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EFFICIENT AND OPTIMAL PROGRAMS WHEN INVESTMENT IS IRREVERSIBLE

A Duality Theory

Tapan MITRA and Debraj RAY*

Cornell University, Ithaca, NY 14853, USA

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An attempt is made in this paper to formulate a satisfactory duality theory of efficient and optimal programs in intertemporal models with irreversible inveatment. The introduction of the constraint that depreciated capital stock cannot be used for present consumption makes the meaningful choice and interpretation of dual variables a more difficult problem, as is pointed out by means of an example. A new definition of a competitive program is introduced, and this is seen to lead to useful characterizations of efficient and optimal programs.

1. Introduction

In this paper, an attempt is made to formulate a satisfactory 'duality' theory of efficient and optimal programs in intertemporal models with irreversible investment. While the introduction of a 'depreciation constraint', as the requirement that depreciated capital stock cannot be used for present consumption, does not destroy the usual convexity properties of the model, it does make the meaningful choice and interpretation of dual variables a more difficult problem. These difficulties are pointed out by means of an example (see Example 4.1). A new definition of a competitive program, which we feel to be more appropriate in this situation, is introduced [see the relations (2.4), (2.4′) and (2.5)]. This concept is discussed in some detail below (section 4), and leads to a useful characterization of optimal programs.

A brief review of the literature will serve to place this paper in perspective. Two of the major themes in intertemporal economics have been the analysis of efficient and optimal consumption programs. Consider, first, the study of efficient programs. The first important issue in this area is a 'direct' characterization of efficient programs, motivated by the need to construct a test for the efficiency (or inefficiency) of a given feasible program by studying

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the feasible program *alone*. Complete characterizations of this kind have been obtained in models (where investment is not irreversible) by Cass (1972), Benveniste (1976), and Mitra (1979b), to name a few. A complete characterization (of the 'direct' kind) in a model with irreversibility of investment was obtained by Mitra (1978).

A second issue in the study of efficient programs was motivated by the need to study the qualitative properties of these programs. In particular, the following question may be asked. Suppose that a sequence of 'prices' can be proposed for a given feasible program, so that a 'competitive producer' imitating the input–output sequence along this program maximizes profit at each date relative to all feasible input–output pairs. If a feasible program 'maximizes' (loosely speaking) the total value of consumption, evaluated at these prices, relative to all other feasible programs, call it consumption-value-maximizing. Now, does the set of efficient programs coincide with that of the consumption-value-maximizing programs?

An affirmative answer, in the case where investment is 'reversible', was obtained by Cass and Yaari (1971). When investment is irreversible, however, this result does *not* hold, in general. Example 3.1 describes an efficient program which is not consumption-value-maximizing. Theorems 3.1 and 3.2 go on to provide a complete characterization of efficient programs, using the properties of consumption-value-maximization and an additional condition. While the new sequence of prices is used to arrive at the result, it is not at all necessary in the analysis of efficient programs *per se*. In fact, the 'traditional' definition of competitive prices agrees with the new definition proposed in section 2. However, this is not the case in the study of optimal programs, to which we now turn.

Some of the major issues[1] in the theory of optimal allocation are (a) the monotonicity of optimal stocks with respect to a change in the initial or final stocks, and the 'sensitivity' analysis of large but finite-horizon optimal programs, (b) the 'turnpike' properties of infinite-horizon optimal programs, and (c) the characterization of optimal programs using appropriate dual variables (the so-called 'price-characterization of optimal programs').

The issues summarized under (a) were first analyzed by Brock (1971), in the case where investment is reversible. These results extend (with some modifications) to the irreversibility model; the reader may consult Majumdar and Nermuth (1981) and Mitra (1981) for the relevant details. The 'turnpike' issues in (b) are too familiar to be restated here; it suffices to mention their extension to the irreversibility case by Majumdar and Nermuth (1981).

This paper concentrates on the problems raised by (c), and provides a first step towards a duality theory for the case of irreversible investment. This is

[1]We could, of course, add to this list. An important question is that of the existence of optimal programs. For some results in the case or irreversible investment, see Majumdar and Nermuth (1981).

done in section 5. Example 4.1 is constructed to show that the traditional definition of competitive prices does not capture the nature of optimal programs. A pair of theorems (4.1 and 4.2) provide a complete characterization of finite-horizon optimal programs using an alternative definition of competitive prices. Theorems 4.3 and 4.4 characterize the infinite-horizon case with varying utility and production functions (albeit incompletely); and the results are tightened in subsequent theorems to a complete characterization in the 'stationary' case of invariant utility and production functions. The characterization is broadly in terms of competitiveness and efficiency, an approach first used by Brock (1971). An alternative characterization of optimal programs (in the 'stationary' case) in terms of the competitive conditions and an additional transversality condition is also provided.

2. The feasible set

In this section, we shall spell out the features of our model, and define some terms.

The *technology* is given by a sequence $\langle g_t \rangle_0^\infty$ of *net-output functions* and a sequence $\langle \delta_t \rangle_0^\infty$ of *depreciation rates*. Throughout the paper, these will be assumed to satisfy

(T.1) $g_t: \boldsymbol{R}_+ \to \boldsymbol{R}_+, \qquad t \geqq 0,$

(T.2) g_t increasing, continuous and concave, $\qquad t \geqq 0,$

(T.3) $0 \leqq \delta_t \leqq 1, \qquad t \geqq 0.$

Additional assumptions will be made as we go along. Oberve that the sequence $\langle g_t, \delta_t \rangle_0^\infty$ implies a sequence of gross-output or *production functions*, given by

$$f_t(x) = g_t(x) + \delta_t x, \quad x \geqq 0 \quad \text{for} \quad t \geqq 0. \tag{2.1}$$

Throughout, x will denote (capital) stocks, z will denote (gross) investment, and c consumption. Subscripts refer to time dates. The initial stock $\boldsymbol{x} > 0$ is given. Consumption commences at date 1.

A *feasible program* $\langle x, c, z \rangle$ (or briefly, $\langle x, c \rangle$) is a sequence $\{(x_t)_0^\infty, (c_t)_1^\infty, (z_t)_1^\infty\}$ satisfying

$$x_0 = \boldsymbol{x},$$

$$z_{t+1} = x_{t+1} - \delta_t x_t, \qquad t \geqq 0,$$

$$c_{t+1}=f_t(x_t)-x_{t+1}=g_t(x_t)-z_{t+1}, \qquad t\geqq 0, \tag{2.2}$$

$$(x_t, c_{t+1}, z_{t+1})\geqq 0, \qquad t\geqq 0.$$

The requirement $z_t \geqq 0$ captures the *irreversibility of investment.*[2]

In one of the sections, we shall be concerned with finite-horizon feasible programs. We define a T-period feasible program with final stocks $b\geqq 0$, or briefly, a *T-program to b* as a sequence $\{(x_t)_0^T, (c_t)_1^T, (z_t)_1^T\}$ satisfying (2.2) for $t=0,\ldots, T-1$, and, in addition

$$x_T \geqq b. \tag{2.2$'$}$$

The *welfare objectives* of the planner, or of society, is given by a sequence $\langle u_t\rangle_1^\infty$ of *utility functions*. Throughout the paper, these will be taken to satisfy

(U.1) $u_t\colon \boldsymbol{R}_+ \to \boldsymbol{R}, \qquad t\geqq 1,$[3]

(U.2) u_t increasing, continuous and concave, $\qquad t\geqq 1.$

Additional assumptions will be made when necessary. In a later section the sequence $\langle u_t\rangle_1^\infty$ will be taken as $u_t=\rho^t u$, $t\geqq 1$, where $0<\rho\leqq 1$ is the *discount factor*, and u satisfies (U.1) and (U.2).

We now introduce some central terms. A feasible program $\langle x, c\rangle$ is *inefficient* if there exists a feasible program $\langle x', c'\rangle$ such that $c'_t \geqq c_t$ for all $t\geqq 1$, with $c'_s > c_s$ for some $s\geqq 1$. Otherwise, the feasible program $\langle x, c\rangle$ is said to be *efficient.*

A feasible program $\langle x, c\rangle$ is *optimal* if for every feasible program $\langle x', c'\rangle$

$$\liminf_{T\to\infty} \sum_{t=1}^{T} [u_t(c'_t)-u_t(c_t)]\leqq 0. \tag{2.3}$$

A T-program $\langle x, c\rangle$ to $b\geqq 0$ is said to be *optimal* if for every feasible T-program to b, $\langle x', c'\rangle$,

$$\sum_{t=1}^{T} u_t(c'_t)\leqq \sum_{t=1}^{T} u_t(c_t). \tag{2.3$'$}$$

[2]Below, we shall refer occasionally to a model where $z_t\geqq 0$ as an *irreversibility* model. The same framework, but with z_t not constrained to be non-negative, will be referred to as a reversibility model.

[3]This means that $u_t(0)$ is finite. The case $u_t(0)=-\infty$ can be incorporated in our analysis by allowing the range of the utility functions to be the extended real line. Apart from some minor changes in the steps of the proofs, the analysis remains unaffected.

A feasible program $\langle x,c\rangle$ is said to be *competitive* (*in production*) if there exists a sequence of non-null, non-negative *prices* $\langle p_t\rangle_1^\infty$, $\langle r_t\rangle_0^\infty$ such that

(a) $p_{t+1}g_t(x_t)+r_{t+1}\delta_t x_t - r_t x_t \geqq p_{t+1}g_t(x)+r_{t+1}\delta_t x - r_t x$ for all $x\geqq 0$, (2.4)

(b) $p_{t+1}\geqq r_{t+1}$ for all $t\geqq 0$,

(c) $p_{t+1}>r_{t+1}$ implies $x_{t+1}=\delta_t x_t$, $t\geqq 0$.

A T-program to b, $\langle x,c\rangle$ is *competitive* (*in production*) if there exists a sequence of non-null, non-negative prices $\langle p_t\rangle_1^T$, $\langle r_t\rangle_0^T$, such that (2.4) is satisfied for $t=0,\dots,T-1$, and in addition,[4]

$$r_T(x_T-b)=0. \tag{2.4'}$$

A feasible program $\langle x,c\rangle$ is competitive (*in production and utility*) if there exists a sequence of non-null, non-negative prices $\langle p_t\rangle_1^\infty$, $\langle r_t\rangle_0^\infty$, satisfying (2.4), and in addition,

$$u_t(c_t)-p_t c_t \geqq u_t(c)-p_t c, \qquad t\geqq 1. \tag{2.5}$$

A T-program to b, $\langle x,c\rangle$, is competitive (*in production and utility*) if there exists a sequence of non-null, non-negative prices $\langle p_t\rangle_1^T$, $\langle r_t\rangle_0^T$, satisfying (2.4) for $t=0,\dots,T-1$, (2.4′) and (2.5) for $t=1,\dots,T$.

Often, when no confusion is possible, we shall speak of *competitive programs* when referring to programs competitive in production, or programs competitive in production and utility.

Remark. In a model where production functions are differentiable and $\delta_t>0$ for all t, we can simply define (noting that all feasible programs must have $x_t>0$) $p_t=r_t=q_t$ for $t\geqq 1$, $r_0=q_0$, where $\langle q_t\rangle_0^\infty$ is constructed as

$$\begin{aligned} &q_0=K \quad \text{for some} \quad K>0,\\ &q_{t+1}=q_t/f'(x_t), \qquad t\geqq 0. \end{aligned} \tag{2.6}$$

The sequence $\langle p_t\rangle_1^\infty$, $\langle r_t\rangle_0^\infty$ constructed in this way is easily seen to satisfy (2.4), and so all infinite-horizon feasible programs are seen to be competitive in production. However, the presence of double sequences $\langle p_t\rangle_1^\infty$, $\langle r_t\rangle_0^\infty$ become crucial for a satisfactory duality theory of optimal programs under

[4]This condition is, of course, a familiar transversality condition, but deserves special attention since $x_T>b$ is possible even for optimal programs.

irreversible investment. This we shall see below. An analogous definition is therefore used for programs competitive in production, to facilitate an easier analysis of the interactions between efficient and optimal programs.

A feasible program $\langle x, c \rangle$ is said to be *regular* if $x_t > 0$ for all $t \geqq 0$, and

$$\liminf_{t \to \infty} z_t/x_t > 0. \tag{2.7}$$

A feasible program $\langle x, c \rangle$ which is competitive (in production, or in production and utility) with prices $\langle p_t \rangle_1^\infty$, $\langle r_t \rangle_0^\infty$ is *consumption-value-maximizing* if

$$\liminf_{T \to \infty} \sum_{t=1}^{T} p_t(c_t' - c_t) \leqq 0 \quad \text{for all feasible programs} \quad \langle x', c' \rangle. \tag{2.8}$$

Finally, some additional notation. Recall that for a concave function $h(x)$ defined on $\boldsymbol{R}^+$, the *right-hand derivative* is defined for $x \geqq 0$, and this will be denoted by $h^+(x)$. Moreover, the *left-hand derivative* is defined for $x > 0$, and this will be denoted by $h^-(x)$. Clearly, $h^-(x) \geqq h^+(x)$ for all $x > 0$, and if $x < y$, then $h^+(x) \geqq h^-(y)$. When h is differentiable we shall denote its derivative by h'; when twice differentiable, its second derivative by h''.

3. Consumption-value-maximization and efficiency

One of the intuitive implications of efficiency is the feature of consumption-value-maximization, i.e., efficient programs should 'maximize' consumption value, or, precisely, in the sense of (2.8), not be 'overtaken' in value by another feasible program. The converse statement is also worthy of conjecture: a feasible program which is consumption-value-maximizing is efficient.

Cass and Yaari (1971) proved, in a model where investment is *not* irreversible, that efficient programs coincide exactly with the set of consumption-value-maximizing programs. In this section, we show by means of an example that this complete characterization of efficient programs by means of consumption-value-maximizing properties fails to hold when investment is irreversible. This is followed by two theorems completely characterizing efficient programs using the consumption-value-maximizing property and an additional condition.

Example 3.1. (*Efficiency need not imply consumption-value-maximization*)

Let $g_t = g$ for all $t \geqq 0$, $\delta_t = \delta \in (0, 1)$, $t \geqq 0$. Let g be differentiable and strictly concave, and let $f(x) = g(x) + \delta x$ have the following properties:

(i) there exists $x^*>0$ such that $f'(x^*)=1$,

(ii) there exists $\hat{x}>0$ such that $f(\hat{x})=\hat{x}$,

(iii) there exists $\bar{x}$, $\underline{x}$ such that

(a) $\hat{x}>\bar{x}>\underline{x}>x^*$,

(b) $f(x^*)\geqq\underline{x}$,

(c) $f(\underline{x})=\bar{x}$, $\delta\bar{x}=\underline{x}$ and $[f'(\bar{x})+1][f(x^*)-x^*]>[f(\bar{x})-\underline{x}]$.

Such functions exist. For instance, consider $g(x)=x^{1/8}$, $\delta=\frac{1}{2}$, $\bar{x}=2(\frac{2}{3})^{8/7}$, $\underline{x}=(\frac{2}{3})^{8/7}$, $x^*=(\frac{1}{4})^{8/7}$, $\hat{x}=(2)^{8/7}$.

Let the initial stock be given by $\mathbf{x}=x^*$. Define a program $\langle x,c\rangle$ by $x_0=\mathbf{x}$, $x_t=\underline{x}$ if t is odd, $x_t=\bar{x}$ if t is even, for all $t\geqq 1$, $c_{t+1}=f(x_t)-x_{t+1}$ for all $t\geqq 0$. It is easy to check that (i) $(c_t,z_t)\geqq 0$ for all $t\geqq 1$, and (ii) $z_t=0$ for all $t>2$ and odd. By (i), $\langle x,c\rangle$ is feasible. It follows from (ii) and Proposition (3.1) in Mitra (1978) that $\langle x,c\rangle$ is efficient.

Clearly the sequence $r_0=1$, $p_t=r_t=r_{t-1}/f'(x_{t-1})$, $t>1$, form competitive prices for $\langle x,c\rangle$. [These are also the prices used by Cass and Yaari]. Define $a\equiv 1/f'(\underline{x})$, $b\equiv 1/f'(\bar{x})$. Clearly, $b>a>1$. Then $p_1=1$, and for $t>1$ and even, $p_t=(1/b)a^{t/2}b^{t/2}$. For $t>1$ and odd, $p_t=(1/ab)^{\frac{1}{2}}a^{t/2}b^{t/2}$. Now $c_1=f(x^*)-\underline{x}$. For $t>1$ and even, $c_t=0$. For $t>1$ and odd, $c_t=f(\bar{x})-\underline{x}$.

Define a feasible program $\langle x',c'\rangle$ by $x'_t=x^*$ for all $t\geqq 0$, $c'_t=f(x'_{t-1})-x'_t$ for all $t\geqq 1$.

Now,

$$p_1[c'_1-c_1]=f(x^*)-x^*-f(x^*)+\underline{x}=\underline{x}-x^*>0,$$

$$p_2[c'_2-c_2]=a[f(x^*)-x^*]>0.$$

For $t\geqq 2$ and odd,

$$E_t\equiv p_t[c'_t-c_t]+p_{t+1}[c'_{t+1}-c_t]$$

$$=\left(\frac{1}{ab}\right)^{\frac{1}{2}}a^{t/2}b^{t/2}[\{f(x^*)-x^*\}-\{f(\bar{x})-\underline{x}\}]+\frac{1}{b}a^{(t+1)/2}b^{(t+1)/2}[f(x^*)-x^*]$$

$$=\left(\frac{1}{ab}\right)^{\frac{1}{2}}a^{t/2}b^{t/2}[\{f(x^*)-x^*\}-\{f(\bar{x})-\bar{x}\}-\{\bar{x}-\underline{x}\}]+\left(\frac{a}{b}\right)^{\frac{1}{2}}a^{t/2}b^{t/2}[f(x^*)-x^*]$$

$$\geqq\left(\frac{a}{b}\right)^{\frac{1}{2}}a^{t/2}b^{t/2}[f(x^*)-x^*]-\left(\frac{1}{ab}\right)^{\frac{1}{2}}a^{t/2}b^{t/2}[\bar{x}-\underline{x}]$$

$$\geqq \left(\frac{1}{ab}\right)^{\frac{1}{2}} a^{t/2}b^{t/2}[\{f(x^*)-x^*\}-\{\bar{x}-\underline{x}\}]=\left(\frac{1}{ab}\right)^{\frac{1}{2}} a^{t/2}b^{t/2}[\{f(x^*)-x^*\}-\{f(\underline{x})-\underline{x}\}]$$

$$\geqq \varepsilon\left(\frac{1}{ab}\right)^{\frac{1}{2}} a^{t/2}b^{t/2} \quad \text{where} \quad \varepsilon>0.$$

Similarly, $E_t \geqq \varepsilon' \geqq 0$ for $t \geqq 2$ and even.

Hence

$$\liminf_{T\to\infty} \sum_{t=1}^{T} p_t(c_t'-c_t)=\infty,$$

so that the efficient program $\langle x,c\rangle$ is not consumption-value-maximizing.

It should be clear from the example that additional conditions are required if one is to completely characterize efficient programs in a framework of consumption-value-maximization. Before presenting the main theorems which accomplish this, we note a variant of the basic Cass lemma,[5] stated for the case of irreversible investment.

Lemma 3.1. Let $\langle x,c\rangle$ be a feasible program. Suppose that there exists a positive integer T and a sequence $(\varepsilon_t)_T^\infty$ satisfying

$$\begin{aligned} &\text{(a)} \quad 0<\varepsilon_t \leqq z_t, \quad t\geqq T,\\ &\text{(b)} \quad \varepsilon_{t+1} \geqq f_t(x_t)-f_t(x_t-\varepsilon_t), \quad t \geqq T. \end{aligned} \tag{3.1}$$

Then $\langle x,c\rangle$ is inefficient.

Proof. Define a program $\langle x',c'\rangle$ by $x_t'=x_t$, $0\leqq t\leqq T-1$, $x_t'=x_t-\varepsilon_t$, $t\geqq T$, $c_{t+1}'=f_t(x_t')-x_{t+1}'$, $t\geqq 0$.

Observe that for all $t \geqq 0$,

$$x_{t+1}'-\delta_t x_t' = x_{t+1}-\delta_t x_t'-\varepsilon_{t+1} \geqq x_{t+1}-\delta_t x_t-\varepsilon_{t+1}=z_{t+1}-\varepsilon_{t+1}\geqq 0.$$

Hence $z_t' \geqq 0$ for all $t \geqq 1$.

Also,

$$c_T'=f_{T-1}(x_{T-1}')-x_T'=f_{T-1}(x_{T-1})-x_T+\varepsilon_T=c_T+\varepsilon_T>c_T.$$

[5]See Cass (1972).

For $t \leqq T-1$, $c'_T = c_T$. For $t \geqq T$,

$$c'_{t+1} = f_t(x'_t) - x'_{t+1} = f_t(x_t - \varepsilon_t) - x_{t+1} + \varepsilon_{t+1} = f_t(x_t - \varepsilon_t) - f_t(x_t) + c_{t+1} + \varepsilon_{t+1}.$$

So

$$c'_{t+1} - c_{t+1} = \varepsilon_{t+1} - [f_t(x_t) - f_t(x_t - \varepsilon_t)] \geqq 0.$$

Hence, $\langle x', c' \rangle$ is a feasible program satisfying $c'_t \geqq c_t$, $t \geqq 1$, and $c'_T > c_T$. Therefore, $\langle x, c \rangle$ is inefficient.

We now present a set of sufficient conditions for inefficiency.

Theorem 3.1. Suppose that $\langle x, c \rangle$ *is competitive in production [with prices* $(p_t)_1^\infty$, $(r_t)_0^\infty$*], and satisfies the following property: There exists a feasible program* $\langle x', c' \rangle$ *such that*

$$\text{(a)} \qquad \liminf_{T \to \infty} \sum_{t=1}^{T} p_t(c'_t - c_t) > 0,$$

$$\text{(b)} \qquad \liminf_{T \to \infty} z_T / (x_T - x'_T) > 0. \tag{3.2}$$

Then $\langle x, c \rangle$ *is inefficient.*

Proof. Let a competitive program $\langle x, c \rangle$ satisfying (3.2) for some feasible program $\langle x', c' \rangle$ be given. Then there exists an integer $\hat{T}$ and $\beta > 0$ such that for all $T \geqq \hat{T}$,

$$\sum_{t=1}^{T} p_t(c'_t - c_t) \geqq \beta. \tag{3.3}$$

Define for $t \geqq 0$,

$$\begin{aligned} W_t &\equiv p_{t+1} g_t(x_t) + r_{t+1} \delta_t x_t - r_t x_t, \\ W'_t &\equiv p_{t+1} g_t(x'_t) + r_{t+1} \delta_t x'_t - r_t x'_t. \end{aligned} \tag{3.4}$$

Clearly, by (2.4),

$$W_t \geqq W'_t \quad \text{for all} \quad t \geqq 0.$$

We have, for $t \geqq 1$,

$$\begin{aligned} p_t(c'_t - c_t) &= p_t[g_{t-1}(x'_{t-1}) - z'_t] - p_t[g_{t-1}(x_{t-1}) - z_t] \\ &= [p_t g_{t-1}(x'_{t-1}) + r_t \delta_{t-1} x'_{t-1} - r_{t-1} x'_{t-1}] \\ &\quad - [p_t g_{t-1}(x_{t-1}) + r_t \delta_{t-1} x_{t-1} - r_{t-1} x_{t-1}] \end{aligned}$$

$$+[r_t\delta_{t-1}x_{t-1}-r_{t-1}x_{t-1}+p_tx_t-p_t\delta_{t-1}x_{t-1}]$$

$$-[r_t\delta_{t-1}x'_{t-1}-r_{t-1}x'_{t-1}+p_tx'_t-p_t\delta_{t-1}x'_{t-1}]$$

$$=(W'_{t-1}-W_{t-1})+(p_t-r_t)(x_t-\delta_{t-1}x_{t-1})-(r_{t-1}x_{t-1}-r_tx_t)$$

$$-(p_t-r_t)(x'_t-\delta_{t-1}x'_{t-1})+[r_{t-1}x'_{t-1}-r_tx'_t]$$

$$\leqq(W'_{t-1}-W_{t-1})+(r_tx_t-r_{t-1}x_{t-1})-(r_tx'_t-r_{t-1}x'_{t-1})$$

[by (2.4)].

Summing from 1 to T, we have

$$\sum_{t=1}^{T} p_t(c'_t-c_t)\leqq \sum_{t=0}^{T-1} W'_t-\sum_{t=0}^{T-1} W_t+r_T(x_T-x'_T). \tag{3.5}$$

Note that by (3.5) and 3.1),

$$x_T>x'_T \quad \text{for all} \quad T\geqq \hat{T}.$$

By (3.2b), there exists an interger $\hat{\hat{T}}$ and $\hat{\alpha}>0$ such that for all $T\geqq \hat{\hat{T}}$,

$$z_T/(x_T-x'_T)\geqq\hat{\alpha}. \tag{3.6}$$

Let $T^*=\max(\hat{T},\hat{\hat{T}})$. Define $\alpha=\min(\hat{\alpha},1)$. For $T\geqq T^*$, define ε_T by

$$p_T\varepsilon_T=\alpha\left[\left(\sum_{t=0}^{T-1} W_t-\sum_{t=0}^{T-1} W'_t\right)+\beta\right], \tag{3.7}$$

and η_T by

$$\eta_T=\alpha[x_T-x'_T]. \tag{3.8}$$

Observe that

$$p_Tz_T\geqq\alpha p_T(x_T-x'_T)=\alpha r_T(x_T-x'_T)\geqq\alpha\left[\left(\sum_{t=0}^{T-1} W_t-\sum_{t=0}^{T-1} W'_t\right)+\beta\right]$$

$$=p_T\varepsilon_T\geqq\beta>0$$

[using (2.4), (3.3) and (3.5)]. Using the last inequality to get $p_T>0$, we have

$$0<\varepsilon_T\leqq\eta_T\leqq z_T \quad \text{for} \quad T\geqq T^*. \tag{3.9}$$

Now, for $T \geqq T^*$,

$$
\begin{aligned}
& p_{T+1}\varepsilon_{T+1} \\
& = p_T\varepsilon_T + \alpha(W_T - W'_T) \\
& = p_T\varepsilon_T + \alpha[(p_{T+1}g_T(x_T) + r_{T+1}\delta_T x_T - r_T x_T) - (p_{T+1}g_T(x'_T) + r_{T+1}\delta_T x'_T - r_T x'_T)] \\
& = p_T\varepsilon_T + \alpha[(p_{T+1}f_T(x_T) - p_T x_T) - (p_{T+1}f_T(x'_T) - p_T x'_T)] \\
& = p_T\varepsilon_T + \alpha[p_{T+1}\{f_T(x_T) - f_T(x_T - \eta_T/\alpha)\}] - p_T\eta_T \\
& \geqq \alpha[p_{T+1}\{f_T(x_T) - f_T(x_T - \eta_T/\alpha)\}] \qquad (3.10)
\end{aligned}
$$

[using $p_T = r_T$ for all $T \geqq T^*$, by (2.4) and (3.6), and $\eta_T \geqq \varepsilon_T$].

Now

$$
\begin{aligned}
\alpha[f_T(x_T) - f_T(x_T - \eta_T/\alpha)] &= [f_T(x_T) - f_T(x_T - \eta_T/\alpha)]/\cdot\eta_T/(\eta_T/\alpha) \\
&> [f_T(x_T) - f_T(x_T - \varepsilon_T)]/\varepsilon_T \cdot \varepsilon_T,
\end{aligned}
$$

since $\eta_T/\alpha \geqq \eta_T \geqq \varepsilon_T$.

Using this in (3.10), we have

$$p_{T+1}\varepsilon_{T+1} \geqq p_{T+1}[f_T(x_T) - f_T(x_T - \varepsilon_T)],$$

and since $p_T > 0$ for all $T \geqq T^*$, we have

$$\varepsilon_{T+1} \geqq f_T(x_T) - f_T(x_T - \varepsilon_T) \quad \text{for} \quad T \geqq T^*. \qquad (3.11)$$

Using (3.9), (3.11) and Lemma 3.1, $\langle x, c \rangle$ is inefficient.

Remark. This technique is easily modified to provide a simple proof of the Cass–Yaari theorem (the non-trivial 'necessity' direction) in the case of reversible investment.

Here by the basic lemma of Cass (1972), we only need establishish the existence of an integer T^* and a sequence $\langle \varepsilon_t \rangle_{T^*}^{\infty}$ such that $0 < \varepsilon_t \leqq x_t$, $t \geqq T^*$, and that (3.1b) is satisfied. Define ε_T and η_T as in the proof, with α set equal to 1. (3.5b) is verified in exactly the same way, and so is $\varepsilon_T \leqq \eta_T$ for all $T \geqq T^*$. Therefore $\varepsilon_T \leqq x_T - x'_T \leqq x_T$, and this establishes the result.

For the converse to Theorem 3.1, we will make the following additional assumption on the technology:

(T.4) *There exists* $\alpha > 0$ *such that for all feasible programs* $\langle x, c\rangle, g_t^+(x_t) \geqq \alpha$ *for all* $t \geqq 0$.

Remark. Observe that this does not require $\inf_{x \geqq 0} g_t^+(x) > 0$ for all t.

Take for example, the stationary case, with $g_t = g$ for all $t \geqq 0$, and $\delta_t = \delta \in (0,1)$ for all $t \geqq 0$. In this case, it is easy to check (using $\delta < 1$) that (T.5) is satisfied, even if $\inf_{x \geqq 0} g^+(x) = 0$.

Theorem 3.2. Suppose that (T.4) holds. Let $\langle x, c\rangle$ *be an inefficient program which is competitive in production, with* $p_t > 0$ *for all* $t > 1$. *Then there exists a feasible program such that*

$$\text{(a)} \qquad \liminf_{T\to\infty} \sum_{t=1}^{T} p_t(c_t' - c_t) > 0,$$

$$\text{(b)} \qquad \liminf_{T\to\infty} z_T/(x_T - x_T') > 0. \tag{3.12}$$

Proof. Since $\langle x, c\rangle$ is inefficient, there exists a feasible program $\langle x'', c''\rangle$ such that $c_t'' \geqq c_t$ for all $t \geqq 1$, and $c_t'' > c_t$ for some $t \geqq 1$. Let s be the first period in which $c_s'' > c_s$. Define a program $\langle x', c'\rangle$ by $x_t' = x_t''$, $0 \leqq t \leqq s$, and $x_{t+1}' = f_t(x_t') - c_{t+1}$ for $t \geqq s$; $c_t' = c_t''$ for $1 \leqq t \leqq s$, and $c_t' = c_t$ for $t \div s$. Clearly, $c_t' = c_t$ for $t \neq s$, and $c_t' > c_t$ for $t = s$.

To check that this program is feasible, first note that

$$c_{t+1}' = f_t(x_t') - x_{t+1}' \quad \text{for} \quad t \geqq 0.$$

It remains to check that $z_t' \geqq 0$ for all $t \geqq 1$. To show this, we first claim that

$$x_t' \geqq x_t'' \quad \text{for} \quad t \geqq 0.$$

This is certainly true for $0 \leqq t \leqq s$. Also

$$x_{s+1}' = f_s(x_s') - c_{s+1} \geqq f_s(x_s') - c_{s+1}'' = f_s(x_s'') - c_{s+1}'' = x_{s+1}''.$$

Now suppose that the claim is true for $T = s + t$, so $t \geqq 1$. Then

$$x_{T+1}' = f_T(x_T') - c_{T+1} \geqq f_T(x_T'') - c_{T+1}'' = x_{T+1}''.$$

Hence

$$x_t' \geqq x_t'' \quad \text{for all} \quad t \geqq 0.$$

Therefore, for $t \geqq 0$,

$$z_{t+1}' = g_t(x_t') - c_{t+1} \geqq g_t(x_t'') - c_{t+1}'' = z_{t+1}'' \geqq 0.$$

Now, $p_s > 0$. Thus $\alpha \equiv p_s(c'_s - c_s) > 0$. For all $T > s$,

$$\sum_{t=1}^{T} p_t(c'_t - c_t) \geqq p_s(c'_s - c_s) = \alpha > 0. \tag{3.13}$$

This verifies (3.12a). To verify (3.12b), first observe that $x_t > x'_t$ for all $t \geqq s$, using (3.13), (3.5), and $W_t \geqq W'_t$ for all $t \geqq 0$. Define $\varepsilon_t \equiv x_t - x'_t$ for all $t \geqq s$. Clearly, for $t \geqq s$, $\varepsilon_t > 0$ and

$$\begin{aligned} z_{t+1} - z'_{t+1} &\geqq g_t(x_t) - g_t(x'_t) \\ &= [f_t(x_t) - f_t(x'_t)] - \delta_t[x_t - x'_t] \\ &= [x_{t+1} - x'_{t+1}] - \delta_t[x_t - x'_t] \\ &= \varepsilon_{t+1} - \delta_t \varepsilon_t \end{aligned}$$

So

$$z_{t+1}/\varepsilon_{t+1} \geqq 1 - \delta_t(\varepsilon_t/\varepsilon_{t+1}) \quad \text{for} \quad t \geqq s. \tag{3.14}$$

By (3.14), $z_t/(x_t - x'_t)$ is defined for all $t > s$. Now, for $t \geqq s$,

$$\varepsilon_{t+1} = x_{t+1} - x'_{t+1} = f_t(x_t) - c_{t+1} - f_t(x'_t) + c'_{t+1} = f_t(x_t) - f_t(x_t - \varepsilon_t),$$

so that by concavity of f_t,

$$\varepsilon_{t+1} \geqq f_t^+(x_t)\varepsilon_t = [g_t^+(x_t) + \delta_t]\varepsilon_t, \qquad t \geqq s. \tag{3.15}$$

Using this in (3.14), we have for $t \geqq s$,

$$\begin{aligned} z_{t+1}/(x_{t+1} - x'_{t+1}) = z_{t+1}/\varepsilon_{t+1} &\geqq 1 - \{\delta_t/[g_t^+(x_t) + \delta_t]\} \\ &\geqq 1 - [1/\{g_t^+(x_t) + 1\}] \geqq 1 - [1/\{\alpha + 1\}] > 0. \end{aligned}$$

This establishes (3.12b).

Remark. Theorems 3.1 and 3.2 provide a complete characterization of all inefficient programs in a framework which takes as its central feature the concept of value-maximization. Additional conditions are needed because an efficient program with the irreversibility model need not be efficient within the reversibility model. This by itself does not imply that the given program fails to be a consumption-value-maximizing program, for the set of programs which 'overtake' it in consumption value in the reversibility model may be excluded by the irreversibility conditions. However, one might suspect that this does not occur, in general, and this suspicion is confirmed by Example 3.1.

It is useful to have a set of conditions which, apart from the consumption-value criteria, focus on the feasible program *alone* (such conditions are not given by the theorems). Such conditions are given in Corollary 3.1, and will be used in a later section.

Corollary 3.1. Suppose that a feasible program $\langle x,c\rangle$ *is competitive in production [with prices* $(p_t)_1^\infty$, $(r_t)_0^\infty$*] and satisfies the following properties:*

(a) *There exists a feasible program* $\langle x',c'\rangle$ *such that*

$$\liminf_{T\to\infty} \sum_{t=1}^{T} p_t(c_t'-c_t)>0, \tag{3.16}$$

(b) $\langle x,c\rangle$ *is regular.*

Then $\langle x,c\rangle$ *is inefficient.*

Proof. If (3.16a) is satisfied, there exists an integer T^* and $\alpha>0$ such that for all $T\geqq T^*$,

$$r_T(x_T-x_T')\geqq \sum_{t=1}^{T} p_t(c_t'-c_t)\geqq \alpha>0,$$

and so

$$x_T>x_T' \quad \text{for all} \quad T\geqq T^*.$$

Hence $z_t/(x_t-x_t')$ is defined for all $T\geqq T^*$, and

$$z_t/(x_t-x_t')\geqq z_t/x_t.$$

So

$$\liminf_{t\to\infty} z_t/(x_t-x_t')>0,$$

and conditions (3.2a) and (3.2b) of Theorem 3.1 are satisfied. Therefore $\langle x,c\rangle$ is inefficient.

One might, perhaps, feel that the converse of Corollary 3.1 is true. This would be very useful, but is unfortunately not true, as the following example demonstrates.

Example 3.2. (An inefficient program need not be regular)

Consider the technology described in Example 3.1. Part (iiib) is not necessary for the argument here, and the condition $f(\underline{x})=\bar{x}$ may be weakened to $f(\underline{x})\geqq\bar{x}$.

Let the initial stock $\boldsymbol{x}=\bar{x}$. Consider a program $\langle x,c\rangle$ defined by $x_0=\boldsymbol{x}=\bar{x}$, $z_t=(1-\delta)\bar{x}/t$, t odd, $z_t=\bar{x}-\delta[\delta\bar{x}+(1-\delta)\bar{x}/(t-1)]$, t even, for $t\geqq 1$; $x_{t+1}=\delta x_t+z_{t+1}$ for $t\geqq 0$; $c_{t+1}=f(x_t)-x_{t+1}$ for $t\geqq 0$.

To check feasibility, we need only verify that (i) $z_t\geqq 0$ for $t\geqq 1$, and (ii) $c_t\geqq 0$ for $t\geqq 1$. It is clear that (i) holds.

To verify (ii), note that for $t\geqq 0$, $x_t=\bar{x}$ if t is even, and $x_t=\delta\bar{x}+(1-\delta)\bar{x}/t$, if t is odd. To see this, observe first that $x_0=\bar{x}$. Now let $x_s=\bar{x}$ for some $s\geqq 0$ and even. Then

$$x_{s+1}=\delta x_s+z_{s+1}=\delta\bar{x}+(1-\delta)\bar{x}/(s+1),$$

and so

$$\begin{aligned} x_{s+2}&=\delta x_{s+1}+z_{s+2}\\ &=\delta[\delta\bar{x}+(1-\delta)\bar{x}/(s+1)]+[\bar{x}-\delta\{\delta\bar{x}+(1-\delta)\bar{x}/(s+1)\}]=\bar{x}. \end{aligned}$$

This shows that $x_t=\bar{x}$ for even t. That $x_t=\delta\bar{x}+(1-\delta)\bar{x}/t$ for odd t follows immediately. To show, now, that $c_t\geqq 0$ for all $t\geqq 1$, it suffices to show that $g(x_{t-1})\geqq z_t$ for all $t\geqq 1$. If $t\geqq 1$ is odd,

$$g(x_{t-1})=g(\bar{x})=f(\bar{x})-\delta\bar{x}\geqq\bar{x}-\delta\bar{x}\geqq(1-\delta)\bar{x}/t=z_t$$

(using here the fact that $\hat{x}>\bar{x}$). If $t\geqq 1$ is even,

$$\begin{aligned} g(x_{t-1})&=g(\delta\bar{x}+(1-\delta)\bar{x}/t)\geqq g(\delta\bar{x})=g(\underline{x})\geqq\bar{x}-\delta\underline{x}\\ &=\bar{x}-\delta[\delta\bar{x}]\geqq\bar{x}-\delta[\delta\bar{x}+(1-\delta)\bar{x}/(t-1)]=z_t. \end{aligned}$$

Note that $x_t\geqq\underline{x}$ for all $t\geqq 0$. It follows that

$$q_0\equiv 1,\qquad q_{t+1}\equiv\left[\prod_{s=0}^{t}f'(x_s)\right]^{-1}\geqq\left[\prod_{s=0}^{t}f'(\underline{x})\right]^{-1}=h^t,$$

where $h>1$ (since $x^*<\underline{x}$). Thus

$$q_{t+1}z_{t+1}\geqq h^t[(1-\delta)\bar{x}/(t+1)],\quad t \text{ odd},$$

and

$$q_{t+1}z_{t+1}\geqq h^t[\bar{x}-\delta\{\delta\bar{x}+(1-\delta)\bar{x}/t\}],\quad t \text{ even}.$$

Thus

$$\liminf_{t\to\infty} q_t z_t > 0 \quad (\text{in fact} = \infty).$$

This verifies condition (4.1) in Mitra (1978). Condition (4.2) in Mitra (1978) is verified by noting that $x_t \geqq \underline{x} > x^*$, hence $\langle x_t \rangle$ is inefficient in the reversibility model, so that by Cass (1972), $\sum_{t=0}^{\infty} 1/q_t < \infty$. Therefore by Mitra (1978, theorem 4.1), $\langle x, c \rangle$ is inefficient.

But $z_t/x_t \leqq (1-\delta)\bar{x}/t\underline{x}$ for t odd, showing that

$$\liminf_{t\to\infty} z_t/x_t = 0.$$

4. Competition, efficiency and optimality

In this section, we shall invoke the techniques of duality theory in an attempt to characterize the class of optimal programs. Utility functions and the technology are allowed to vary over time. While this generally forces us to sacrifice a complete characterization (complete characterizations, however, will be obtained in the 'stationary' models), it is not the central obstacle to a satisfactory application of duality techniques. Irreversibility of investment, surely a realistic phenomenon, is the key problem here. While irreversibility does not destroy any 'convexity' features of the model, thus still permitting the use of separation arguments, the nature of the dual variables, or their meaningful economic interpretation as 'competitive prices', is by no means clear without careful analysis. Certainly, the 'traditional' competitive prices[6] will not do, and this is shown in Example 4.1.

Intuitively, we may proceed in the following way. Consider an optimal program, and two time periods t, $t+1$. There will exist 'prices' such that utility-maximization occurs in each of these dates, relative to these prices and budgets equal to the value of consumption. However, these prices may necessitate profit-maximizing behavior for 'each producer' which is infeasible for the economy, if the depreciation constraint is binding. Each producer may want to move to a lower capital stock in the interests of profit-maximization, whereas the economy as a whole cannot. One would expect the price of capital to *fall* in this case. As a result, the price of consumption exceeds that of the capital stock, and this in fact would be a signal that the depreciation constraint is binding. With this verbal argument in mind, examine the 'competitive conditions' (2.4). Regard p_t as the (shadow) price of new output, and r_t as the price of capital. The condition $p_t \geqq r_t$ simply expresses the intuitive notion that there should be no reason for capital

[6]These are the all too familiar prices defined below, see (4.1).

stocks to be at a premium along an optimal program, since the depreciation constraint only hinders the *disposal* of capital. The condition '$p_t > r_t$ implies $x_t = \delta_{t-1} x_{t-1}$' expresses the verbal argument we have advanced above: consumption is at a premium only if the depreciation constraint is binding. Finally, the profit-maximizing condition may be interpreted thus: revenue in period $t+1$ is simply the quantity of new output, evaluated at price p_{t+1}, plus the quantity of depreciated capital, evaluated at *its* price r_{t+1}. Costs in period t are written simply as $r_t x_t$. This is because if the depreciation constraint were binding in the *previous* period, $x_t = \delta_{t-1} x_{t-1}$, the appropriate price is r_t. If it were not, then x_t would consist of a 'previously depreciated' stock $\delta_{t-1} x_{t-1}$ and a fresh investment $z_t > 0$. The former would be evaluated at r_t, the latter at p_t. However, since the constraint was not binding, we have $p_t = r_t$ and so costs are still expressible as $p_t z_t + r_t \delta_{t-1} x_{t-1} = r_t x_t$.

This insight will help us develop a duality theory for optimal programs. First, we recall the standard definition of a program competitive in production and utility. The definition is given for infinite-horizon programs; the obvious modifications hold for finite-horizon programs.[7] Henceforth, we will refer to a program which is competitive in production and utility as a competitive program (where no confusion is possible).

The usual definition describes a program as being competitive if there exists a non-null non-negative sequence of prices $\langle q_t \rangle_0^\infty$ such that

$$\begin{aligned} &u_t(c_t) - q_t c_t \geqq u_t(c) - q_t c \quad \text{for} \quad c \geqq 0, \qquad t \geqq 1, \\ &q_{t+1} f_t(x_t) - q_t x_t \geqq q_{t+1} f_t(x) - q_t x \quad \text{for} \quad x \geqq 0, \qquad t \geqq 0. \end{aligned} \tag{4.1}$$

That these conditions do not characterize optimality, even with a finite horizon, is shown by the following example.

Example 4.1. (*Optimal programs need not be competitive in the sense of* (*4.1*))

Let $\boldsymbol{x} = 1$, $g_t(x) = \frac{1}{2}x$, $\delta_t = \frac{1}{2}$, $b = 0$, $u_t(c) = c^{\frac{1}{2}}$. Let $T = 2$. Any T-program $\langle x, c \rangle$ then satisfies: $c_1 = \frac{1}{2} - \varepsilon$, $c_2 = \frac{1}{4} + \frac{1}{2}\varepsilon$, where $0 \leqq \varepsilon \leqq \frac{1}{2}$.

Now,

$$\tfrac{1}{2}^{\frac{1}{2}} - (\tfrac{1}{2} - \varepsilon)^{\frac{1}{2}} \geqq \tfrac{1}{2}^{\frac{1}{2}} \varepsilon,$$

and

$$\tfrac{1}{4}^{\frac{1}{2}} - (\tfrac{1}{4} + \tfrac{1}{2}\varepsilon)^{\frac{1}{2}} \geqq \tfrac{1}{2}(\tfrac{1}{4}^{-\frac{1}{2}})[-\varepsilon/2],$$

Thus,

$$[\tfrac{1}{2}^{\frac{1}{2}} + \tfrac{1}{4}^{\frac{1}{2}}] - [(\tfrac{1}{2} - \varepsilon)^{\frac{1}{2}} + (\tfrac{1}{4} + \tfrac{1}{2}\varepsilon)^{\frac{1}{2}}] \geqq \varepsilon[1/\sqrt{2} - \tfrac{1}{2}].$$

[7]The finite horizon 'transversality condition', $q_t(x_t - b) = 0$, must be added.

Hence, a T-optimal program $\langle x^*, c^*\rangle$ must satisfy

$$c_1^* = \tfrac{1}{2}, \qquad c_2^* = \tfrac{1}{4}.$$

We claim that $\langle x^*, c^*\rangle$ is not competitive, in the sense of (4.1). Suppose, on the contrary, it is. Then $u_1'(c_1^*) = q_1$, $u_2'(c_2^*) = q_2$, and $f_1'(x_1^*) = q_1/q_2$. Since $q_1 \neq q_2$, so $f_1'(x_1^*) \neq 1$. But, since $f_t(x) = g_t(x) + \delta_t x = \frac{1}{2}x + \frac{1}{2}x = x$.
So, $f_1'(x_1^*) = 1$, a contradiction. This establishes our claim.

The following theorems (4.1 and 4.2) completely characterize optimal programs in a finite horizon model.

Theorem 4.1. Suppose that a T-program to b, $\langle x^*, c^*\rangle$ is competitive. Then it is optimal.

Proof. Let $\langle x^*, c^*\rangle$ be competitive with prices $\langle p_t^*\rangle_1^T$, $\langle r_t^*\rangle_0^T$. Then, using the competitive conditions (2.4) and (2.5), and (3.5), we have, for any feasible program $\langle x', c'\rangle$,

$$\sum_{t=1}^{T} [u_t(c_t') - u_t(c_t^*)] \leqq \sum_{t=1}^{T} p_t^*(c_t' - c_t^*)$$

$$\leqq r_T^*(x_T^* - x_T') \leqq r_T^*(x_T^* - b) = 0 \quad [\text{by } (2.4')].$$

Theorem 4.2. Suppose that there exists a feasible program $\langle \hat{x}, \hat{c}\rangle$ with $\hat{c}_t > 0$ for some $t = 1, \ldots, T$. Then any optimal program to b, $\langle x^*, c^*\rangle$, is competitive.

Proof. Define

$$\mathscr{C} = \{(c_1, \ldots, c_T, x_1 - \delta_0 x_0, \ldots, x_T - \delta_{T-1} x_{T-1}) \in \boldsymbol{R}^{2T}:$$

$$x_t \geqq 0, 0 \leqq t \leqq T, c_{t+1} \leqq f_t(x_t) - x_{t+1}, 0 \leqq t \leqq T, \ x_0 = \boldsymbol{x}, x_T \geqq b\},$$

and

$$\mathscr{D} = \left\{(c_1, \ldots, c_T, d_1, \ldots, d_T) \in \boldsymbol{R}_+^{2T} : \sum_1^T u_t(c_t) > \sum_1^T u_t(c_t^*)\right\}.$$

It is obvious that $\mathscr{C}$ and $\mathscr{D}$ are non-empty and convex, and that $\mathscr{C} \cap \mathscr{D} = \Phi$. By the Minkowski Separation Theorem, there exists $(p_t, q_t)_1^T$, non-null, and $\alpha \in \boldsymbol{R}$ such that

$$\sum_1^T p_t c_t + \sum_1^T q_t(x_t - \delta_{t-1} x_{t-1}) \leqq \alpha \qquad (4.2)$$

for all $((c_t)_1^T, (x_t - \delta_{t-1} x_{t-1})_1^T) \in \mathscr{C}$,

$$\sum_1^T p_t c_t + \sum_1^T q_t d_t \geqq \alpha \quad \text{for all} \quad (c_t, d_t)_1^T \in \mathscr{D}. \tag{4.3}$$

Since

$$\{(c_t^*)_1^T, (x_t^* - \delta_{t-1} x_{t-1}^*)_1^T\} \in \mathscr{C},$$

$$\sum_1^T p_t c_t^* + \sum_1^T q_t (x_t^* - \delta_{t-1} x_{t-1}^*) \leqq \alpha. \tag{4.4}$$

By choosing an appropriate sequence in $\mathscr{D}$ [using the continuity of $(u_t)_1^T$], and passing to the limit in (4.3),

$$\sum_1^T p_t c_t^* + \sum_1^T q_t (x_t^* - \delta_{t-1} x_{t-1}^*) \geqq \alpha. \tag{4.5}$$

Combining (4.2), (4.4) and (4.5), we have for all elements on $\mathscr{C}$,

$$\alpha = \sum_1^T p_t c_t^* + \sum_1^T q_t (x_t^* - \delta_{t-1} x_{t-1}^*) \geqq \sum_1^T p_t c_t + \sum_1^T q_t (x_t - \delta_{t-1} x_{t-1}). \tag{4.6}$$

Observe that $\langle p_t, q_t \rangle_1^T$ is non-negative. For if q_s (resp. p_k) were negative for some s (resp. k), we could choose a suitable element of $\mathscr{D}$ to violate (4.3). Define $r_t \equiv p_t - q_t$. We verify that $r_t \geqq 0$, $1 \leqq t \leqq T$. Suppose, on the contrary, that $r_s < 0$ for some s. Then $p_s < q_s$. Define $\langle x', c' \rangle$ by

$$x_0' = \boldsymbol{x}, x_{t+1}' - \delta_t x_t' = x_{t+1}^* - \delta_t x_t^* \quad \text{for all} \quad t \neq s,$$

$$x_s' = x_s^* + \varepsilon \quad \text{for some} \quad \varepsilon > 0,$$

and

$$c_t' = f_{t-1}(x_{t-1}') - \delta_{t-1} x_{t-1}'.$$

Since $x_t' \geqq 0$ for all $1 \leqq t \leqq T$, $x_0' = \boldsymbol{x}$ and $x_T' \geqq b$,

$$(c_t', x_t' - \delta_{t-1} x_{t-1}')_1^T \in \mathscr{C}.$$

We claim that $c_t' \geqq c_t^*$, $s \neq t$, and $c_s' = c_s^* - \varepsilon$. The second relation is immediate. As for the first, observe that $c_t' = c_t^*$, $1 \leqq t \leqq s-1$, and that $x_t' \geqq x_t^*$ for all $1 \leqq t \leqq T$. For $t \geqq s$,

$$c_{t+1}' = f_t(x_t') - x_{t+1}' = g_t(x_t') + (\delta_t x_t' - x_{t+1}')$$

$$= g_t(x_t') + (\delta_t x_t^* - x_{t+1}^*) \geqq g_t(x_t^*) + \delta_t x_t^* - x_{t+1}^* = f_t(x_t^*) - x_{t+1}^* = c_{t+1}^*.$$

This verifies the claim. Now note that

$$\sum_1^T p_t c_t' + \sum_1^T q_t(x_t' - \delta_{t-1} x_{t-1}') \geqq \sum_1^T p_t c_t^* + \sum_1^T q_t(x_t^* - \delta_{t-1} x_{t-1}^*) + (q_s - p_s)\varepsilon$$

$$= \alpha + (q_s - p_s)\varepsilon > \alpha,$$

which contradicts (4.2). Hence $r_t \geqq 0$ for all $t = 1, \ldots, T$.

This also shows that $p_t \geqq q_t$ for all $t = 1, \ldots, T$. Since $(p_t, q_t)_1^T$ is non-null, it therefore follows that $(p_t, r_t)_1^T$ is (non-negative and) non-null, *We have also verified* (since $q_t \geqq 0$) (2.4b), $p_t \geqq r_t$, $1 \leqq t \leqq T$.

Now we verify (2.4c). Suppose, on the contrary, that $(p_s - r_s)(x_s^* - \delta_{s-1} x_{s-1}^*) > 0$ for some $1 \leqq s \leqq T$. Since $q_t(x_t^* - \delta_{t-1} x_{t-1}^*) \geqq 0$ for all t, this implies

$$\sum_1^T q_t(x_t^* - \delta_{t-1} x_{t-1}^*) = \theta > 0.$$

We know that $p_k > 0$, for some k. Define $(c_t'', d_t'')_1^T$ by $c_t'' = c_t^*$, $t \neq k$, $c_k'' = c_k^* + \theta/2p_k$, $d_t'' = 0$, $1 \leqq t \leqq T$. Clearly $(c_t'', d_t'')_1^T \in \mathscr{D}$. But

$$\sum_1^T p_t c_t'' + \sum_1^T q_t d_t'' = \sum_1^T p_t c_t^* + \frac{\theta}{2} < \sum_1^T p_t c_t^* + \theta = \sum_1^T p_t c_t^* + \sum_1^T q_t[x_t^* - \delta_{t-1} x_{t-1}^*] = \alpha,$$

contradicting (4.3). This verifies (2.4c).

Next, we verify (2.5). Define

$$\mathscr{E} = \Bigg\{ (\alpha, \beta) \in \boldsymbol{R}^2 \,\Big|\, \text{for some } \langle c_t \rangle_1^T, c_t \geqq 0,$$

$$\sum_1^T [u_t(c_t) - u_t(c_t^*)] \geqq \alpha \text{ and } \sum_1^T [p_t c_t^* - p_t c_t] \geqq \beta \Bigg\}.$$

Clearly, $\mathscr{E}$ is non-void and convex. We claim that $\mathscr{E} \cap \boldsymbol{R}_{++}^2 = \Phi$. If this is not true, then there exists $(c_t)_1^T$, with corresponding $(\alpha, \beta) \gg 0$. Define $(\tilde{c}_t, \tilde{d}_t) \in \mathscr{D}$ by $\tilde{c}_t = c_t$ for all $t = 1, \ldots, T$, $\tilde{d}_t = 0$ for all $t = 1, \ldots, T$. Then

$$\alpha = \sum_1^T p_t c_t^* + \sum_1^T q_t[x_t^* - \delta_{t-1} x_{t-1}^*] = \sum_1^T p_t c_t^* \geqq \beta + \sum_1^T p_t \tilde{c}_t$$

$$= \beta + \sum_1^T p_t \tilde{c}_t = \beta + \sum_1^T p_t \tilde{c}_t + \sum_1^T q_t \tilde{d}_t, \qquad \text{contradicting (4.3).}$$

So by the Minkowski Separation Theorem, there is $(m,n)\neq 0$ and $\theta\in \boldsymbol{R}$ such that

$$m\alpha+n\beta\geqq\theta \quad \text{for all} \quad (\alpha,\beta)\in \boldsymbol{R}^2_{++}, \tag{4.7}$$

$$m\alpha+n\beta\leqq\theta \quad \text{for all} \quad (\alpha,\beta)\in \mathscr{E}. \tag{4.8}$$

Now, $\theta\geqq 0$, since $(0,0)\in\mathscr{E}$. Also, $\theta\leqq 0$, otherwise we could contradict (4.7) by choosing $(\alpha,\beta)\gg 0$ sufficiently small; so $\theta=0$. Clearly, $(m,n)\geqq 0$, otherwise we could pick $(\alpha,\beta)\in \boldsymbol{R}^2_{++}$ to violate (4.7). We claim, now, that $m>0$. Suppose not, then certainly $n>0$. By (4.8), and the fact that $\theta=0$, $n\beta\leqq 0$ for all $(\alpha,\beta)\in\mathscr{E}$, i.e., $\beta\leqq 0$ for all $(\alpha,\beta)\in\mathscr{E}$. For any feasible program $\langle x,c\rangle$, $(c_t, x_t-\delta_{t-1}x_{t-1})_1^T\in\mathscr{C}$.

Using (4.6), and property (2.4c) (which we have already proved),

$$\sum_1^T p_t c_t^* \geqq \sum_1^T p_t c_t + \sum_1^T q_t(x_t-\delta_{t-1}x_{t-1}) \geqq \sum_1^T p_t c_t. \tag{4.9}$$

Since there exists a feasible program with consumption positive at some date, it is easy to check that there exists a feasible program $(\hat{x},\hat{c})$ with $\hat{c}_t>0$ for all $t=1,\dots,T$. Further more $p_k>0$ for some k. Therefore, using (4.9),

$$\sum_1^T p_t c_t^* \geqq \sum_1^T p_t\hat{c}_t \geqq p_k\hat{c}_k=\gamma>0.$$

Now pick $c_t=c_t^*/2$, $t=1,\dots,T$. Then

$$\beta=\sum_1^T p_t c_t^* \Big/ 2\geqq\gamma/2>0,$$

violating $\beta\leqq 0$ for all $(\alpha,\beta)\in\mathscr{E}$. This establishes the claim. So

$$\alpha+\mu\beta\leqq 0 \quad \text{for all} \quad (\alpha,\beta)\in\mathscr{E}, \quad \text{where} \quad \mu\equiv n/m. \tag{4.10}$$

Clearly $\mu>0$, otherwise $\alpha\leqq 0$ for all $(\alpha,\beta)\in\mathscr{E}$, which violates the assumption that u_t is an increasing function for all $t=1,\dots,T$.

Now, for any t, and $c\geqq 0$, define $(c_t')_1^T$ by $c_s'=c_s^*$, $s\neq t$, $c_t'=c$,

$$\alpha=\sum_1^T [u_t(c_t')-u_t(c_t^*)], \qquad \beta=\sum_1^T [p_t c_t^* - p_t c_t'].$$

Cancelling common terms,

$$u_t(c)-u_t(c_t^*)\leqq\mu p_t(c-c_t^*).$$

Define $p_t^* = \mu p_t$, $q_t^* = \mu q_t$, $r_t^* = \mu r_t$, $t = 1, \ldots, T$. Note that $(p_t^*, r_t^*)_1^T$ satisfies all the properties verified so far. Furthermore,

$$u_t(c_t^*) - p_t^* c_t^* \geqq u_t(c) - p_t^* c, \qquad c \geqq 0, \qquad t = 1, \ldots, T. \tag{4.11}$$

Also, by the fact that u_t is increasing, $p_t^* > 0$ for all t. Summarizing, we have so far shown that properties (2.4b), (2.4c) and (2.5) are satisfied by the non-null non-negative sequence $\langle p_t^*, r_t^* \rangle_1^T$.

Now we check that (2.4′) holds. For any $x \geqq b$, define $\langle x_t' \rangle_0^T$ by $x_t' = x_t^*$, $0 \leqq t \leqq T-1$; $x_T' = x$, $c_{t+1}' = f_t(x_t') - x_{t+1}'$, $0 \leqq t \leqq T-1$. Then $(c_t', x_t' - \delta_{t-1} x_{t-1}')_1^T \in \mathscr{C}$. Using (4.6) with (p_t^*, q_t^*), and cancelling common terms,

$$p_T^* c_T^* + q_T^*(x_T^* - \delta_{T-1} x_{T-1}^*) \geqq p_T^* c_T' + q_T^*(x - \delta_{T-1} x_{T-1}^*),$$

or

$$p_T^*[f_{T-1}(x_{T-1}^*) - x_T^*] + q_T^* x_T^* \geqq p_T^*[f_{T-1}(x_{T-1}^*) - x] + q_T^* x,$$

or

$$r_T^* x_T^* \leqq r_T^* x,$$

or

$$r_T^*(x_T^* - b) \leqq r_T^*(x - b). \tag{4.12}$$

If $r_T^*(x_T^* - b) > 0$, we contradict (4.12) by choosing $x = b$. Therefore $r_T^*(x_T^* - b) = 0$ and this verifies (2.4′).

Finally, we verify (2.4a). First, r_0^* has to be defined appropriately. To this end, let

$$\mathscr{A} = \{(x, y) \in \boldsymbol{R}_+^2 : y \leqq g_0(x)\},$$

and

$$\mathscr{B} = \{(x, y) \in \boldsymbol{R}^2 : x < x_0, y > g_0(x_0)\}.$$

Since g_0 is increasing, $\mathscr{A} \cap \mathscr{B} = \Phi$. Moreover, both $\mathscr{A}$ and $\mathscr{B}$ are non-empty and convex. A reapplication of the Separation Theorem yields the existence of scalars $(u, v) \neq 0$ and γ such that

$$ux + vy \geqq \gamma \quad \text{for all} \quad (x, y) \in \mathscr{B}, \tag{4.13}$$

$$ux + vy \leqq \gamma \quad \text{for all} \quad (x, y) \in \mathscr{A}, \tag{4.14}$$

Using a sequence in $\mathscr{B}$ converging to $(x_0, g(x_0))$, we conclude, passing to the limit in (4.13), that

$$ux_0 + vg_0(x_0) \geqq \gamma. \tag{4.15}$$

Since $(x_0, g_0(x_0)) \in \mathscr{A}$,

$$ux_0 + vg_0(x_0) \leqq \gamma. \tag{4.16}$$

Combining (4.14), (4.15) and (4.16), we have

$$vg_0(x_0) + ux_0 \geqq vy + ux \quad \text{for all} \quad (x, y) \in \mathscr{A}. \tag{4.17}$$

Clearly $v \geqq 0$. If $v < 0$, then we could violate (4.13) by appropriate choice of $(x, y) \in \mathscr{B}$. Similarly $u \leqq 0$. For if $u > 0$, we would pick $(x, 0) \in \mathscr{A}$ with x large enough to violate (4.14).

We now claim that $v > 0$. If $v = 0$, then $u < 0$. Thus, using (4.17), $x_0 \leqq x$ for all $(x, y) \in \mathscr{A}$, a contradiction since $x_0 > 0$ and $(0, 0) \in \mathscr{A}$.

Recall that $p_1^* > 0$. Multiplying by an appropriate scalar, we can rewrite (4.17) as

$$p_1^* g_0(x_0) + \hat{u} x_0 \geqq p_1^* g_0(x) + \hat{u} x \quad \text{for all} \quad x \geqq 0. \tag{4.18}$$

Now define $r_0^* \equiv r_1^* \delta_0 - \hat{u}$. Clearly $r_0^* \geqq 0$. Also, using this in (4.18), we have

$$p_1^* g_0(x_0) + r_1^* \delta_0 x_0 - r_0^* x_0 \geqq p_1^* g_0(x) + r_1^* \delta_0 x - r_0^* x. \tag{4.19}$$

Now fix $t \geqq 1$, and some $x \geqq 0$. Define $(x_t')_0^T$ by $x_s' = x_s^*$, $s \neq t$, $x_t' = x$; $(c_t')_1^T$ by $c_{t+1}' = f_t(x_t') - x_{t+1}'$. Clearly $(c_t', x_t' - \delta_{t-1} x_{t-1}')_1^T \in \mathscr{C}$. Using (4.6),

$$\sum_1^T p_t^* c_t^* + \sum_1^T q_t^* (x_t^* - \delta_{t-1} x_{t-1}^*) \geqq \sum_1^T p_t^* c_t' + \sum_1^T q_t^* [x_t' - \delta_{t-1} x_{t-1}'].$$

Cancelling common terms, we have

$$p_t^* c_t^* + p_{t+1}^* c_{t+1}^* + q_t^* (x_t^* - \delta_{t-1} x_{t-1}^*) + q_{t+1}^* (x_{t+1}^* - \delta_t x_t^*)$$

$$\geqq p_t^* c_t' + p_{t+1}^* c_{t+1}' + q_t^* (x - \delta_{t-1} x_{t-1}^*) + q_{t+1}^* (x_{t+1}^* - \delta_t x),$$

or

$$p_t^* [f_{t-1}(x_{t-1}^*) - x_t^*] + p_{t+1}^* [f_t(x_t^*) - x_{t+1}^*] + q_t^* [x_t^* - \delta_{t-1} x_{t-1}^*] + q_{t+1}^* [x_{t+1}^* - \delta_t x_t^*]$$

$$\geqq p_t^* [f_{t-1}(x_{t-1}^*) - x] + p_{t+1}^* [f_t(x) - x_{t+1}^*] + q_t^* [x - \delta_{t-1} x_{t-1}^*] + q_{t+1}^* [x_{t+1}^* - \delta_t x].$$

Again cancelling common terms,

$$p^*_{t+1}f_t(x^*_t)-q^*_{t+1}\delta_t x^*_t-r^*_t x^*_t \geqq p^*_{t+1}f_t(x)-q^*_{t+1}\delta_t x-r^*_t x,$$

or

$$p^*_{t+1}g_t(x^*_t)+r^*_{t+1}\delta_t x^*_t-r_t x^*_t \geqq p^*_{t+1}g_t(x)+r^*_{t+1}\delta_t x-r^*_t x,$$

which verifies (2.4a), and proves the theorem.

We now turn to a characterization of infinite-horizon optimal programs. There are two possible routes. One way is to attempt a characterization by means of the competitive conditions and some additional transversality condition.[8] Another route, perhaps of greater tractability in 'non-stationary' models, is to characterize optimal programs as the class coinciding precisely with that of the competitive and efficient programs.[9] In this section, we will employ the second method. In the next section, Theorems 4.3 and 4.4 will be used to provide complete characterizations of optimal programs in the stationary case. Characterizations by the first route will also be provided.

Theorem 4.3. Suppose that a regular program $\langle x,c\rangle$ is competitive and efficient. Then it is optimal.

Proof. Suppose that $\langle x,c\rangle$ is not optimal. Then there exists a feasible program $\langle x',c'\rangle$, an integer T^* and $\alpha>0$ such that for all $T\geqq T^*$,

$$\alpha \leqq \sum_{t=1}^{T} [u_t(c'_t)-u_t(c_t)]. \tag{4.20}$$

Since $\langle x,c\rangle$ is competitive, there exists $\langle p_t\rangle$ such that

$$\sum_{t=1}^{T} [u_t(c'_t)-u_t(c_t)] \leqq \sum_{t=1}^{T} p_t(c'_t-c_t), \qquad T\geqq 1. \tag{4.21}$$

Combining (4.20) and (4.21), we have

$$\sum_{t=1}^{T} p_t(c'_t-c_t)\geqq \alpha, \qquad \forall\, T\geqq T^*, \tag{4.22}$$

i.e.,

$$\liminf_{T\to\infty} \sum_{t=1}^{T} p_t(c'_t-c_t)>0.$$

[8]This is, of course, the usual approach; see, for example, Peleg and Ryder (1972).
[9]This is the approach taken by Brock (1971).

Observe that $x_T - x'_T > 0$ for $T \geqq T^*$, since $r_T(x_T - x'_T) \geqq \sum_{t=1}^{T} p_t(c'_t - c_t) > \alpha$. Using this fact, the efficiency of $\langle x, c \rangle$ and Theorem 3.2 yields

$$\liminf_{T\to\infty} z_T/(x_T - x'_T) = 0.$$

But for $T \geqq T^*$,

$$z_T/(x_T - x'_T) \geqq z_T/x_T.$$

So

$$\liminf_{T\to\infty} z_T/x_T = 0,$$

contradicting the fact that $\langle x, c \rangle$ is regular.

We now provide a partial converse of Theorem 4.3.

Theorem 4.4. Any infinite horizon optimal program is competitive and efficient.

Proof. Let $\langle x, c \rangle$ be an optimal program. Clearly it is efficient. To prove that $\langle x, c \rangle$ is competitive, observe first that there exists a first $\tau \geqq 1$ such that $c_\tau > 0$. Consider, now the feasible T-programs to x_T given by $x_t^T = x_t$, $0 \leqq t \leqq T$, $c_t^T = c_t$, $1 \leqq t \leqq T$. For $T \geqq \tau$, Theorem 4.2 is applicable, and we may assert, noting (by the 'Principle of Optimality') that $(x_t^T, c_t^T)_1^T$ is in fact optimal to $b = x_T$, that there exists competitive prices for this T-program, $(p_t^T)_1^T$, $(r_t^T)_0^T$ satisfying the finite-horizon competitive conditions (2.4) and (2.4′).

We claim now, that $\|p_\tau^T, r_\tau^T\|_{T=\tau}^{\infty}$ is bounded (where $\|\cdot\|$ denotes the sum-norm). Since $p_\tau^T \geqq r_\tau^T$, it suffices to exhibit the boundedness of p_τ^T. Suppose, on the contrary, that $p_\tau^T \to \infty$ along a subsequence of T. By the competitive conditions, we have

$$u_\tau(c_\tau) - u_\tau(c) \geqq p_\tau^T(c_\tau - c) \quad \text{for all} \quad c \geqq 0 \quad \text{and} \quad T \geqq \tau.$$

Since $c_\tau > 0$, pick $c = c_\tau/2 > 0$. Then

$$u_\tau(c_\tau) - u_\tau(c_\tau/2) \geqq p_\tau^T c_\tau/2, \qquad T \geqq \tau,$$

which, however, cannot hold for large T if $\lim_{T\to\infty} \sup p_\tau^T = \infty$.

Hence

$$\|p_\tau^T, r_\tau^T\| \leqq M(1) < \infty.$$

We now demonstrate that $\|p_t^T, r_t^T\|_{T=t}^{\infty}$ is bounded for all $t \geqq \tau$. Let $s > \tau$ be the first period in which $\lim_{T\to\infty} \sup \|p_s^T, r_s^T\| = \infty$. Pick $x > x_s^T$, and define

$$\lambda = \min\,[\delta_s(x - x_s^T), \{g_s(x) - g_s(x_s^T)\}] \quad \text{if} \quad \delta_s > 0.$$

Let $e = (1, 1)'$. Then, for $T \geqq s$,

$$\lambda \|p_s^T, r_s^T\| = (p_s^T, r_s^T)\lambda e$$

$$\leqq p_s^T\{g_s(x) - g_s(x_s^T)\} + r_s^T\{\delta_s x - \delta_s x_s^T\}$$

$$\leqq r_{s-1}^T(x - x_s^T) \leqq r_{s-1}^T x$$

$$\leqq (p_{s-1}^T + r_{s-1}^T)x \leqq M(s-1)x < \infty.$$

But the left-hand side goes to ∞ for a subsequence of T, a contradiction. So $\|p_s^T, r_s^T\|_{T=s}^{\infty}$ is bounded for $s \geqq \tau$. (The case $\delta_s = 0$ is easily checked.)

We will now show that r_0^T is bounded in T, and that $\|p_t^T, r_t^T\|_{T=\tau}^{\infty}$ is bounded for all $t \geqq 1$. Suppose that this is not true for some t; clearly $t < \tau$. If $t = 0$, this means that $\lim_{T\to\infty} \sup r_0^T = \infty$. If $t > 1$, $\lim_{T\to\infty} \sup \|p_t^T, r_t^T\| = \infty$. In this case, too, $\lim_{T\to\infty} \sup r_t^T = \infty$, since $p_t^T = r_t^T$ [by (2.4c), since $c_t^T = 0$, hence $z_t^T > 0$]. Observing that $q_t(0) \geqq 0$, we have by the competitive conditions,

$$p_{t+1}^T g_t(x_t^T) + r_{t+1}^T \delta_t x_t^T - r_t^T x_t^T \geqq p_{t+1}^T g_t(0) \geqq 0.$$

Since $x_t^T > 0$, it follows that $\lim_{T\to\infty} \sup \|p_{t+1}^T, r_{t+1}^T\| = \infty$. If $t + 1 = \tau$, this is a contradiction. Otherwise, $c_{t+1}^T = 0$, so $p_{t+1}^T = r_{t+1}^T$ and hence $\lim_{T\to\infty} \sup r_{t+1}^T = \infty$. Noting that $x_{t+1}^T > 0$, repeat this procedure. In a finite number of steps, we get $\|p_\tau^T, r_\tau^T\| \to \infty$ along a subsequence of T, which is a contradiction.

So we have established that $r_0^T \leqq M(0)$ and $\|p_t^T, r_t^T\| \leqq M(t)$, where $M(t) < \infty$, for all $t \to 0$. By the Cantor Diagonal method, we can extract a subsequence T_n such that $(p_t^{T_n}, r_t^{T_n}) \to (p_t, r_t)$ as $T_n \to \infty$ and $r_0^{T_n} \to r_0$ as $T_n \to \infty$. The sequences $(p_t)_1^{\infty}$, $(r_t)_0^{\infty}$ are easily seen to be infinite horizon competitive prices.

Theorems 4.3. and 4.4, taken together, do *not* provide a complete characterization of optimal programs. Example 4.3 below demonstrates that the 'regularity' assumption on the sifficiency side is not superfluous, by constructing a competitive and efficient program which is not optimal. The example rests heavily on the feature of varying utility functions. Below, in an analysis of the stationary case, we shall see that Theorems 4.3 and 4.4 can indeed be tightened to completely characterize the class of optimal programs. Indeed, in one case, a complete characterization is possible without forsaking the generality of $\langle u_t \rangle_1^{\infty}$.

Example 4.3. (A competitive and efficient program which is not optimal)

Consider Example 3.1, and the feasible program defined there, with its associated prices $\langle p_t\rangle_1^\infty$, $\langle r_t\rangle_0^\infty$. Suppose that the utility functions $\langle u_t\rangle_1^\infty$ are of the form

$$u_t(c)=p_tc, \qquad t\geqq 1.$$

Consider the 'golden rule' program $\langle x',c'\rangle$ with $x'_t=x^*$ for $t\geqq 0$, $c'_t=f(x^*)-x^*$, $t\geqq 1$. For this program

$$\liminf_{T\to\infty}\sum_{t=1}^{T}[u_t(c'_t)-u_t(c_t)]=\liminf_{T\to\infty}\sum_{t=1}^{T}p_t(c'_t-c_t)=\infty.$$

Hence, $\langle x,c\rangle$ is not optimal. However, it is certainly competitive and, by the analysis in Example 3.1, efficient.

This result does *not* depend on the linearity of $\langle u_t\rangle_1^\infty$, which is simply assumed for convenience. The reader may construct a similar example with a strictly concave u_t, of sufficiently small curvature, satisfying $u'_t(c_t)=p_t$ (take u_t to be differentiable). The crucial issue here is the time-dependence of the utility functions.

5. Characterizing optimal programs: The stationary case

In this section, we tighten the results obtained in the general case by considering a model with stationary technology. We shall take $g_t=g$ for all $t\geqq 0$ and $\delta_t=\delta\in(0,1)$ for all $t\geqq 0$. Consider, first, a situation where $u_t=\rho^t u$, $0<\rho\leqq 1$.

We shall make use of the following lemma more than once, which is true under (T.1) and (T.2):

Lemma 5.1. Suppose that $\langle f_t\rangle$ is a sequence of differentiable functions. Consider a program $\langle x,c\rangle$ with $x_t>0$ for all $t\geqq 0$, competitive relative to prices $\langle p_t\rangle_1^\infty$, $\langle r_t\rangle_0^\infty$. Then $r_t\leqq q_t$ for all $t\geqq 0$, where q_t is defined by (2.6) with $K=r_0$.

Proof. Clearly, $r_0\leqq q_0$. Suppose, now, that $r_s\leqq q_s$ fro some $s\geqq 0$. Since $x_s>0$, we have, using the competitive condition (2.4a),

$$p_{s+1}g'_s(x_s)+r_{s+1}\delta_s=r_s. \tag{5.1}$$

Suppose that $r_{s+1}>q_{s+1}$. By (2.4b), $p_{s+1}>q_{s+1}$. Using this in (5.7), we have

$$q_{s+1}f'_s(x_s)<p_{s+1}g'_s(x_s)+r_{s+1}\delta_s=r_s\leqq q_s,$$

contradicting the definition of q_{s+1}. Hence $r_{s+1} \leqq q_{s+1}$, and so $r_t \leqq q_t$ for all $t \geqq 0$.

For Theorem 5.1, we shall assume:

(T.5) *$f(x)=g(x)+\delta x$ is continuously differentiable, with $f(0)=0$, $f'(x)>0$, $[\lim_{x \downarrow 0} f'(x)]\rho > 1$, and f is concave.*

(U.3) *u is continuously differentiable, with $u'(0)>0$, and u is strictly concave.*

Theorem 5.1. Under (T.5) and (U.3), a program $\langle x, c \rangle$ is optimal if and only if it is competitive, efficient and regular.

Proof (Sufficiency). Follows from Theorem 4.3.

(Necessity). First note that $\langle x, c \rangle$ is competitive and efficient by Theorem 4.4. From the analysis in Majumdar and Nermuth (1981), we may conclude that optimal programs exhibit either (a) $x_{t+1} \geqq x_t$ for all $t \geqq 0$, or (b) $x_{t+1} \leqq x_t$ for all $t \geqq 0$. In case (b), we will rule out the situation $\lim_{t \to \infty} x_t = 0$.

Use the fact that $x_t \geqq \delta^t \mathbf{x} > 0$ for all $t \geqq 0$, and competitive condition (2.4a) to get

$$p_{t+1} g'(x_t) + r_{t+1}\delta - r_t = 0, \qquad t \geqq 0. \tag{5.2}$$

Use (2.5) to get

$$p_{t+1} \geqq \rho^{t+1} u'(c_{t+1}), \qquad t \geqq 0. \tag{5.3}$$

Combining (5.2) and (5.3), noting that $r_{t+1} \geqq 0$, and applying Lemma 5.1,

$$\rho^{t+1} u'(c_{t+1}) g'(x_t) \leqq p_{t+1} g'(x_t) \leqq r_t \leqq q_t, \tag{5.4}$$

where q_t is defined in Lemma 5.1.

Now

$$q_{t+1} = r_0 \Big/ \prod_{s=0}^{t-1} (f'(x_s)), \qquad t \geqq 0. \tag{5.5}$$

Hence by (5.4) and (5.5),

$$u'(c_{t+1}) g'(x_t) \leqq (1/\rho)^{t+1} \cdot r_0 \Big/ \prod_{s=0}^{t-1} (f'(x_s)), \qquad t \geqq 1. \tag{5.6}$$

Suppose, now, that $x_t \to 0$ as $t \to \infty$. Then

$$f'(x_t) \to f'(0) > 1/\rho.$$

It is therefore easy to see that the right-hand side of (5.6) tends to zero as $t \to \infty$. Since $g'(x_t) \geqq g'(x) > 0$, it follows that

$$u'(c_t) \to 0 \quad \text{as} \quad t \to \infty. \tag{5.7}$$

Therefore $\lim_{t\to\infty} \inf c_t > 0$. Since $f(0)=0$, this contradicts the supposition that $\lim_{t\to\infty} x_t = 0$. Thus, in case (b), $\lim_{t\to\infty} x_t = \tilde{x} > 0$. So $z_t \to \tilde{z} > 0$. In this case,

$$\liminf_{t\to\infty} z_t/x_t \geqq \liminf_{t\to\infty} z_t/x = \tilde{z}/x > 0.$$

Return now to case (a). In this case,

$$z_{t+1} = x_{t+1} - \delta x_t = x_{t+1} - x_t + (1-\delta)x_t \geqq (1-\delta)x_t.$$

So

$$z_{t+1}/x_{t+1} \geqq (1-\delta)(x_t/f(x_t)) \geqq (1-\delta)(x/f(x)) > 0.$$

Therefore $\lim_{t\to\infty} \inf z_t/x_t > 0$, proving the theorem.

The strongly productive case actually admits of a complete characterization (of this type) in situations of far greater generality. Such a characterization is also instructive, because it shows that in the special cases studied above regularity of a feasible program is *implied* by competitiveness and efficiency when the technology is strongly productive. Continue to assume that $g_t = g$ for all $t \geqq 0$, $\delta_t = \delta \in (0,1)$ for all $t \geqq 0$, but retain the generality of $\langle u_t \rangle_1^\infty$. We shall assume

(T.6) *$f(x) = g(x) + \delta x$ is differentiable and concave, with $\inf_{x \geqq 0} f'(x) = a > 1$, and $f(0)=0$.*

We now state:

Theorem 5.2. Under (T.6), a program $\langle x, c \rangle$ is optimal if and only if it is competitive and efficient.

Proof (Necessity). Follows from Theorem 4.4.

(Sufficiency). Suppose, on the contrary, that $\langle x, c \rangle$ is competitive and efficient, but not optimal. Therefore there exists a feasible program $\langle x', c' \rangle$ an integer T^*, and $\alpha > 0$ such that

$$\sum_{t=1}^{T} [u_t(c_t') - u_t(c_t)] \geqq \alpha \quad \text{for} \quad T \geqq T^*. \tag{5.8}$$

Since for $\langle x', c' \rangle$,

$$r_T x_T \geqq r_T(x_T - x_T') \geqq \sum_{t=1}^{T} [u_t(c_t') - u_t(c_t)], \tag{5.9}$$

it follows that $\lim_{t\to\infty} \inf r_t x_t > 0$. Now given $\delta \in (0,1)$, we must have $x_t > 0$ for all $t \geqq 0$. Hence Lemma 5.1 applies, and $r_t \leqq q_t$ for all $t \geqq 0$ (with $\langle q_t \rangle$ defined as in the Lemma). Therefore

$$\liminf_{t\to\infty} q_t x_t > 0, \tag{5.10}$$

and since $q_t > 0$, $x_t > 0$ for $t \geqq 0$, it follows that

$$\inf_{t\to\infty} q_t x_t = \beta > 0. \tag{5.11}$$

Now,

$$q_t x_t = q_{t+1} f'(x_t) x_t \leqq q_{t+1} f(x_t) = q_{t+1} c_{t+1} + q_{t+1} x_{t+1}$$

$$= q_{t+1} c_{t+1} + q_{t+1} z_{t+1} + q_{t+1} \delta x_t \leqq q_{t+1} c_{t+1} + q_{t+1} z_{t+1} + (\delta/a) q_t x_t.$$

Hence, for all $t \geqq 0$,

$$q_{t+1} z_{t+1} \geqq (1 - \delta/a) q_t x_t - q_{t+1} c_{t+1}$$

$$\geqq (1 - \delta/a)\beta - q_{t+1} c_{t+1}$$

$$= \eta - q_{t+1} c_{t+1} \quad \text{where} \quad \eta > 0. \tag{5.12}$$

It is a property of the strongly-productive technology that $\sum_{t=1}^{\infty} q_t c_t < \infty$ [see, for example, Mitra (1979a)], and so $q_t c_t \to 0$ as $t \to \infty$. Using this in (5.12), we have

$$\liminf_{t\to\infty} q_{t+1} z_{t+1} > 0. \tag{5.13}$$

Another feature of the strongly productive case is that $q_t x_t \leqq M < \infty$ [see for example, Benveniste (1976)]. Therefore

$$z_t / x_t = q_t z_t / q_t x_t \geqq q_t z_t / M,$$

and using (5.13),

$$\liminf_{t\to\infty} z_t/x_t > 0.$$

But this implies $\langle x,c\rangle$ is regular. Hence, by Theorem 4.3, it is optimal, which contradicts our original supposition. This proves the theorem.

The final task of this section (and the paper) is to explore, as promised, an alternative route to characterizing infinite-horizon optimality. This involves obtaining a complete characterization of optimal programs by means of the competitive conditions and a transversality criterion.

Consider the discounted case of the stationary model, with $\rho<1$.

We shall assume

(T.7) *$f(x)=g(x)+\delta x$ is twice continuously differentiable, with $f(0)=0$, $f'(x)>0$, $f''(x)<0$, and $\lim_{x\downarrow 0} f'(x)=\infty$.*

(U.4) *u is twice continuously differentiable, with $u(0)=0$, $u'(c)>0$, $u''(c)<0$, and $\lim_{c\downarrow 0} u'(c)=\infty$.*

Theorem 5.3. Under (T.7) and (U.4) with $\rho<1$, a feasible program $\langle x,c\rangle$ is optimal if and only if (i) *it is competitive,* (ii) $\lim_{t\to\infty}\inf r_t x_t=0$.

Proof (Sufficiency). For any feasible program $\langle x',c'\rangle$, recall that we may write, for $T\geqq 1$,

$$\sum_{t=1}^{T} \rho^t[u(c_t')-u(c_t)] \leqq r_T x_T.$$

Since $\lim_{T\to\infty}\inf r_T x_T=0$, we are done.

(Necessity). Optimal programs are competitive, by Theorem 4.4.

In the case where $\inf_{x\geqq 0} f'(x)=a\leqq 1$ follow the analysis of Majumdar and Nermuth (1981) to obtain $x_t\to\tilde{x}$ as $t\to\infty$, where $\tilde{x}$ satisfies $\rho f'(\tilde{x})=1$. Since $\rho<1$, $f'(\tilde{x})>1$, and so $q_t\to 0$ as $t\to\infty$. Thus $\lim_{t\to\infty}\inf q_t x_t=\lim_{t\to\infty} q_t\tilde{x}=0$. Since $x_t>0$ for all $t\geqq 0$, Lemma 5.1 applies, and $q_t\geqq r_t$. Hence $\lim_{t\to\infty}\inf r_t x_t =0$.

In the case where $\inf_{x\geqq 0} f'(x)=a>1$, proceed in the following way.[10] We

[10]The rest of the proof is due to K. Sengupta. Observe that a result of the argument is that efficient programs imply $\lim_{t\to\infty}\inf q_t x_t=0$. It is well known [see for example, Mitra (1979a, corollary 1)] $\lim_{t\to\infty}\inf q_t x_t=0$ implies efficiency. Thus efficient programs in the 'strongly productive' case ($a>1$) are completely characterized by the condition $\lim_{t\to\infty}\inf q_t x_t\doteq 0$. It can be shown, therefore, that consumption-value-maximization *completely characterizes* efficiency in the strong productive case. We are indebted to Mr. Sengupta for this argument.

will show that $\lim_{t\to\infty}\inf q_t x_t = 0$. Suppose, on the contrary, that this is not true; then there exists $\alpha > 0$ and T^* such that $q_t x_t \geqq \alpha$ for $t \geqq T^*$. Now observe that, for $t \geqq 0$,

$$g(x_t) \geqq g'(x_t)x_t \geqq (a-\delta)x_t. \tag{5.14}$$

We have

$$g(x_t) \geqq z_{t+1} = x_{t+1} - \delta x_t,$$

so using (5.14), we get

$$\beta g(x_t) \geqq x_{t+1} \quad \text{where} \quad \beta = a/(a-\delta) > 0. \tag{5.15}$$

Multiplying both sides of (5.15) by q_{t+1}, we get, for $t > T^*$,

$$q_{t+1}g(x_t) \geqq (1/\beta)q_{t+1}x_{t+1} \geqq \alpha/\beta > 0. \tag{5.16}$$

Now we know that $\inf_{x \geqq 0} f'(x) > 1$ implies [see, for example, Mitra (1979b)]

$$\sum_{t=1}^{\infty} q_t c_t < \infty. \tag{5.17}$$

Using (5.16) and (5.17), we have

$$\sum_{t=0}^{\infty} q_{t+1}c_{t+1}/q_{t+1}g(x_t) < \infty. \tag{5.18}$$

Define

$$m_T = \sum_{t=T}^{\infty} q_{t+1}c_{t+1}/q_{t+1}g(x_t).$$

Clearly, $m_T > 0$ for $T \geqq 1$, otherwise $\langle x, c\rangle$ is inefficient, hence not optimal. Also, $m_T \to 0$ as $T \to \infty$.

Observe now that $z_t > 0$ along a subsequence of t, otherwise $\lim_{t\to\infty}\inf q_t x_t = 0$ (since $q_t \to 0$ as $t \to \infty$). Therefore, there exists T such that $m_t < 1$ for $t \geqq T$ and $z_T > 0$.

Define a program $\langle x', c'\rangle$ by $x'_0 = x$, $z'_t = z_t$, $0 \leqq t < T$, $z'_t = m_t z_t$ for $t \geqq T$, $x'_{t+1} = z'_{t+1} + \delta x'_t$, $t \geqq 0$, $c'_{t+1} = f(x'_t) - x'_{t+1}$, $t \geqq 0$. Clearly, $z'_t \geqq 0$ for all t. If we can show that $c'_t > c_t$ for $t = T$, and $c'_t \geqq c_t$ for all $t \geqq 1$, we will establish the inefficiency of $\langle x, c\rangle$. This we will now do:

$$c'_T = g(x'_{T-1}) - z'_T > g(x_{T-1}) - z'_T/m_T$$

$$= g(x_{T-1}) - z_T = c_T \quad (\text{using } z'_T = m_T z_T > 0).$$

For $t<T$, $c_t' \geqq c_t$. For $t \geqq T$, we first show that $x_t' \geqq m_t x_t$. For $t=T$,

$$x_t' = z_t' + \delta x_{t-1} \geqq m_t[z_t + \delta x_{t-1}] = m_t x_t.$$

Now suppose that $x_s' \geqq m_s x_s$, for some $s \geqq T$. Then

$$x_{s+1}' = z_{s+1}' + \delta x_s \geqq m_{s+1} z_{s+1} + m_s \delta x_s \geqq m_{s+1}[z_{s+1} + \delta x_s]$$

$$= m_{s+1} x_s \quad [\text{using } m_s \geqq m_{s+1}].$$

So $x_t' \geqq m_t x_t$ for $t \geqq T$. Therefore, for $t \geqq T$,

$$c_{t+1}' = g(x_t') - z_{t+1}' \geqq g(m_t x_t) - m_{t+1} z_{t+1}$$

$$\geqq m_t g(x_t) - m_{t+1} z_{t+1} \geqq g(x_t)[m_t - m_{t+1}] \quad [\text{since } g(x_t) \geqq z_{t+1}].$$

Therefore,

$$c_{t+1}' \geqq g(x_t) \cdot q_{t+1} c_{t+1} / q_{t+1} g(x_t) = c_{t+1},$$

which shows that $\langle x, c \rangle$ is inefficient.

But $\langle x, c \rangle$ is optimal, and therefore efficient, which is a contradiction. Hence $\lim_{t \to \infty} \inf q_t x_t = 0$, and so by Lemma 5.1, $\lim_{t \to \infty} \inf r_t x_t = 0$.

References

Benveniste, L., 1976, Two notes on the Malinvaud condition for efficiency of infinite horizon programs, Journal of Economic Theory 12, 338–346.

Brock, W., 1971, Sensitivity of optimal growth paths with respect to a change in target stocks, Zeitschrift für Nationalökonomie, suppl. 1, 73–89.

Cass, D., 1972, On capital overaccumulation in the aggregative neo-classical model of economic growth: A complete characterization, Journal of Economic Theory 4, 200–223.

Cass, D. and M. Yaari, 1971, Present values playing the role of efficiency prices in the one-good growth model, Review of Economic Studies 38, 331–339.

Majumdar, M. and M. Nermuth, 1981, Dynamic optimization in non-convex models with irreversible investment: Sensitivity and turnpike resuls, Mimeo. (Department of Economics, Cornell University, Ithaca, NY).

Mitra, T., 1978, A note on efficient growth with irreversible investment and the Phelps–Koopmans theorem, Journal of Economic Theory 18, 216–223.

Mitra, T., 1979a, On the value-maximizing property of infinite-horizon efficient programs, International Economic Review 20, 635–642.

Mitra, T., 1979b, Identifying inefficiency in smooth aggregative models of economic growth: A unifying criterion, Journal of mathematical Economics, 85–111.

Mitra, T., 1981, Sensitivity of optimal programs with respects to changes in target stocks: The case of irreversible investment, Journal of Economic Theory, forthcoming.

Peleg, B. and H. Ryder 1972, On optimal consumption plans in a multi-sector economy, Review of Economic Studies 39, 159–169.