

By Lemma 1, we are done if we exhibit a $z \in Z^o$ that solves itself. This will occupy the rest of the proof.

Our next lemma establishes that if the policy and indirect utility employed by succeeding generations are usc and nondecreasing, then the best response of the current generation can be chosen to have the same properties.

LEMMA 2. *Let $z \in Z$. Then there is a unique $z^* \in Z$ such that z^* solves z.*

Proof. Given $z = (v, s) \in Z$, $\hat{V}_k(\cdot)$ as defined by (5) is usc for each k. Hence the objective function in (7) is usc on a compact set $[0, y]$ (for each $y \in [0, \hat{Y})$), so a maximum exists. Define $v^*(y)$ by (8) for $y \in [0, \hat{Y})$, and $v^*(\hat{Y}) = M$. Clearly, $v^* \in K''$; we will show that $v^* \in K$.

Note that because $v \in K$ and by (u.4), $v^*(y) \leqslant M$, $y \in [0, \hat{Y}]$. Also by (u.1), $v^*(\cdot)$ is clearly nondecreasing (in fact, strictly increasing). Finally, to show that v^* is usc, fix $y \in [0, \hat{Y})$ and take a sequence $y^n \to y$. Let x^n be a sequence of maximizers for (7), for each y^n, and without loss of generality (w.l.o.g.) let $x^n \to x$. Then

$$v^*(y_n) = u(y_n - x_n, \hat{V}(x_n)).$$

Taking limits and using the fact that $\hat{V}$ is usc,

$$\begin{aligned} \lim_{n \to \infty} v^*(y_n) &\leqslant u(y - x, \hat{V}(x)) \\ &\leqslant v^*(y). \end{aligned}$$

The claim that $v^* \in K$ is established.

Define for each $y \in [0, \hat{Y})$,

$$\Phi(y) = \{\arg\max_{0 \leqslant x \leqslant y} u(y - x, \hat{V}(x))\}.$$

We now prove that if $y_1 > y_2$, $x_1 \in \Phi(y_1)$, and $x_2 \in \Phi(y_2)$, then $x_1 \geqslant x_2$. Suppose this is not the case for some $y_1 > y_2$, $x_1 \in \Phi(y_1)$ and $x_2 \in \Phi(y_2)$. Observe that by definition of Φ,

$$u(y_1 - x_1, \hat{V}(x_1)) \geqslant u(y_1 - x_2, \hat{V}(x_2)) \tag{10}$$

and

$$u(y_2 - x_2, \hat{V}(x_2)) \geqslant u(y_2 - x_1, \hat{V}(x_1)). \tag{11}$$

Adding both sides of (10) and (11) and transposing,

$$\begin{aligned} &u(y_1 - x_1, \hat{V}(x_1)) - u(y_2 - x_1, \hat{V}(x_1)) \\ &\quad \geqslant u(y_1 - x_2, \hat{V}(x_2)) - u(y_2 - x_2, \hat{V}(x_2)). \end{aligned} \tag{12}$$

By (u.3), the premise that $x_1 < x_2$, and the fact that $\hat{V}_k(x)$ is nondecreasing (inspect (5)),

$$\begin{aligned} &u(y_1 - x_1, \hat{V}(x_1)) - u(y_2 - x_1, \hat{V}(x_1)) \\ &\quad \leqslant u(y_1 - x_1, \hat{V}(x_2)) - u(y_2 - x_1, \hat{V}(x_2)). \end{aligned} \tag{13}$$

Combining (12) and (13),

$$u(y_1-x_1, \hat{V}(x_2))-u(y_2-x_1, \hat{V}(x_2))$$
$$\geqslant u(y_1-x_2, \hat{V}(x_2))-u(y_2-x_2, \hat{V}(x_2)). \tag{14}$$

But (14) contradicts the strict concavity of u in c, because $y_2-x_1>y_2-x_2$.

Next, let y_n be a sequence in $[0, \hat{Y})$ with $y_n \to y$, $y_n \geqslant y$; all n. Let $x_n \in \Phi(y_n)$. Then if x is a limit point of x_n, I claim that $x \in \Phi(y)$. To show this, assume w.l.o.g. that $x_n \to x$ and proceed by contradiction. Then there is $x^* \in [0, y]$ with

$$u(y-x^*, \hat{V}(x^*))>u(y-x, \hat{V}(x)). \tag{15}$$

By our earlier results, because $y_n \downarrow y$, it must be that $x_n \downarrow x$. So, given that $\hat{V}_k(\cdot)$ is usc, nondecreasing, and therefore rightcontinuous,

$$u(y_n-x_n, \hat{V}(x_n)) \to u(y-x, \hat{V}(x)). \tag{16}$$

Note that because $y_n \downarrow y$, $x^* \in [0, y_n]$. But this, together with (15) and (16) is a contradiction to the fact that $x_n \in \Phi(y_n)$, all n, for with large n,

$$u(y_n-x^*, \hat{V}(x^*))>u(y_n-x_n, \hat{V}(x_n)). \tag{17}$$

This argument establishes in particular that $\Phi(y)$ is compact for each y. Define $s^*(y) \equiv \max \Phi(y)$, $y \in [0, \hat{Y})$, and $s^*(\hat{Y})=\hat{Y}$. By our result on Φ, s^* must be nondecreasing. It remains to show that s^* is usc. To do so, let $y_n \to y$, suppose, w.l.o.g., that $\lim_n s^*(y_n)$ exists, and assume, contrary to the claim, that

$$\lim_n s^*(y_n)>s^*(y). \tag{18}$$

Then it must be the case that for all large n, $y_n>y$ (otherwise nondecreasingness of $s^*(\cdot)$ is violated). But in this case, by the argument above, $\lim_n s^*(y_n) \in \Phi(y)$, and so (18) violates the construction of $s^*(\cdot)$.

It remains to show that $s^*(\cdot)$ is the only usc selection from $\Phi(\cdot)$. Suppose there is some other $\hat{s} \in S$ such that $\hat{s}(y) \in \Phi(y)$, $y \in [0, \hat{Y})$. Then for some $\hat{y} \in [0, \hat{Y})$, $s^*(\hat{y})>\hat{s}(\hat{y})$. Let $y^n \downarrow \hat{y}$; since by assumption, $\hat{s}$ is usc, $\hat{s}(\hat{y}) \geqslant \limsup_n \hat{s}(y^n)$. Thus, there exists N such that $s^*(\hat{y})>\hat{s}(y^N)$ while $y^N>\hat{y}$. But this contradicts the monotonicity property of Φ established earlier.

We have therefore established that $s^* \in S$, and earlier that $v^* \in K$. And by construction, it is clear that $z^*=(v^*, s^*) \in Z$ solves $z=(v, s)$. Q.E.D.

Thus, for all $z \in Z$, we can find a unique $z^* \in Z$ that solves z. So we have a mapping $H: Z \to Z$, which yields, for every $z \in Z$, a "best response" $H(z)$ for the current generation.

Endow the space S and K with the topology of weak convergence (see Billingsley [11, p. 22]) and Z with the product topology.

LEMMA 3. *Every continuous mapping from Z into itself has a fixed point.*

Proof. S and K are compact by Helly's selection theorem (see Billingsley [11, p. 227]) therefore so is Z. Endowed with the weak topology, these are also convex subsets of a locally convex vector space, so Z also has this property. Thus by the Schauder–Tychonoff theorem (Smart [30, p. 15]), Z has the fixed point property. Q.E.D.

Before proceeding to the main continuity result (Lemma 6) we establish

LEMMA 4. *Suppose that $z^n = (v^n, s^n) \to z = (v, s) \in Z$. Define $\hat{V}_k^n$ by (5) for (v^n, s^n), and $\hat{V}_k$ by (v, s) for each k. Then $\hat{V}_k^n(x) \to \hat{V}_k(x)$ at all points of continuity of $\hat{V}_k$.*

Proof. Take any point x such that $\hat{V}_k(x)$ is continuous at x. Then it must be that s is continuous at all points of the form $[f \circ s]^m \circ f(x)$, $m = 0, \ldots, k-2$. Suppose not. Let n be an integer in $0, \ldots, k-2$ such that s is discontinuous at $[f \circ s]^n \circ f(x)$. Then, given f and s are usc and nondecreasing, if $x_q \uparrow x$, $w_q \equiv s \circ [f \circ s]^n \circ f(x_q)$, then $\lim_q w_q < w \equiv s \circ [f \circ s]^n \circ f(x)$. So $\lim_q \hat{V}_k(x_q) \equiv \lim_q v \circ [f \circ s]^{k-n-2} f(w_q) < v \circ [f \circ s]^{k-n-2} f(w) = \hat{V}_k(x)$, using here the fact that f is strictly increasing. This contradicts our supposition.

Given this, it is also easy to show that v must be continuous at $[f \circ s]^{k-1} \circ f(x)$.

The weak convergence of $\hat{V}_k(\cdot)$ now follows directly. Q.E.D.

LEMMA 5. *Let $\langle a_k^n, b_k^n \rangle$ be a double sequence of real-valued, usc, nondecreasing functions on $[0, \hat{Y}]$ such that for each k, $a_k^n \to a_k$, $b_k^n \to b_k$ weakly, where a_k, b_k are also usc and nondecreasing, $k \geqslant 1$. Let x^*, $y^* \in [0, \hat{Y})$, $x^* \leqslant y^*$, be given. Then there is a sequence $\langle x_q, y_q \rangle_1^\infty$ with $x_q \downarrow x^*$, $y_q \downarrow y^*$, $0 \leqslant x_q \leqslant y_q$, and a subsequence n_q of n such that, as $q \to \infty$,*

$$a_k^{n_q}(x_q) \to a_k(x^*), \qquad k \geqslant 1. \tag{19}$$

$$b_k^{n_q}(y_q) \to b_k(y^*), \qquad k \geqslant 1. \tag{20}$$

Proof. By the assumptions of the lemma, each a_k, b_k has at most countably many discontinuities. So the a_k's *together* have at most countably many discontinuities on $[0, \hat{Y}]$ and so have the b_k's. We can therefore pick a sequence from the continuity points of the a_k's, x_q, with $x_q \downarrow x^*$. Likewise, pick a sequence $y_q \downarrow y^*$, on the continuity points of the b_k's, taking care that $y_q \geqslant x_q$, all q.

Perform a recursive construction as follows. Define $n(0, q) = n(k, 0) = 1$ for all $(k, q) \geqslant 0$. For $(k, q) \geqslant 1$, suppose that $n(s, t)$ is defined for all $s = 0, ..., k-1$ and $t = 0, ..., q-1$. Then define $n(k, q)$ with $n(k, q) > n(k-1, q)$ and $n(k, q) > n(k, q-1)$ so that for all $n \geqslant n(k, q)$,

$$|a_k^n(x_q) - a_k(x_q)| \leqslant \frac{1}{q},$$

and

$$|b_k^n(y_q) - b_k(y_q)| \leqslant \frac{1}{q}. \tag{21}$$

Define $n_q \equiv n(q, q)$. Then $n_q = n(q, q) > n(q, q-1) > n(q-1, q-1) = n_{q-1}$, so that n_q is a well-defined subsequence of n. Now fix k. Then for large q, $n_q = n(q, q) > n(k, q)$. So, by the construction of $n(k, q)$,

$$|a_k^{n_q}(x_q) - a_k(x_q)| \leqslant \frac{1}{q} \to 0 \qquad \text{as} \quad q \to \infty. \tag{22}$$

Now,

$$|a_k^{n_q}(x_q) - a_k(x^*)| \leqslant |a_k^{n_q}(x_q) - a_k(x_q)| + |a_k(x_q) - a_k(x^*)|.$$

It is easy to check that a_k is right-continuous, given the assumptions of the lemma. Using this and (22), the RHS of the inequality above tends to 0 as $q \to \infty$. A similar argument holds for the b_k's, so we are done. Q.E.D.

It remains to establish the continuity of H with respect to the topology on Z. This is done in

LEMMA 6. *$H: Z \to Z$ is continuous.*

Proof. Let $z^n \to z$ in Z. W.l.o.g., let $\hat{z}^n \equiv H(z^n) \to \hat{z}$ in Z. We are to show that $\hat{z} = H(z)$. Using (5) and (6), define $\hat{V}_k^n$ and $\hat{V}^n$ by the sequence z^n, and $\hat{V}_k$, $\hat{V}$ by z. Let $\hat{z} = (\hat{v}, \hat{s})$; first, we show that $\hat{s}(y)$ solves (7) for all $y \in [0, \hat{Y})$, given z.

Let s^* be the unique policy in S that solves (7), given z. It clearly suffices to show that $s^*(y) = \hat{s}(y)$ at all points of continuity of the former, given that s^* and $\hat{s}$ are both usc and nondecreasing. Suppose there is some $y^* \in [0, \hat{Y})$ such that $s^*(y^*) \neq \hat{s}(y^*)$. Then, letting $x^* \equiv s^*(y^*)$,

$$u(y^* - x^*, \hat{V}(x^*)) > u(y^* - \hat{s}(y^*), \hat{V}(\hat{s}(y^*))), \tag{23}$$

the strict inequality in (19) following from the fact that if s^* is continuous at y^*, $\Phi(y^*)$ must be a singleton, otherwise the monotonicity property of Φ established in the proof of Lemma 2 will be violated.

Now regard $\hat{V}_k^n$ as the sequence a_k^n in Lemma 5 with limit $\hat{V}_k = a_k$. Similarly, identify $(\hat{s}^n, \hat{V}_k^n \circ \hat{s}^n)$ with the sequence b_k^n, and their limits with the b_k's. Given x^*, y^*, and Lemma 4, all the conditions of Lemma 5 are satisfied. So there is a sequence $(x_q, y_q) \downarrow (x^*, y^*)$ and a subsequence n_q such that

$$\hat{V}_k^{n_q}(x_q) \to \hat{V}_k(x^*), \qquad k \geqslant 1, \tag{24}$$

$$\hat{V}_k^{n_q}(\hat{s}^{n_q}(y_q)) \to \hat{V}_k(\hat{s}(y^*)), \qquad k \geqslant 1, \tag{25}$$

and

$$\hat{s}^{n_q}(y_q) \to \hat{s}(y^*). \tag{26}$$

Combining (23)–(26) and using the continuity of u, we have, for large q,

$$u(y_q - x_q, \hat{V}^{n_q}(x_q)) > u(y_q - \hat{s}^{n_q}(y_q), \hat{V}^{n_q}(\hat{s}^{n_q}(y_q))), \tag{27}$$

but (27) contradicts the fact that $\hat{s}^{n_q}$ solves (7) for each $y \in [0, \hat{Y})$, given $z^{n_q} = (v^{n_q}, s^{n_q})$. This establishes that $\hat{s} = s^*$.

Finally, we must prove that $\hat{v}$ is the indirect utility arising from the maximization of (7), given (v, s).

As a first step, I claim that if $x_n \to x$ in $[0, \hat{Y})$, then for each k,

$$\limsup_n \hat{V}_k^n(x_n) \leqslant \hat{V}_k(x). \tag{28}$$

Suppose that this is not true for some k, x, and $x_n \to x$. W.l.o.g., let $\hat{V}_k^n(x_n)$ converge; then there is N and $\delta > 0$ such that

$$\hat{V}_k^n(x_n) - \hat{V}_k(x) \geqslant \delta, \qquad n \geqslant N. \tag{29}$$

Using the right continuity of $\hat{V}_k$, pick $x' > x$ such that x' is a point of continuity of $\hat{V}_k$, and

$$\hat{V}_k(x') - \hat{V}_k(x) < \delta/2. \tag{30}$$

Now pick $m \geqslant N$ such that $x_m < x'$ and

$$\hat{V}_k^m(x') - \hat{V}_k(x') < \delta/2. \tag{31}$$

Combining (29)–(31), we have

$$\begin{aligned}\hat{V}_k^m(x') - \hat{V}_k^m(x_m) &= (\hat{V}_k^m(x') - \hat{V}_k(x')) + (\hat{V}_k(x') - \hat{V}_k(x)) \\ &\quad + (\hat{V}_k(x) - \hat{V}_k(x_m)) < \delta/2 + \delta/2 - \delta = 0,\end{aligned}$$

which contradicts the fact that $\hat{V}_k^m$ is nondecreasing. So (28) must be true.

Denote by v^* the indirect utility from (8), given (v, s). It suffices to show that $v^* = \hat{v}$ at all points of continuity of the former.

Observe first that for any $y \in [0, \hat{Y})$, using (28),

$$\begin{aligned}\limsup_{n \to \infty} \hat{v}^n(y) &= \limsup_{n} u(y - x_n, \hat{V}_k^n(x^n)) \\ &\leqslant u(y - x, \hat{V}(x)) \\ &\leqslant v^*(y), \end{aligned} \tag{32}$$

where the x_n's solve (7) for z^n, given y.

Let y^* be a continuity point of $\hat{v}$. W.l.o.g. let $v^n(y^*)$ converge. Suppose

$$\hat{v}(y^*) = \lim_{n \to \infty} \hat{v}^n(y^*) < v^*(y^*). \tag{33}$$

Let $x^* = \hat{s}(y^*)$. By Lemma 5, there is a sequence $y_q \downarrow y^*$, $x_q \downarrow x^*$, $x_q \leqslant y_q$, and a subsequence n_q of n such that

$$\hat{V}_k^{n_q}(x_q) \to \hat{V}_k(x^*), \qquad k \geqslant 1 \tag{34}$$

$$\hat{v}^{n_q}(y_q) \to \hat{v}(y^*). \tag{35}$$

So, using (34),

$$u(y_q - x_q, \hat{V}^{n_q}(x_q)) \to v^*(y^*). \tag{36}$$

But then for large q, using (33), (35), and (36)

$$u(y_q - x_q, \hat{V}^{n_q}(x_q)) > \hat{v}^{n_q}(y_q), \tag{37}$$

which contradicts the definition of $\hat{v}^n$. Q.E.D.

Proof of Theorem A. By Lemmas 3 and 6, there is $(v^*, s^*) \in Z$ such that $(v^*, s^*) = H(v^*, s^*)$; i.e., such that (v^*, s^*) solves itself. Denote by $(\tilde{v}, \tilde{s})$ the restriction of (v^*, s^*) to $[0, Y] \times [0, Y]$. Then by Lemma 1, $(\tilde{v}, \tilde{s})$ is an equilibrium. And $(\tilde{v}, \tilde{s})$ clearly meets the other requirements of Theorem A. Q.E.D.

Proof of Theorem B. Let (v, s) be an equilibrium. We show that s must be nondecreasing. Pick y_1, y_2 in $[0, Y]$ with $y_1 > y_2$. For $i = 1, 2$, $j \neq i$,

$$w(y_i - s(y_i)) + W(\hat{V}(s(y_i))) \geqslant w(y_i - s(y_j)) + W(\hat{V}(s(y_j))). \tag{38}$$

Summing the two inequalities in (38) and rearranging,

$$w(y_1 - s(y_1)) - w(y_2 - s(y_1)) \geqslant w(y_1 - s(y_2)) - w(y_2 - s(y_2)). \tag{39}$$

Given $y_1 > y_2$ and w strictly concave, (39) is satisfied only if $s(y_1) \geqslant s(y_2)$. Q.E.D.

Proof of Theorem C. Let (v, s) be an equilibrium. Fix $\mathbf{y} \in [0, Y]$. The sequence of outputs then generated is given by

$$y_0 = \mathbf{y} \tag{40}$$

$$y_{t+1} = f \circ s(y_t), \qquad t \geqslant 0. \tag{41}$$

We claim that either $y_t \leqslant y_{t+1}$, all t, or $y_t \geqslant y_{t+1}$, all t. Suppose $y_0 \leqslant y_1$. Then by Theorem B, $s(y_0) \leqslant s(y_1)$. So using (41) and (F.1), $y_1 \leqslant y_2$. Extend this argument for all t. A similar line of reasoning applies if $y_0 \geqslant y_1$.

So $\langle y_t \rangle$ is a monotone bounded sequence. Consequently $\langle x_t \rangle$ is a monotone bounded sequence, converging to x^*, say. We have

$$x_{t+1} = s \circ f(x_t), \qquad t \geqslant 0. \tag{42}$$

Passing to the limit in (42) easily yields that x^* is a fixed point of $\Phi(s)$ under f.

Now let $\hat{x}$ be a fixed point of s under f. Take $\mathbf{y} \in s^{-1}(\hat{x})$. The resulting equilibrium program trivially converges to $\hat{x}$.

Finally, let $\langle x, y, c \rangle$ (resp. $\langle x', y', c' \rangle$ be an equilibrium path from $\mathbf{y}$ (resp. $\mathbf{y}'$), and let $\mathbf{y} \geqslant \mathbf{y}'$. Then the fact that s is nondecreasing can be easily shown to imply $x_t \geqslant x'_t$, $y_t \geqslant y'_t$ for all t. Q.E.D.

Proof of the Claim in Section IIc. Suppose that all succeeding generations employ (v, s), where

$$v(y) = by^a, \qquad b, a > 0,$$

and

$$s(y) = dy, \qquad 1 > d > 0,$$

for all $y \in [0, Y]$. The current generation then solves problem (2). This can be simply written, for each $y \in [0, Y]$, as

$$\max_{0 \leqslant x \leqslant y} (y - x)^{\delta} (bx^{\alpha a})^{\beta} \{b(dx^{\alpha})^{\alpha a}\}^{\gamma} \tag{43}$$

which is equivalent to

$$\max_{0 \leqslant x \leqslant y} (y - x)^{\delta} x^{\eta}, \qquad \text{where} \quad \eta = \alpha a(\beta + \alpha\gamma). \tag{44}$$

First-order conditions for (44) are clearly necessary and sufficient. These are

$$\delta(y - x)^{\delta - 1} x^{\eta} = \eta(y - x)^{\delta} x^{\eta - 1}$$

or, simplifying,

$$\frac{x}{y} = \frac{\eta}{\eta + \delta} \equiv \hat{d}. \tag{45}$$

So the optimal savings policy for the current generation is $s^*(y) = \hat{d}y$, with $\hat{d}$ given by (45).

Now we calculate $v^*(y)$, the indirect utility that results from (43). This is easily computed to be

$$\begin{aligned} v^*(y) &= \left\{ \left[\frac{\delta}{\eta + \delta}\right]^{\delta} b^{\beta + \gamma} d^{a\alpha\gamma} \left[\frac{\eta}{\eta + \delta}\right]^{\eta} \right\} y^{\eta + \delta} \\ &\equiv \hat{b} y^{\hat{a}}. \end{aligned} \tag{46}$$

To compute the equilibrium, we set $(v^*, s^*) = (v, s)$; i.e.,

$$d = \hat{d}, \qquad b = \hat{b}, \qquad \text{and} \qquad a = \hat{a}. \tag{47}$$

These yield

$$d = \frac{\eta}{\eta + \delta} \tag{48}$$

$$b^{1 - \beta - \gamma} = \left[\frac{\delta}{\eta + \delta}\right]^{\delta} \left[\frac{\eta}{\eta + \delta}\right]^{\alpha a \gamma + \eta} \tag{49}$$

and

$$a = \eta + \delta = a\alpha[\beta + \alpha\gamma] + \delta. \tag{50}$$

Equation (49) has a solution for b provided $\beta + \gamma \neq 1$. For (50) to yield a solution with $a > 0$, it must be that $\alpha[\beta + \alpha\gamma] < 1$. These are both given in the statement of the claim.

Using (50), we obtain from (48) that

$$d = \alpha[\beta + \alpha\gamma]. \tag{51}$$

It is now easily seen, either by direct computation, or by applying the Cass criterion for efficiency of a feasible program (Cass [12]), that the equilibrium program is inefficient if and only if $\beta + \alpha\gamma > 1$.

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