Monetary equilibrium in an overlapping generations model with productive capital⋆

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Summary. We study perfect foresight competitive equilibrium in an overlapping generations model with productive capital and a fixed nominal stock of money. We obtain almost-complete characterizations of (a) the existence of a monetary equilibrium from an arbitrary initial capital stock, and (b) the existence of an *efficient* monetary equilibrium from an arbitrary initial capital stock. When the initial capital stock is no larger than the golden rule stock, the necessary and sufficient condition for both (a) and (b) is the dynamic inefficiency (in the sense of Malinvaud) of the autarkic (or nonmonetary) equilibrium from the same initial stock. However, this condition, though necessary, is *not sufficient* for the existence of a monetary equilibrium when the initial stock exceeds the golden rule stock (and still more conditions are needed for an *efficient* monetary equilibrium to exist). We provide characterizations for these cases, and as corollaries obtain examples in which (a) the nonmonetary equilibrium is inefficient but no monetary equilibrium exists, and (b) monetary equilibria exist but no efficient monetary equilibrium does.

1 Introduction

We study an overlapping generations economy with productive capital and a fixed nominal stock of money. Our objectives are

(i) to characterize the conditions under which a monetary equilibrium exists, and
(ii) to characterize the conditions under which a Pareto-optimal monetary equilibrium exists.

The fact that productive capital is available as a potential store of wealth makes the analysis of these questions fundamentally different from that of the standard exchange economy model.

Samuelson [15] was the first to observe that a competitive equilibrium *without* money may fail to generate a Pareto-optimal distribution of goods across

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generations. He then suggested that the introduction of a fixed nominal supply of money would be sufficient to ensure that a Pareto-optimal competitive equilibrium exists.

Samuelson's suggestion about the role of money has been made sharper in subsequent investigations, notably in the important contribution of Cass, Okuno and Zilcha [8]. The need for sharper statements arose for the following reasons. First, the introduction of a *supply* of outside money does not guarantee that money will actually be *used* in a competitive equilibrium. This raises the need, in the first place, for establishing the existence of a competitive equilibrium in which money commands a *strictly positive price*, i.e., the existence of a *monetary* equilibrium. Secondly, even when the outcome is indeed a monetary equilibrium, this does not guarantee the Pareto-optimality of such an equilibrium. However, in Samuelson's model, the following propositions can be established (see, e.g., [8]):

(i)′ There is a monetary equilibrium if and only if no non-monetary equilibrium is Pareto-optimal.
(ii)′ A Pareto-optimal monetary equilibrium exists whenever a monetary equilibrium does.
(iii)′ There is at most one Pareto-optimal monetary equilibrium.

Observe that (i)′ and (ii)′ serve as "answers" to the questions raised by (i) and (ii). In fact, whenever there is a monetary equilibrium there is actually a continuum of monetary equilibria relative to any given nominal stock of money. Associated with a particular monetary equilibrium is a particular initial price of money (in terms of goods). As is common for perfect foresight equilibria, the "choice" of this initial price is not determined by the model. Thus (iii)′ implies that the government must get the initial price of money exactly right if its one-shot intervention is to result in Pareto-optimality.

Continuing in the context of an exchange economy with a single perishable good but giving up the assumption of identical tastes, the general validity of these propositions was examined in [8]. The results of their careful scrutiny were counterexamples to (i)′ and (ii)′ (see also [7]). These examples rely on heterogeneity among consumers of the same generation, or on the non-stationarity of tastes over time, or on the violation of convexity or monotonicity in preferences. Similar counterexamples in a stationary one-good exchange economy with identical tastes have also been obtained by Mitra [14], but the price paid for simplicity in this direction is the essential reliance of such examples on a time-varying nominal stock of money. On the other hand, the study by Benveniste and Cass [3] is more supportive of Samuelson's original insight that an appropriately priced fixed quantity of money is sufficient to restore Pareto-optimality. The authors extend Samuelson's model to a multidimensional commodity space and demonstrate that there always exists *some* Pareto-optimal stationary equilibrium if individuals are permitted to issue bonds.

The above papers, as also the important contributions of Gale [10] and Balasko and Shell [1], consider a pure exchange economy with given endowments of perishable goods. Except for [3], in these papers money is the *only* possible asset, the unique vehicle for transferring consumption across time. Then it may not be

surprising that certain Pareto-improving moves, that were not attainable through the market in the absence of *any* money, are made accessible once *some* money is made available. But what if there is an alternative asset that is essentially distinct from money? Would there still be room enough for money to effect further improvements? We feel that investigating these questions would help bring into sharper focus the *distinctive* role of money *vis-a-vis* other stores of value. A simple step in this direction is to enrich Samuelson's framework by allowing the holding of productive capital as an asset.

An early extension of the Samuelsonian model to include production with durable capital is the well known piece by Diamond [9]. His questions were, however, different. He studied competitive programs, giving an explicit example of a (non-monetary) competitive program that is inefficient, and focussed on tax-financed, interest-bearing redeemable public debt. Wilson [17] also studied the structure of competitive programs in an overlapping generations model with capital, obtaining some results on the optimality of steady states and on the long-run behaviour of competitive programs. Recently, Galor and Ryder [11] have studied the existence, uniqueness and stability of non-monetary equilibrium in the Diamond model, which is a special case of the model of this paper. But these papers do not study monetary equilibria.

Tirole [16] studies asset bubbles in an overlapping generations model with productive capital, which is equivalent to studying the existence of a monetary equilibrium. Tirole's model includes rent-bearing assets; when these are removed, his model reduces to a special case of ours.[1] Moreover, Tirole does not examine the question of existence of an *efficient* monetary equilibrium.

Apart from the introduction of capital, our model is strictly Samuelsonian: the structure of the economy in respect of tastes and technology is stationary, consumers are identical except for their dates of birth, there is only one good, and the rate of population growth is constant. Equilibrium outcomes are much more varied, though. Instead of one non-monetary equilibrium for the model, there is one for each initial capital stock. Similarly, the set of monetary equilibria is parametrized by the initial capital stock. Finally, the failure of the "invisible hand" can be detected without any reference at all to the distribution of goods and hence to the concept of Pareto-optimality, but by observing whether or not there is *overaccumulation of capital* in the long run – a feature first highlighted by Malinvaud [13].

Overaccumulation of capital signals *inefficiency*. It is possible, then, to obtain a larger supply of *aggregate* consumption in some period without less in any other. This appears to be a more preliminary problem in the sense that with monotonic preferences, no feasible program that is inefficient can hope to be Pareto-optimal, and the converse need not be true. However, for *competitive* programs, the requirements of efficiency and Pareto-optimality have been shown to be *exactly* the same (see [4]).[2] Therefore, in addressing issues of Pareto-optimality, it is necessary

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[1] Specifically, assumptions are made in his paper that guarantee that the limit interest rate along a *nonmonetary* competitive program is independent of the initial capital stock. In our model, the limit behavior depends on initial conditions and leads to new results in the characterization.

[2] This link was further explored in [12] for a multi-sector closed model of production.

and sufficient to focus on the efficiency of competitive programs (see Section 2 below). We use this to work towards a characterization of the existence of monetary and efficient monetary programs.

It turns out that inefficiency (or what is equivalent here, the lack of Pareto-optimality) is necessary but *not* sufficient for the existence of a monetary equilibrium (Theorem 3.1). Moreover, additional conditions are needed to characterize the existence of an *efficient* monetary equilibrium (Theorem 3.2).

These results have the following immediate implication. Unlike the counter-examples obtained by Cass, Okuno and Zilcha [8], Cass [7] and Mitra [14], in our model there are counterexamples to the analogues of (i)′ and (ii)′ even though we assume stationary, identical, monotonic and convex preferences, a stationary technology and a constant nominal stock of money. In fact these counterexamples can arise in our model even when the production function satisfies strong restrictions such as the Inada conditions, or is Cobb–Douglas. In this sense our counterexamples are simpler.

Section 2 sets up the basic model and derives the reduced form appropriate for our purpose. This section is lengthy but unavoidable: in order to state the theorem on the equivalence of efficient competitive programs and Pareto-optimal competitive programs, we have to use the "extensive" rather than the reduced form of the model. The core of the paper is Section 3 which presents the main results and the counterexamples to (i)′ and (ii)′. Proofs for Section 3 are given in Section 4.

2 The model and preliminaries

We describe here the model of the paper, together with all the assumptions and definitions to be used in Sections 3 and 4. Next, we justify the model as a reduced form representation of an explicit overlapping generations economy with productive capital and a constant nominal stock of money. Finally we restate a known result (Proposition 2.1) on the equivalence between the requirements of efficiency and Pareto-optimality for competitive equilibria. This enables us, in Section 3, to state our main result using only the concept of efficiency and to simply assert that these results directly address the issue of Pareto-optimality.

2.1 The model

The working model of this paper can be expressed in terms of a pair of real valued functions (S, f) and a pair of difference equations that chalk out sequences $\{k_t, m_t\}$ such that for all $t \geq 0$, $(k_t, m_t) \in R_+^2$ satisfies

$$k_0 = \mathbf{k} > 0, \quad m_0 \geq 0 \tag{2.1}$$

$$f(k_t) \geq k_{t+1} \geq 0 \tag{2.2}$$

$$f'(k_{t+1})m_t = m_{t+1} \tag{2.3}$$

$$k_{t+1} + m_t = S(k_t, k_{t+1}) \tag{2.4}$$

Here k_t is the capital-labour ratio and m_t is an index of the real stock of money in

period t. $f(k)$ is a normalized production function and $S(k, k')$ is a normalized savings function where k is the "current" capital-labour ratio, and k' the capital-labour ratio in the "next period". The following assumptions will be maintained throughout.

(A.1) $f(k)$ is twice continuously differentiable, with $f'(k) > 0$ and $f''(k) < 0$ for $k > 0$, and with $f(k) = 0$ for $k = 0$.

(A.2) $\lim_{k \to 0} f'(k) > 1 > \lim_{k \to \infty} f'(k)$.

(A.3) $S(k, k')$ is continuous in (k, k') and non-decreasing in k for $k \geq 0$.

(A.4) $S(k, 0) > 0$ for $k > 0$, and for each $m \in [0, S(k, 0)]$, there is a unique value of $k' \geq 0$ such that $m + k' = S(k, k')$.

(A.5) If $S(k^*, k^*) = k^*$, where $f'(k^*) = 1$, then for all sequences $(k, k', k'') \to (k^*, k^*, k^*)$ with $k' \neq k''$,

$$\liminf_{(k, k', k'') \to (k^*, k^*, k^*)} \frac{S(k, k') - S(k, k'')}{k' - k''} > -\infty.$$

Note that by (A.1) and (A.2) there exists a unique positive number k^* such that $f'(k^*) = 1$. This capital stock will be called the golden rule stock. (A.1) and (A.2) enables us to utilize the beautiful Cass criterion for identifying inefficiency ([6], see Lemma 4.1 below). (A.4) is indispensable in order to rule out multiple equilibria from the same initial condition (k_0, m_0). Admitting these would destroy the simple form in which the characterizations are obtained here. Finally, (A.5) is a mild additional restriction; if S is continuously differentiable in (k, k') then all that (A.5) amounts to is the condition that in case $S(k^*, k^*) = k^*$, then the partial derivative of S with respect to its second argument must not be $-\infty$ when evaluated at (k^*, k^*).

Nonnegative sequences satisfying (2.1)–(2.4) will be called (competitive) programs. In Section 2.2 we explain why such programs are equivalent to intertemporal competitive equilibria. A program will be called monetary (resp. non-monetary) if $m_t > 0$ (resp. $m_t = 0$) for $t \geq 0$. By virtue of (A.1) and (2.3), a program is either monetary (if $m_0 > 0$) or non-monetary (if $m_0 = 0$). It also follows from (A.4) that for each $\mathbf{k} > 0$, there is a unique non-monetary program $\{k_t, 0\}$ with $k_0 = \mathbf{k}$.

A program $\{k_t, m_t\}$ is called inefficient if there is a sequence $\{\tilde{k}_t\}$ satisfying $\tilde{k}_0 \leq k_0$ and (2.2) for all $t \geq 0$, such that $f(\tilde{k}_t) - \tilde{k}_{t+1} \geq f(k_t) - k_{t+1}$ for all $t \geq 0$, with strict inequality for some $t \geq 0$. A program is efficient if it is not inefficient.

2.2 The underlying extensive form

We focus on intertemporal perfect foresight competitive equilibrium from some arbitrary initial date.

Consumers

Each consumer lives for two periods and at the beginning of every period t, N_t consumers are born. A consumer born in period t will be called a 't-person'. A t-person has the ability to supply one unit of labour when young and h units of labour when old, where $h \geq 0$. There are no bequests. A t-person consumes x_t (when young) and y_{t+1} (when old) of a single good, and derives a lifetime utility level u_t where $u_t = u(x_t, y_{t+1})$ is a real valued function on R_+^2 satisfying

(U.1) $u(x, y)$ *is continuous, strictly quasi-concave and strictly increasing in each argument.*

A t-person faces given non-negative market prices, (w_t, w_{t+1}) for labour and (q_t, q_{t+1}) for money – all four expressed in units of the produced good – and also a given real return factor, R_{t+1}, on capital investment. The t-person then responds by choosing a consumption vector (x_t, y_{t+1}) and a portfolio (a_t, b_t) of capital and money so as to maximize $u(x, y)$ subject to

$$x + a + q_t b = w_t \tag{2.5}$$

$$y = h w_{t+1} + R_{t+1} a + q_{t+1} b \tag{2.6}$$

$$(x, y, a, b) \geq 0 \tag{2.7}$$

Note that given (U.1), no (x_t, y_{t+1}) can be utility-maximizing unless the price of the produced good is strictly positive in each period. Thus it is legitimate to express the prices of labour and money in terms of the produced good as (2.5) and (2.6) do. Also, given (U.1) there is no loss of generality in expressing the budget constraints as equalities.

Production

A single good is produced by a constant-returns-to-scale production function. In any period, aggregate *net* output (excluding the capital stock used) is given by $Lg(k)$, where L is the aggregate labour employed, k is the capital-labour ratio ($K \equiv Lk$ is aggregate capital) and $g: R_+ \to R_+$ satisfies

(F.1) $g(k)$ *is twice continuously differentiable with* $g(0) = 0$, *and with* $g(k) > 0$, $g'(k) > 0$, *and* $g''(k) < 0$ *for all* $k > 0$.

In every period, firms maximize profits, given the wage rate (w_t) and rental rate (R_t). Define $w(k) \equiv g(k) - kg'(k)$ and $r(k) \equiv g'(k)$. Note that if $k_t > 0$, then $w_t = w(k_t)$ and $R_t = 1 + r(k_t)$.

Market-clearing exchanges

As an expository device consider the following narrative. In any period there are three spot markets. Two of these are factor markets, one for the services of labour, another for the services of capital. These exchange for output. Finally there is the money market in which money exchanges for the produced good. The two factor markets open up first with firms settling their dues with instantly produced output. With the clearing of the factor markets, the entire net output will have been distributed as factor payments and at this point firms call it a day as far as the current period goes. The money market is activated now. The elders offer their entire stock of money to the young in exchange for the produced good and consume this together with whatever factor incomes they have received in this period and whatever capital they may have accumulated in their youth. The young, after having exchanged part of their wages for money, decide to consume part of the residual. What remains with them of the produced good after that constitutes the capital stock available to the economy for the next round of production.

Recall that N_t is the number of young persons in period t (i.e., the number of t-persons). Since at the beginning of the period the entire stock of capital is owned by the elders (i.e., $(t-1)$-persons), the supply of capital services is $N_{t-1}a_{t-1}$. The demand for capital in the service market is K_t and this comes from firms. The demand and supply of labour services is L_t and $(N_t + hN_{t-1})$ respectively. Finally, the demand for money is $N_t b_t$ and its supply is $N_{t-1}b_{t-1}$, there being no money creation or destruction by the government after $t = 0$.

We now describe the market clearing conditions that will later enter into our description of competitive equilibrium. As money is intrinsically useless, we must allow for an excess supply of nominal money and a consequent zero price in equilibrium.

The supply-demand balances for any $t \geq 1$ are:

$$L_t = N_t + hN_{t-1} \tag{2.8}$$

$$K_t = N_{t-1}a_{t-1} \tag{2.9}$$

$$N_t b_t \leq N_{t-1}b_{t-1}, \quad q_t(N_t b_t - N_{t-1}b_{t-1}) = 0 \tag{2.10}$$

and at the initial date we have

$$L_0 = N_0 + \mathbf{N} \tag{2.11}$$

$$K_0 = \mathbf{K} \tag{2.12}$$

$$N_0 b_0 \leq \mathbf{M}, \quad q_0(N_0 b_0 - \mathbf{M}) = 0 \tag{2.13}$$

Here $\mathbf{N} > 0$ and $\mathbf{K} > 0$ are historically given supplies of labour and capital processed by the elders and $\mathbf{M} > 0$ is an exogenously supplied nominal stock of money, either brought forward by the elders or supplied by the "government".

Competitive equilibrium

Given a strictly positive sequence $\{N_t\}$ such that $N_{t+1} = nN_t$ for $t \geq 0$ and $n > 0$, and strictly positive numbers $\mathbf{M}, \mathbf{K}$ and $\mathbf{N}$, a *competitive equilibrium* is a non-negative sequence $\{x_t, y_t, a_t, b_t, L_t, K_t, k_t, w_t, q_t, R_t\}$ such that for each $t \geq 0$

  (i) k_t, w_t, R_t are strictly positive.
 (ii) (x_t, y_{t+1}, a_t, b_t) maximizes $u(x, y)$ subject to (2.5)–(2.7).
(iii) (k_t, L_t) maximizes $L[g(k) + k - w_t - R_t k]$ over $(k, L) \geq 0$; $K_t = L_t k_t$
 (iv) (a_t, b_t, L_t, K_t) satisfy (2.8)–(2.13).

We shall now observe certain simple properties of a competitive equilibrium and, making use of them, pass on to the more economical reduced form. The first of these properties is

PROPERTY 2.1 *In a competitive equilibrium, for each* $t \geq 0$,
 (i) $q_{t+1} = R_{t+1}q_t$, *and*
(ii) $q_t > 0$ *implies* $N_t b_t = \mathbf{M}$

Property 2.1 is very intuitive. Part (i) says that the rates of return to holding capital and money should be equalized in competitive equilibrium. Part (ii) says

that if money has a positive equilibrium price, then its demand equals its supply. The proof of these observations is omitted.

In view of Property 2.1, we are justified in replacing the money market equilibrium condition (2.10) by

$$N_t b_t \leq \mathbf{M}, \quad q_t(N_t b_t - \mathbf{M}) = 0 \tag{2.14}$$

Multiplying both sides of condition (i) of Property 2.1 by $\mathbf{M}$ and noting that $N_{t+1} = nN_t$ we obtain, using (2.14):

$$nq_{t+1}b_{t+1} = R_{t+1}q_t b_t \tag{2.15}$$

relating the real value of money in successive periods to the ratio between the real return factor and the population growth factor.

We are now in a position to derive the reduced form (2.1)–(2.4). In the light of Property 2.1, a t-person may be assumed to select (x_t, y_{t+1}) so as to maximize $u(x, y)$ subject to $(x, y) \geq 0$, $x \leq w_t$, and the life-time budget constraint

$$R_{t+1}x + y = R_{t+1}w_t + hw_{t+1} \tag{2.16}$$

Given (w, w', R'), let $s(w, w', R') \equiv \arg\max u(w - s, R's + hw')$ and let $\bar{s}(k, k') \equiv s(w(k), w(k'), 1 + r(k'))$. In a competitive equilibrium, total savings by an individual, s_t, must equal $\bar{s}(k_t, k_{t+1})$. Next, note from (2.5) that $s_t = a_t + q_t b_t$, and that given $N_{t+1} = nN_t$, (2.8) and (2.9) imply $a_t = (n + h)k_{t+1}$. Therefore, defining

$$S(k, k') \equiv \bar{s}(k, k)/(n + h)$$

and

$$m_t \equiv q_t b_t/(n + h),$$

we obtain (2.4).

Define $f(k) \equiv [g(k) + k]/n$. Then $nf'(k) = 1 + g'(k)$ and so in a competitive equilibrium, $nf'(k_t) = R_t$. Using this and the definition of m_t in (2.15) immediately gives (2.3). For (2.2), simply note that $w(k) = n[f(k) - kf'(k)] \leq nf(k) \leq (n + h)f(k)$ since $h \geq 0$, hence $(n + h)k_{t+1} \leq s_t \leq w_t \leq (n + h)f(k_t)$. Finally (2.1) simply incorporates the initial condition $(\mathbf{K}, N_0 + \mathbf{N}, \mathbf{M}) \gg 0$ and $q_0 \geq 0$.

We are now in a position to comment on the assumptions of the reduced form. The assumptions of continuity and differentiability derive from those on g. Assumption (F.1) on g implies all conditions required by (A.1). (A.2) is equivalent to the existence of positive numbers v_1 and v_2 such that $g'(v_1) > n - 1 > g'(v_2)$, where $n - 1$ is the population growth rate. This standard requirement for the existence of a golden rule stock requires no comment. For (A.3), the continuity of S derives from the continuity and curvature properties of u and f. The assumption that $S(k, k')$ is nondecreasing in k is equivalent to second-period consumption being non-inferior. (A.4), which concerns the response of second-period consumption to changes in k', is admittedly more troublesome. If the change in k' is translated into the corresponding change in the rental rate, the reaction in first-period savings is composed of the familiar income and substitution effects, and there is no theoretical reason to presuppose any particular direction of change. However, our assumption is

equivalent to assuming that $S(k, k') - k'$ is decreasing in k'^{3} and therefore allows $S(k, k')$ to be ambiguous with respect to k', though within bounds. Some standard utility functions (e.g. Cobb-Douglas) comfortably satisfy (A.4).

2.3 Pareto-optimality and efficiency

It will be convenient to express the extensive form above in per-capita terms.

A *feasible program* from $(\mathbf{k}, \mathbf{y})$ is a non-negative sequence $\{k_t, x_t, y_t, u_t\}$ such that

$$u_t = u(x_t, y_{t+1}),\tag{2.17}$$

$$f(k_t) = k_{t+1} + \frac{nx_t + y_t}{n(n + h)},\tag{2.18}$$

for $t \geq 0$, and

$$k_0 = \mathbf{k}, \quad y_0 = \mathbf{y}\tag{2.19}$$

This per-capita version of the extensive model, is easy to derive from the balance equations:

(a) $L_t g(k_t) = K_{t+1} - K_t + C_t$
(b) $L_t = N_t + hN_{t-1}$
(c) $C_t = N_t x_t + N_{t-1} y_t$

where C is aggregate consumption.

Note that a competitive equilibrium $\{x_t, y_t, a_t, b_t, L_t, K_t, k_t, w_t, q_t, R_t\}$ generates a feasible program in the components $\{k_t, x_t, y_t, u(x_t, y_t)\}$, where $\mathbf{k} \equiv K/N_0 + h\mathbf{N}$, and $\mathbf{y} \equiv \dfrac{[1 + r(k_0)]\mathbf{K} + q_0\mathbf{M}}{\mathbf{N}}$. Call such a (generated) feasible program a *competitive program*.

A feasible program $\{k_t, x_t, y_t, u_t\}$ is *Pareto-optimal* if there is no feasible program $\{\tilde{k}_t, \tilde{x}_t, \tilde{y}_t, \tilde{u}_t\}$ with $\tilde{k}_0 \leq k_0$ and $\tilde{y} \geq y_0$, such that $\tilde{u}_t \geq u_t$ for all $t \geq 0$, with strict inequality for some $t \geq 0$.

Our characterizations will relate the existence of monetary programs from an arbitrary $\mathbf{k} > 0$ to the inefficiency of the non-monetary program from $\mathbf{k}$. But even though our theorems do not explicitly invoke the concept of Pareto-optimality, whereas existing results from the pure exchange model do, our results can be related to the relevant literature by noting the equivalence between the requirements of efficiency and Pareto-optimality for competitive programs. This is what we do now. Bose [4] established equivalence for the above overlapping generations economy. His result was extended to a multi-commodity, closed model of production by Majumdar, McFadden and Mitra [12]. For completeness we state the result here (Proposition 2.1).[4] We strengthen (U.1) to

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[3] Even this can be relaxed but at the cost of further computations.

[4] In [4] this result was established under the needlessly strong hypothesis of a separable utility function but it was mentioned there that this could easily be dispensed with. Also, the result was stated in a form that invoked the idea of "short-run Pareto-optimality". For the present it is sufficient to note that a competitive program is necessarily short-run Pareto-optimal, so the result is stated here for competitive programs.

(U.1)′ *u(x, y) is twice differentiable with $u_i > 0$ for $i = 1, 2$, $u(x, y)$ is strictly concave and $[u_{ij}]$ is negative definite when evaluated at a positive vector.*[5]

PROPOSITION 2.1 *Given (U.1)′, (A.1) and (A.2), suppose that $\{k_t, x_t, y_t, u_t\}$ is a competitive program such that $\inf_{t \geq 0}(k_t, x_t, y_t) > 0$. Then this program is Pareto-optimal if and only if it is efficient.*

Therefore in our central theorems (Theorems 3.1 and 3.2), the word "inefficient" ("efficient") may be replaced by the word "Pareto-inoptimal" ("Pareto-optimal"), on the understanding that (U.1)′ is valid. This is because, under (A.1)–(A.4), along inefficient non-monetary programs and efficient monetary programs, the sequence $\{k_t, x_t, y_t\}$ will necessarily be bounded away from zero.[6]

From this point on we shall concentrate only on the efficiency or inefficiency of programs and make no further reference to the concept of Pareto-optimality.

3 Monetary equilibrium and efficiency

In this central section of our paper, we address the two issues discussed in Section 1. First, is the existence of a monetary program linked to the inefficiency of non-monetary programs? Second, is the existence of an *efficient* monetary program guaranteed under the same conditions that permit *some* monetary program to exist?

3.1 Monetary equilibrium

Our principal result concerning the existence of monetary programs is given by:

THEOREM 3.1 *Suppose that a monetary program exists from $\mathbf{k} > 0$. Then the following conditions hold:*

(i) *The non-monetary equilibrium from $\mathbf{k}$ is inefficient, and*
(ii) *If $\mathbf{k} > k^*$, then there exists $\tilde{k} \in [k^*, \mathbf{k})$ such that $S(\tilde{k}, \tilde{k}) \geq \tilde{k}$.*

Conversely, suppose that for some $\mathbf{k} > 0$,

(i)′ *The non-monetary equilibrium from $\mathbf{k}$ is inefficient, and*
(ii)′ *If $\mathbf{k} > k^*$, then there exists $\tilde{k} \in [k^*, \mathbf{k})$ such that $S(\tilde{k}, \tilde{k}) > \tilde{k}$.*

Then a monetary program from $\mathbf{k}$ exists.

This theorem represents an almost-complete characterization of existence of monetary programs. A complete characterization, while certainly desirable, appears to be not without substantial technical difficulty. However, the result is certainly tight enough to yield some sharp implications. We consider two cases.

Case I: $\mathbf{k} \leq k^*$. In this case, the following observation is an immediate consequence of Theorem 3.1.

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[5] Alternatively, Benveniste's assumption of Q-strictness could be invoked; see [2].
[6] Note that from Lemmas 4.3 and 4.5, on such programs, $\lim k_t \geq k^* > 0$ as $t \to \infty$. The assertion now follows easily in view of (U.1)′, (A.1) and (A.2).

COROLLARY 3.1 *If* $k \leq k^*$, *a monetary program from* k *exits if and only if the non-monetary program from* k *is inefficient.*

In this case, then, the implication is clear. The presence or absence of a productive asset is *not* the crucial test of existence of a monetary program. Rather, it is the "social non-productivity" of capital in the long-run, i.e., the inefficiency of the non-monetary competitive equilibrium from k, that is necessary and sufficient for existence.

Case II: $k > k^*$. In this case, the existence of a monetary program certainly implies that the non-monetary program is inefficient. But the converse is not, in general, true.

Example 3.1 [The non-monetary equilibrium is inefficient, but no monetary program exists]: Let $S(k, k')$ satisfy $S(k^*, k^*) < k^*$, and $S(k, k) \geq k$ for some $k \geq k^*$. It is easy to write down conditions on the primitives of our model (that is, the utility and production functions) such that $S(\cdot, \cdot)$ will satisfy the requirement above. We omit these for brevity.

Let $\hat{k} \equiv \min\{k \geq k^* | S(k, k) = k\}$. Then $\hat{k}$ is well-defined, with $\hat{k} > k^*$. The non-monetary program from $k = \hat{k}$ is simply the stationary program $\{\hat{k}_t\}$ with $\bar{k}_t = \hat{k}$ for all $t \geq 0$, and is obviously inefficient. But it is clear from the definition of $\hat{k}$ that $S(k, k) < k$ for all $k \in [k^*, \hat{k})$. Consequently, (ii) of Theorem 3.1 fails and there is no monetary program from $\hat{k}$.

Examples to make the same point are given in [8] and [14]. However, it should be noted that our example is, in a sense, simpler than these. Cass, Okuna and Zilcha [8] rely for their example on the heterogeneity of consumers in each generation. Mitra's example in [14] relies on the non-stationarity of *nominal* monetary stocks. In our model, there is homogeneity of consumers and a fixed nominal monetary stock. Moreover, the inefficient non-monetary program is time-stationary.

Our model suggests, however, that this counterexample may not be very significant from an economic viewpoint. Consider Figure 1.

Observe that by (i)' and (ii)' of Theorem 3.1, the *only* way in which one can obtain a k satisfying the objective of Example 3.1 is to consider functions $S(\cdot, \cdot)$ such that the *first* intersection of the $S(k, k)$ curve with the 45° line in the region (k^*, ∞) is from "below".[7]

Using (i)' and (ii)' again, we see that if the intersection is a "cut" (as in Figure 1a), there can be no more than *one* capital stock ($\hat{k}$ in Figure 1a) satisfying the feature of Example 3.1. The only way in which there *might* be more than one such stock is if the intersection is a "tangency" (as in Figure 1b). For example, in Figure 1b, all initial capital stocks in the interval $[\hat{k}, \bar{k}]$ satisfy the necessary conditions of Theorem 3.1, and may serve as potential examples.[8]

However, it should be obvious that the configuration of Figure 1b is *not* robust to "small" perturbations in the utility and production functions. While making such

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[7] There must be *some* such intersection, otherwise we will not have an inefficient non-monetary program from any capital stock to start with!

[8] We say "potential" because our theorem is silent on whether there exist monetary programs from the stocks. Condition (ii) *is* satisfied, but condition (ii)' is not.
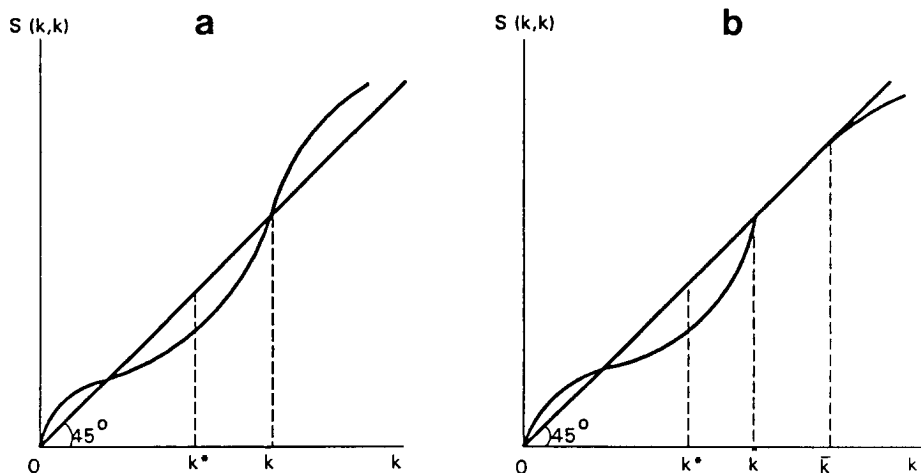
Figure 1 a, b. Inefficiency does not imply existence of monetary program. **a** The "robust" case, **b** a "non-robust" case

an idea absolutely precise is, we feel, harping too much on a relatively unimportant issue, the thrust of all this is quite clear. *Inefficiency of the non-monetary program is sufficient to signal to existence of a monetary program for all but at most one capital stock, for "generic" economies with differentiable technologies.*[9]

3.2 Efficiency monetary equilibrium

We now turn to the question of when there exists an *efficient* monetary program. Recall that by Proposition 2.1 in Section 2, an efficient monetary program is equivalent to a Pareto-optimal monetary program.

We start with an observation on efficient monetary programs.

THEOREM **3.2** *For each* **k** > 0, *there is at most one efficient monetary program.*

This theorem underlines an important fact: that even if an efficient monetary program exists, it is imperative to get the initial price of money exactly right to guarantee that the program will be efficient. While this result is well-known in exchange economy models (at least with one commodity), its counterpart in a model with production does not appear to have been discussed in the literature.

We now state our characterization result on the existence of this (unique) efficient monetary program.

THEOREM **3.3** *Suppose that an efficient monetary program exists from* **k** > 0. *Then*

(i) *the non-monetary program from* **k** *is inefficient, and*

(ii) $S(k^*, k^*) \geq k^*$.

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[9] We use the additional qualification "differentiable technology" because in economies with, say, Leontief technologies this observation is *not* valid (see Bose and Ray [5]).

Conversely, suppose that from some **k** > 0,

(i)′ *the non-monetary program from* **k** *is inefficient, and*
(ii)′ $S(k^*, k^*) > k^*$.

Then there exists an efficient monetary program from **k**.

As in Theorem 3.1, Theorem 3.2 provides an almost-complete characterization of when efficient monetary programs exist. This characterization yields the following observations. Again, consider two cases.

Case 1: k $\leq k^*$. In this case we have the following result.

COROLLARY 3.2 *An efficient monetary program exists from* **k** $\leq k^*$ *if and only if the non-monetary program from* **k** *is inefficient.*

While this corollary is not immediate from Theorem 3.2, it follows from the additional observation that if **k** $\leq k^*$ and autarky is inefficient from **k**, then $S(k^*, k^*) > k^*$. (See Lemma 4.9).

We see, then, *that in this case*, the second question posed in our paper is answered in the affirmative. The existence of *some* monetary program is equivalent to the existence of an *efficient* one. However, the next case yields very different outcomes.

Case II: k $> k^*$. The following corollary of Theorems 3.1 and 3.2 is self-explanatory.

COROLLARY 3.3 *Suppose that for some* **k** $> k^*$, *the non-monetary program from* **k** *is inefficient. If condition (ii) of Theorem 3.1 is satisfied but* $S(k^*, k^*) < k^*$, *then there exists a monetary program but no efficient monetary program from* **k**.

Corollary 3.3 provides a set of sufficient conditions for there to be monetary programs, yet no efficient monetary program. By inspecting Theorems 3.1 and 3.2, it is easy to see that these conditions are "almost" necessary for this phenomenon to occur.

Figure 2 illustrates by displaying a typical $\hat{k}$ where the conditions of Corollary 3.3 are satisfied.

We make the following observations:

1. Indeed, under the conditions, there does not exist an efficient monetary program from *any* initial capital stock in the economy.
2. Unlike Example 3.1, this situation has a strong robustness property. First, whenever it occurs for *some* initial stock **k**, it occurs simultaneously for *every* alternative initial stock above **k**. Second, local perturbations of the given data will *not* alter the outcome. It is sufficient, here, to study the $S(.,.)$ curve in Figure 2. Any small shift of the $S(k, k)$ curve will leave the outcome completely undisturbed: monetary programs exist, but there is none that is efficient.
3. The result of Corollary 3.3 is similar to examples advanced in the literature, notably, those of Cass, Okuno and Zilcha [8] and Mitra ([14], Example 3). Once again, our example (or rather, characterization) appears attractive, because the existing examples rely either on non-stationary preferences and endowments (cf. [8]) or on a non-stationary monetary policy (cf. [14]).
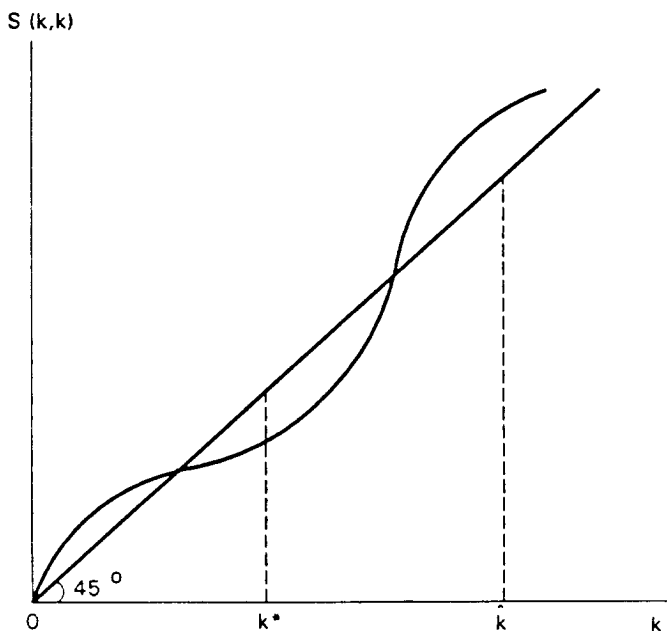
Figure 2. Existence of monetary program does not imply existence of efficient equilibrium.

4 Proofs

Our argument relies heavily on the following theorem, due to Cass [1972].
For $k > 0$ and a program $\{k_t, m_t\}$ from k, define a sequence $\{p_t\}_1^\infty$ by $p_t = [\prod_{s=1}^t f(k_s)]^{-1}$ for all $t \geq 1$.

LEMMA **4.1** *(Cass): Suppose that a program $\{k_t, m_t\}$ from k satisfies* $\inf_{t \geq 0} k_t > 0$. *Then it is efficient if and only if*

$$\sum_{t=1}^\infty (1/p_t) = \infty \tag{4.1}$$

Some additional definitions will be useful. A finite sequence $\{k_t, m_t\}_0^T$ is a *T-program* (from k) if (2.1)–(2.4) holds for all $t = 0, \ldots, T - 1$. A *T*-program is *monetary* if $m_t > 0$ for all $t = 0, \ldots, T - 1$. For ease of writing, we will refer to a *T*-program by simply $\{k_t, m_t\}$. For each $k \geq 0$ and $m \in [0, S(k, 0)]$ define $V(k, m)$ by

$$m + V(k, m) \equiv S(k, V(k, m)) \tag{4.2}$$

Assumptions (A.3) and (A.4) readily yield the following lemma (proof omitted):

LEMMA **4.2** *V is continuous in (k, m), non-decreasing in k, and strictly decreasing in m, on the domain $m \in [0, S(k, 0)]$, $k \in R_+$.*

This observation can then be used to yield

LEMMA **4.3** *Let $\{k_t, m_t\}$ and $\{k_t', m_t'\}$ be programs from $k > 0$, with $m_0 > m_0' \geq 0$.*

Then

$$k_t < k_t' \quad \text{for all } t \geq 1, \tag{4.3}$$

and

$$m_t' < m_t \quad \text{for all } t \geq 0. \tag{4.4}$$

We omit the proof.

LEMMA 4.4 *Consider a (T-)program* $\{k_t, m_t\}$ *such that for some* $t \geq 0, k_{t+1} \leq k^*$. *Then, if* $k_{t+1} \leq k_t$, *we have* $k_{s+1} \leq k_s$, *for all* $s \geq t$.
    On the other hand, suppose that for some $t \geq 0, k_{t+1} \geq k^*$. *Then, if* $k_{t+1} \geq k_t$, *we have* $k_{s+1} \geq k_s$ *for all* $s \geq t$.

Proof: Suppose $k_t \leq k^*$ and $k_{t+1} \leq k_t$. Then $k_{t+2} = V(k_{t+1}, m_{t+1}) \leq V(k_t, m_{t+1}) = V(k_t, f'(k_{t+1})m_t) \leq V(k_t, m_t) = k_{t+1}$.
    Applying this argument repeatedly, we establish the desired result. The other part of the lemma is proved in a similar fashion. ∎

LEMMA 4.5 *If* $\{k_t, m_t\}$ *is a monetary program, then* $(k, m) \equiv \lim_{t \to \infty}(k_t, m_t)$ *exists, and* $k \geq k^*$. *Moreover, if* $k > k^*$ *then* $m = 0$.

Proof: First, we show that $\limsup_{t \to \infty} k_t \geq k^*$. Suppose not. Then there exists $\hat{k} < k^*$ and a time T such that for all $t \geq T, k_t \leq \hat{k}$. But then for all such t, we have $m_{t+1} \geq f'(\hat{k})m_t$, so that $m_t \to \infty$. Now observe that for all $t, k_t \leq \max\{\mathbf{k}, K\}$, where K is such that $f(K) = K$ and $f(x) < x$ for $x > K$. So, using (A.3), $S(k_t, k_{t+1}) \leq \max\{f(\mathbf{k}), f(K)\}$. But now the fact that $m_t \to \infty$ yields a contradiction, using (2.4).
    Next, we show that $\underline{k} \equiv \liminf_{t \to \infty} k_t \geq k^*$. Suppose not. Pick $\epsilon > 0$ such that $\epsilon < (k^* - \underline{k})/3$. Because $\limsup_{t \to \infty} k_t \geq k^*$, there exists t' such that $k_{t'} \geq k^* - \epsilon$. Furthermore, there exists t'' such that $k_{t''} \leq \underline{k} + \epsilon$. So, there must exist $t \in \{t', \ldots, t''\}$ such that $k_{t+1} \leq k_t$ and $k_{t+1} \leq k^*$. But then, by Lemma 4.4, $k_{s+1} \leq k_s$ for all $s \geq t$, and so $\liminf_{t \to \infty} k_t = \limsup_{t \to \infty} k_t$, a contradiction. So $\liminf_{t \to \infty} k_t \geq k^*$.
    Next, we show that $\liminf_{t \to \infty} k_t = \limsup_{t \to \infty} k_t$. Suppose not. Then by an argument similar to that of the preceding paragraph, there exists t such that $k_{t+1} \geq k_t$ and $k_{t+1} \geq k^*$, and this leads to a contradiction.
    Finally, since $f'(k) < 1$ for $k > k^*$, it is obvious that $m = 0$ if $k > k^*$. ∎

LEMMA 4.6 *Let* $\{k_t, m_t\}$ *and* $\{k_t', m_t'\}$ *be two programs from* $\mathbf{k} > 0$, *with* $m_0 > m_0' \geq 0$. *Then* $\{k_t', m_t'\}$ *is inefficient.*

Proof: Define, for each $t \geq 0$,

$$\epsilon_{t+1} \equiv k_{t+1}' - k_{t+1} \tag{4.5}$$

Then, $\epsilon_{t+1} > 0$ for all t, by Lemma 4.3. So Lemma 4.5 implies that $\lim_{t \to \infty} k_t' \geq \lim_{t \to \infty} k_t \geq k^*$. Given this information, it suffices to consider the case $\lim_{t \to \infty} k_t' = \lim_{t \to \infty} k_t = k^*$. For if either of those equalities fail to hold, it is immediate from Lemma 4.1 that $\{k_t', m_t'\}$ is inefficient. Now,

$$\begin{aligned} f'(k_{t+1}') &= f'(k_{t+1} + \epsilon_{t+1}) \\ &= f'(k_{t+1}) + \epsilon_{t+1}f''(\theta_{t+1}) \end{aligned} \tag{4.6}$$

for some $\theta_{t+1} \in [k_{t+1}, k_{t+1}']$.

Because $\{k_t'\}$ and $\{k_t\}$ are positive sequences with positive limits, they are uniformly bounded above and below by positive numbers. So, using (A.2), there exists $a > 0$ such that

$$-\frac{f''(\theta_{t+1})}{f'(\theta_{t+1})} \geq a > 0.$$

for all $t \geq 0$. Using this information together with (4.6), we have

$$\frac{f'(k_{t+1})}{f'(k_{t+1}')} = 1 + \epsilon_{t+1}\left[\frac{-f''(\theta_{t+1})}{f'(k_{t+1}')}\right]$$

$$\geq 1 + \epsilon_{t+1}\left[\frac{-f''(\theta_{t+1})}{f'(\theta_{t+1})}\right]$$

$$\geq 1 + a\epsilon_{t+1} \qquad (4.7)$$

Now define $p_t' \equiv \left[\prod_{s=1}^{t} f'(k_s')\right]^{-1}$ for all $t \geq 1$. Then, using (2.3) and (4.7), we have $p_{t+1}' m_{t+1} \geq (1 + a\epsilon_{t+1})p_t'm_t$ and $p_{t+1}' m_{t+1}' = p_t'm_t'$ for all $t \geq 1$. Consequently, noting that $m_t > m_t'$ for all $t \geq 0$ (Lemma 4.3), we have for all $t \geq 1$,

$$\frac{1}{p_{t+1}'(m_{t+1} - m_{t+1}')} \leq \frac{1}{(1 + a\epsilon_{t+1})p_t'm_t - p_t'm_t'}$$

$$= \frac{1}{p_t'(m_t - m_t')} - \frac{a\epsilon_{t+1}m_t}{(m_t - m_t')\{(1 + a\epsilon_{t+1})p_t'm_t - p_t'm_t'\}}$$

so that

$$\frac{1}{p_{t+1}'(m_{t+1} - m_{t+1}')} \leq \frac{1}{p_1'(m_1 - m_1')} - \sum_{s=1}^{t} \frac{a\epsilon_{s+1}m_s}{(m_s - m_s')\{(1 + a\epsilon_{s+1})p_s'm_s - p_s'm_s'\}} \qquad (4.8)$$

for all $t \geq 1$.

Using (4.8) and noting that ϵ_t is bounded above for all $t \geq 1$,

$$\sum_{t=1}^{\infty} \epsilon_{t+1}/(m_t - m_t')p_t' < \infty \qquad (4.9)$$

Now, let us compare ϵ_{t+1} with $(m_t - m_t')$. We have, using Lemma 4.3,

$$\epsilon_{t+1} = k_{t+1}' - k_{t+1}$$

$$= S(k_t', k_{t+1}') - S(k_t, k_{t+1}) + m_t - m_t'$$

$$\geq (m_t - m_t') + \left[\frac{(S(k_t', k_{t+1}') - S(k_t, k_{t+1}))}{k_{t+1}' - k_{t+1}}\right]\epsilon_{t+1}$$

$$\geq (m_t - m_t') + b\epsilon_{t+1}$$

for some $b \in (-\infty, 0)$. The last inequality above uses (A.5) together with the information that (k_t, k_{t+1}, k_{t+1}') converges to (k^*, k^*, k^*) as $t \to \infty$. Therefore, there exists $\gamma > 0$ such that

$$\epsilon_{t+1} \geq \gamma(m_t - m_t'), \quad t \geq 0 \qquad (4.10)$$

Combining (4.9) and (4.10), we see that

$$\sum_{t=1}^{\infty} (1/p'_t) < \infty$$

so by Lemma 4.1, $\{k'_t, m'_t\}$ is inefficient. ∎

LEMMA 4.7 *Fix* $\mathbf{k} > 0$. *For any* $m \geq 0$, *define a sequence* $\{k_t(m)\}$ *by*

$$k_0(m) = \mathbf{k} \qquad (4.11)$$

$$k_{t+1}(m) = V\left(k_t(m), \min\left\{ S(k_t(m), 0), m \prod_{s=1}^{t} f'(k_s(m)) \right\} \right), \quad t \geq 0 \qquad (4.12)$$

interpreting the product in (4.12) as equal to 1 if $t = 0$. *Then, for each* t, $k_t(m)$ *is a continuous function on* R_+, *and if for some* $m > 0$ *and time* T, $k_T(m) > 0$, *then there exists a monetary* T-*program* $\{k_t, m_t\}$ *from* $\mathbf{k}$ *with* $m_0 = m$, $k_T = k_T(m)$.

Proof: Observe that by (A1)–(A4), S is a continuous function and so is $f'(x)$. So the continuity result follows by simply inspecting (4.12), and by noting that the composition of continuous functions is continuous. If $k_T(m) > 0$ for some $T \geq 0$, then note that by (A.1) and (A.3), we have $k_t(m) > 0$ for $t = 0, \ldots, T$. So, using (A.3) and the definition of V,

$$\min\left\{ S(k_t(m), 0), m \prod_{s=1}^{t} f'(k_s(m)) \right\} = m \prod_{s=1}^{t} f'(k_s(m))$$

for each $t = 0, \ldots, T$. The appropriate construction of a monetary T-program $\{k_t, m_t\}$ with $k_T = k_T(m)$ is now obvious. ∎

LEMMA 4.8 *Suppose that* $\mathbf{k} > k^*$, *and that there exists* $\tilde{k} \in [k^*, \mathbf{k})$ *such that* $S(\tilde{k}, \tilde{k}) > \tilde{k}$. *Define* $\hat{k} \equiv \min\{k' \geq k^*/S(k', k') \geq k'\}$. *Then for each* $T \geq 1$, *there exists a monetary* T-*program* $\{k_t, m_t\}$ *from* $\mathbf{k}$ *with* $k_T = \hat{k}$, *and* $m_0 \geq S(\tilde{k}, \tilde{k}) - \tilde{k}$.

Proof: Consider the sequence $\{k_t(m)\}$ from $\mathbf{k}$ defined in (4.11) and (4.12), with $m = \bar{m} \equiv S(\mathbf{k}, 0)$. Then it is easy to check that $k_t(\bar{m}) = 0$ for $t \geq 1$. Next, define $\{k_t(m)\}$ from $\mathbf{k}$ by (4.11) and (4.12) with $m = \tilde{m} \equiv S(\tilde{k}, \tilde{k}) - \tilde{k}$. We will see that $k_t(\tilde{m}) \geq \tilde{k}$ for all $t \geq 0$. This is certainly true for $t = 0$.

Recursively, suppose that for some $t \geq 0$, $k_s(\tilde{m}) \geq \tilde{k}$ for all $0 \leq s \leq t$. Then, because $\tilde{k} \geq k^*$, $\prod_{s=1}^{t} f'(k_s(\tilde{m})) \leq 1$, so that, by (A.3) and (A.4), $S(k_t(\tilde{m}), 0) \geq S(\tilde{k}, 0) \geq S(\tilde{k}, \tilde{k}) - \tilde{k} = \tilde{m} \geq \prod_{s=1}^{t} f'(k_s(\tilde{m}))\tilde{m}$.
Consequently, $k_{t+1}(\tilde{m}) = V(k_t(\tilde{m}), \prod_{s=1}^{t} f'(k_s(\tilde{m}))\tilde{m}) \geq V(\tilde{k}, \tilde{m}) = \tilde{k}$ completing the recursive argument. So for each $T \geq 1$, $k_T(\bar{m}) \leq \hat{k} \leq \tilde{k} \leq k_T(\tilde{m})$.

By Lemma 4.7, there exists $m \in [\tilde{m}, \bar{m}]$ with $k_T(m) = \hat{k} > 0$. Again by Lemma 4.7, there exists a monetary program $\{k_t(m), m_t\}$ from $\mathbf{k}$ with $m_0 = m$ and $k_T(m) = \hat{k}$. Moreover, $m \geq \tilde{m}$. ∎

LEMMA 4.9 *Suppose that* $\mathbf{k} \leq k^*$, *and the non-monetary program from* $\mathbf{k}$ *is inefficient. Then* $S(k^*, k^*) > k^*$, *and there exists* $\hat{T} \geq 1$ *such that for all* $T \geq \hat{T}$ *there is a monetary* T-*program* $\{k_t, m_t\}$ *from* $\mathbf{k}$ *with* $k_T = k^*$.

Proof: We first prove that $S(k^*, k^*) > k^*$. If this is not true, then it is easy to check, using Lemma 4.2, that $\bar{k}_t \leq k^*$ for all $t \geq 0$, where $\{\bar{k}_t\}$ is the non-monetary program from **k**. By Lemma 4.1, this program must be efficient, a contradiction.

Now, exactly the same argument above, coupled with Lemma 4.2, tells us that if the non-monetary program from $\mathbf{k} \leq k^*$ is inefficient, then there exists $\hat{T} \geq 1$ such that $\bar{k}_t > k^*$ for all $t \geq \hat{T}$. So, noting that (4.11) and (4.12) with $m = 0$ coincide with the definition of the non-monetary program, we see that $\bar{k}_t = k_t(0) > k^*$ for $t \geq \hat{T}$. Moreover, defining $\hat{m} \equiv S(k, 0)$, we note that $k_t(\hat{m}) = 0$ for all $t \geq 1$. Consequently, for each $T \geq \hat{T}$, $k_T(\hat{m}) \leq k^*$ and $k_T(0) > k^*$. Therefore, by the continuity of $k_T(\cdot)$ (Lemma 4.7), there exists $m > 0$ such that $k_T(m) = k^*$. By Lemma 4.7, there exists a monetary T-program $\{k_t, m_t\}$ from **k** with $k_T = k^*$ for each $T \geq \hat{T}$. ∎

Proof of Theorem 3.1: Suppose that there exists a monetary program $\{k_t, m_t\}$ from **k**. Applying Lemma 4.6 with $\{k'_t, m'_t\}$ equal to the non-monetary program from **k**, we get (i). To obtain (ii), observe that we are done [using (A.3)] if $S(\mathbf{k}, \mathbf{k}) > \mathbf{k}$. So let us consider the remaining case: $S(\mathbf{k}, \mathbf{k}) \leq \mathbf{k}$. Suppose, in this case, that (ii) is false. Then for all $k' \in [k^*, \mathbf{k})$, $S(k', k') < k'$.

First, note that along $\{k_t, m_t\}$, $k_1 < \mathbf{k}$ (because $S(\mathbf{k}, \mathbf{k}) \leq \mathbf{k}$ and $m_0 > 0$). Next, note that by our supposition that $S(k', k') < k'$ for $k' \in [k^*, \mathbf{k})$, we have $\limsup_{t \to \infty} \bar{k}_t^1 < k^*$, where $\{\bar{k}_t^1\}_1^\infty$ with $\bar{k}_1^1 = k_1$ is the non-monetary sequence from k_1 starting at date 1. Third, by regarding k_1 as an initial stock and $\{k_t, m_t\}_1^\infty$ as a monetary program from k_1, we have from Lemma 4.3 that $k_t \leq \bar{k}_t^1$ for all $t \geq 1$. Consequently,

$$\limsup_{t \to \infty} k_t \leq \limsup_{t \to \infty} \bar{k}_t^1 < k^*$$

which contradicts Lemma 4.5. Therefore, (ii) is true.

Conversely, suppose that (i)′ and (ii)′ hold, for some **k**. We will demonstrate the existence of a monetary program from **k**. Suppose, first that $\mathbf{k} > k^*$. Now note that the condition of Lemma 4.8 is satisfied, so there exists, for each $T \geq 1$, a monetary T-program $\{k_t^T, m_t^T\}$ from **k** with $k_T^T = \hat{k}$, where $\hat{k}$ is defined as in the proof of Lemma 4.8, and $m_0^T \geq S(\tilde{k}, \tilde{k}) - \tilde{k} > 0$. These programs are bounded uniformly as (t, T) vary, so by a diagonal argument, we can find a subsequence of T (retain notation) such that $m_t^T \to \hat{m}_t$, $k_t^T \to \hat{k}_t$, as $T \to \infty$, for each $t \geq 0$.

It is easy to see that $\{\hat{k}_t, \hat{m}_t\}$ is a program from **k**. Moreover, because $m_0^T \geq S(\tilde{k}, \tilde{k}) - \tilde{k} > 0$ for all T, we have $\hat{m} > 0$. So $\{\hat{k}_t, \hat{m}_t\}$ is a *monetary* program.

If $\mathbf{k} \leq k^*$, then the condition of Lemma 4.9 is satisfied. So using the Lemma, we get T-programs $\{k_t^T, m_t^T\}$ with $k_T^T = k^*$, and by the diagonal argument above, we obtain a program $\{\hat{k}_t, \hat{m}_t\}$ from **k**, as the pointwise limit of these T-programs. Now observe that by Lemma 4.4 and the fact that $k_T^T = k^*$ for all $T \geq \hat{T}$, we have $k_t^T \leq k^*$ for all $T \geq \hat{T}$ and $t = 0, \ldots, T$. Consequently, $\hat{k}_t \leq k^*$ for all $t \geq 0$. Recalling (see e.g., the proof of Lemma 4.9) that $\lim_{t \to \infty} \bar{k}_t > k^*$ for the non-monetary program $\{\bar{k}_t\}$ from **k**, we may conclude that $\{\hat{k}_t, \hat{m}_t\}$ is *not* the non-monetary program from **k**. But then $\{\hat{k}_t, \hat{m}_t\}$ is monetary, and we are done. ∎

Proof of Theorem 3.2: Immediate from Lemma 4.6 and the observation that if two monetary programs $\{k_t, m_t\}$ and $\{k'_t, m'_t\}$ from **k** are distinct, then $m_0 \neq m'_0$. ∎

Proof of Theorem 3.3: Let $\{k_t, m_t\}$ be an efficient monetary program from **k**. Then (i) holds by Theorem 3.1. To prove (ii), combine Lemmas 4.1, 4.5 and the fact that $\{k_t, m_t\}$ is efficient to obtain $\lim_{t \to \infty} k_t = k^*$. So, by the continuity of S,

$$0 \le \lim_{t \to \infty} m_t = S(k^*, k^*) - k^*$$

proving (ii).

Conversely, suppose that (i) and (ii) hold for **k**. We must show that there exists an efficient monetary program from **k**. Suppose, first that $\mathbf{k} > k^*$. The condition of Lemma 4.8 is now satisfied, with $\hat{k}$ as defined in the proof of that lemma equal to k^*. Consequently, for each monetary T-program $\{k_t^T, m_t^T\}$ such that $k_T^T = k^*$ we have, by Lemma 4.5, that $k_t^T \ge k^*$ for $t = 0, \ldots, T$. Therefore,

$$m_{t+1}^T \le m_t^T, \quad t = 0, \ldots, T-1 \tag{4.13}$$

Now,

$$m_T^T = f'(k_T^T) m_{T-1}^T = f'(k^*) m_{T-1}^T = m_{T-1}^T \tag{4.14}$$

and $k^* = V(k_{T-1}^T, m_{T-1}^T)$, where $k_{T-1}^T \ge k^*$.

Consequently, by Lemma 4.2 and (4.14), $m_T^T = m_{T-1}^T \ge m^* \equiv S(k^*, k^*) - k^*$. Combining this information with (4.13),

$$m_t^T \ge m^* \quad \text{for } t = 0, \ldots, T-1$$

So along the limit program $\{\hat{k}_t, \hat{m}_t\}$, $\hat{m}_t \ge m^*$ for all $t \ge 0$. By Lemma 4.5,

$$\lim_{t \to \infty} \hat{k}_t = k^* \quad \text{and} \quad \lim_{t \to \infty} \hat{m}_t = m^*$$

In the other case, $\mathbf{k} \le k^*$. Here, simply follow the same argument in the corresponding section of the proof of Theorem 3.1. We observe, additionally, that because $\hat{k}_t \le k^*$ for all $t \ge 0$, $\hat{m}_{t+1} \ge \hat{m}_t$ for all $t \ge 0$. Consequently, $\lim_{t \to \infty} \hat{m}_t > 0$, and so by Lemma 4.5, $\lim_{t \to \infty}(\hat{k}_t, \hat{m}_t) = (k^*, m^*)$ in this case too. So, in both cases,

$$\lim_{T \to \infty} \hat{m}_0 \prod_{t=1}^{T} f'(\hat{k}_t) = m^* > 0$$

so that

$$\sum_{T=1}^{\infty} \prod_{t=1}^{T} f'(\hat{k}_t) = \infty.$$

Therefore, by Lemma 4.1, $\langle \hat{k}_t, \hat{m}_t \rangle$ is an efficient monetary program, and this completes the proof. ■

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