ALTRUISTIC GROWTH ECONOMIES*

II. Properties of Bequest Equilibria

PART II. Properties of Bequest Equilibria

1. Introduction

In this paper, we study the properties of equilibria in a stationary version of the aggregative growth model with intergenerational altruism introduced in Bernheim and Ray $[1983] \cdot \frac{1}{}$ In this model, each generation is active for a single period. At the beginning of this period it receives an endowment of a single homogeneous good which is the output from a 'bequest investment' made by the previous generation. It divides the endowment between consumption and investment. The return from this investment constitutes the endowment of the next generation. Each generation derives utility from its own consumption and that of its immediate successor. However, since altruism is <u>limited</u>, in the sense that no generation cares about later successors, the interests of distinct agents come into conflict.

Models of this type have been used to analyze a number of issues concerning intergenerational altruism. One line of research, pursued by Arrow [1973] and Dasgupta [1974a] elucidates the implications of Rawl's principle of just savings. Others, beginning with Phelps and Pollak [1968], have addressed the question of how an 'altruistic growth economy' might actually evolve over time. Topics of subsequent investigation have included the asymptotic behavior of capital stocks, the efficiency and optimality of equilibrium programs, and the implications of intergenerational altruism for the distribution of wealth.

In previous efforts to characterize the properties of altruistic growth economies, one of two approaches has been adopted. The first, employed by Phelps and Pollak [1968] and Dasgupta [1974a,b], is to

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simplify the basic structural model by assuming special functional forms. Generality is sacrificed to assure tractability. The second approach, adopted by Iane and Mitra [1981], is to only consider equilibria belonging to a specific class--generations are restricted to choose their strategies from the set of linear consumption functions. As Goldman [1980] has pointed out, in such an equilibrium, agents need not act in their own best interests off the equilibrium path. In general, the set of equilibria in linear strategies and the set of perfect equilibria (in the sense of Selten [1975]) are entirely disjoint. Consequently, this second approach is unsatisfactory.

This previous emphasis on parametric specifications and restricted strategy spaces can perhaps be explained in part by the fact that, until quite recently, no general proof of the existence of perfect equilibria had been exhibited for such models. Indeed, the available existence theorems, adopted from the literature on consistent plans (Peleg and Yaari [1973]), applied only to models where agents were restricted to choose linear strategies, as in Lane and Mitra. Furthermore, Kohlberg [1976] exhibited a disturbing counterexample, for which he demonstrated that no stationary, perfect equilibrium existed in differentiable strategies.

In Bernheim and Ray [1983], we established the general existence of perfect 'bequest equilibria' in a reasonably well behaved class of consumption functions (upper semicontinuous, continuous from the left, with limits on the right). Since this is inherently the most interesting class of equilibria, it is important to characterize the properties

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of associated programs. We concern ourselves here with two classes of properties --- 'positive' and 'normative'.

The positive features of equilibrium programs have received little attention from previous authors. Aside from a few comments by Kohlberg [1976], virtually nothing is known about the asymptotic behavior of capital stocks. In particular, will the long-run capital stock which arises from intergenerational conflict be higher or lower than the 'turnpike' associated with the solution to the optimal planning problem? On <u>a priori</u> grounds, the answer is not clear. Agents who take only a limited interest in the future will tend to bequeath less than those who are far-sighted. However, since each generation views its children's bequest as pure waste, it must bequeath a larger sum to obtain the same consumption value.

In this paper, we obtain steady-state results for equilibrium capital stocks completely analogous to the well-known optimal planning results. By comparing 'steady-states', we show that no limit point of equilibrium capital stocks can exceed the planning turnpike. Under slightly more restrictive conditions, we show that the equilibrium capital stock never exceeds the planning stock in any period. Consequently, limited intergenerational altruism may provide the basis for a theory of chronic capital shortages.

A second set of questions addressed here concern normative issues. In particular, Dasgupta [1974b] has argued that equilibrium programs are never Pareto optimal. Lane and Mitra [1981] demonstrated that, nevertheless, in some cases there exist equilibrium programs which are

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Pareto-optimal in a modified sense. However, as discussed above, these results are not entirely satisfactory, in that they apply only to specific parametrizations of the general model, or require that agents select strategies which imply implausible behavior outside of equilibrium.

In this paper, we extend these previous results to the set of perfect equilibria for altruistic growth economies. Our method of proof differs substantively from that employed in previous work. Other authors have assured tractability by considering equilibria in extremely well behaved strategies (Lane and Mitra [1981] take consumption functions to be linear; Kohlberg [1976] takes them to be differentiable). Since we have established existence in a significantly less tractable class of strategies, this approach is unsatisfactory. Nevertheless, we obtain our results without restricting the class of admissible strategies.

The current paper is organized as follows. Section 2 displays the model, basic assumptions and definitions of equilibria, and reviews some important results presented in Bernheim and Ray [1983]. Positive aspects of equilibrium programs are considered in Section 3; normative aspects are discussed in Section 4. All proofs are deferred to Section 5.

2. The Model

The model is closely related to that of Kohlberg [1976], and corresponds to the stationary altruistic growth economy described in

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Bernheim and Ray [1983]. There is one commodity, which may be consumed or invested. The transformation of capital stock into output takes one period, and is represented by a <u>production function</u> f. In the following sections, certain results require only one weak assumption about f:

(A.1) $f : \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$ is increasing, continuous, and f(0) = 0.

To establish other results, we strengthen this assumption by adding either or both of the following restrictions:

(A.2) f is continuously differentiable and strictly concave.

(A.3) $\lim_{k \to 0} f'(k) > 1$

In each time period, decisions conerning production and consumption are made by a fresh generation. Thus, generation t is endowed with some initial output (y_t) , which it divides between consumption (c_t) and investment $(k_t = y_t - c_t)$. Each generation derives utility from its own consumption, and the consumption of the generation immediately succeeding it. Preferences are represented by a common utility function, u. We assume that u satisfies certain relatively weak conditions:

(A.4) u: $\mathbb{R}^2_+ \rightarrow \mathbb{R}$ is increasing, continuous, and strictly <u>concave</u>. Further, for all c_t , c'_t , c_{t+1} , c'_{t+1} with $c_t \geq c'_t \geq 0$, $c_{t+1} \geq c'_{t+1} \geq 0$,

 $u(c_t, c_{t+1}) - u(c_t', c_{t+1}) \ge u(c_t, c_{t+1}') - u(c_t', c_{t+1}')$.

<u>Remark</u>: The second half of (A.4) is simply an assumption of <u>weak</u> <u>complementarity.</u>²/ For u differentiable, it is equivalent to $\partial^2 u/\partial c_t \partial c_{t+1} \ge 0.$

In most of the relevant literature, a stronger version of (A.4) is employed. In particular, utility is taken to be separable in c_t and c_{t+1} , and to have a rather specific form. At some points, we adopt this more restrictive formulation, in part for technical reasons, and in part to facilitate a comparison of equilibrium and planning programs. Specifically, we occasionally strengthen (A.4) to

(A.5) <u>There exists an increasing</u>, <u>continuously differentiable</u>, <u>strictly concave function</u> v: $\mathbb{R}_+ \to \mathbb{R}$ with $v(c) \to \infty$ as $c \to \infty$ and a <u>discount factor</u> $\delta > 0$ <u>such that</u> $u(c, c') = v(c) + \delta v(c')$ for $(c, c') \in \mathbb{R}^2_+$.

For certain results (particularly those concerning comparisons between equilibrium and planning programs) it will be convenient to assume that agents discount the future at some positive rate. In these cases, we will impose one of the following restrictions on δ .

(A.5.1) $\delta \in (0,1]$ (A.5.2) $\delta \in (0,1)$

(A.5.1) and (A.5.2) can be thought of as the discounted, and strictly discounted cases, respectively.

Finally, to prove certain results, we employ the following assumption concerning the relationship between production and utility.

(A.6)
$$\lim_{k \to \infty} f'(k) < \delta^{-1}$$

<u>Remark</u>: A sufficient condition for (A.6) under either (A.5.1) or (A.5.2) is that the production function eventually cross (and stay below) the 45° line. This assumption rules out the case, lim f'(k) > δ^{-1} , but for most of the results presented here, our $k \rightarrow \infty$ techniques are readily applicable to that situation.³/ See Kohlberg [1976] for a partial analysis of the 'utility-productive' case when f is linear.⁴/

We take the historically given initial output at time zero, y, to lie in some compact interval [0,Y], Y > 0. A program $\langle y_t, c_t, k_t \rangle_0^{\infty}$ is feasible from $y \in [0,Y]$ if

 $y_{0} = y$ $y_{t} = c_{t} + k_{t} , t > 0$ $y_{t+1} = f(k_{t}) , t > 0$ $(y_{t}, c_{t}, k_{t}) > 0 , t > 0$

Denote by $\langle c_t \rangle_0^{\infty}$ the corresponding <u>feasible consumption program</u>. The <u>pure accumulation program</u> is a sequence $\langle \bar{y}_t, \bar{c}_t, \bar{k}_t \rangle_0^{\infty}$ with $\bar{c}_t = 0$ for all $t \ge 0$, $\bar{y}_t = \bar{k}_t$ for all $t \ge 0$, $\bar{y}_{t+1} = f_t(\bar{k}_t)$ for all $t \ge 0$, and $\bar{y}_0 = Y$.

Define C_t as the set of functions $C: [0, \bar{y}_t] \neq [0, \bar{y}_t]$, with $C(y) \leq y$ for all $y \in [0, \bar{y}_t]$. Define $U(c, y, C_{t+1}) = u(c, C_{t+1}(f_t(y - c)))$ for all $C_{t+1} \in C_{t+1}$, and (c,y) > 0 with $c \leq y \leq \overline{y}_t$.

We will impose the behavioral assumption that all generations select perfect Nash strategies (see Selten (1965)). Formally,

<u>Definition</u>: The sequence $\langle C_t^* \rangle_0^{\infty}$, $C_t^* \in C_t$, $t \ge 0$ is a <u>bequest</u> <u>equilibrium</u> (or simply, <u>equilibrium</u>) if for all $t \ge 0$ and $y \in [0, \bar{y}_+]$,

$$\begin{array}{c} C_t^*(y) \in \arg \max \quad U(c, y; C_{t+1}^*) \\ 0 < c < y \end{array}$$

The reader should be aware that although we have restricted attention to the class of strategies for which consumption depends only upon initial endowment, our bequest equilibria continue to be perfect equilibria when all restrictions on strategic choice are removed. See Bernheim and Ray (1983] for a more complete discussion.

A bequest equilibrium is <u>stationary</u> if the equilibrium consumption functions $\langle C_t^* \rangle_0^{\infty}$ satisfy $C_t^*(y) = C_{t+1}^*(y)$ for all $y \in [0, \overline{y}_t]$, $t \ge 0$.

In Bernheim and Ray [1983], we established the existence of a bequest equilibrium under assumptions (A.1) and (A.4), without imposing stationarity on the underlying structure of the model. In addition, we proved that when the underlying model is stationary (the case considered here), a stationary bequest equilibrium exists. Existence is guaranteed within the class of consumption functions which are upper semicontinuous, continuous from the left, with limits on the right.

As an important step in establishing existence, we proved that equilibrium consumption functions satisfy the 'Keynesian property'--the marginal propensity to consume out of endowment never exceeds unity. Formally,

<u>Definition</u>: A consumption function $C_t \in C_t$ satisfies the <u>Keynesian property</u> if for all $y_1, y_2 = [0, \overline{y}_t]$, with $y_1 < y_2, C_t(y_2) - C_t(y_1) \le y_2 - y_1$.

Theorem 2.1: Suppose that for some consumption function $C_{t+1} \in C_{t+1}$ used by generation t + 1, an optimal consumption function for generation $t, C_t \in C_t$, given by

$$C_t(y) \in \underset{0 \leq c \leq y}{\operatorname{arg max}} U(c,y; C_{t+1}) , y \in [0,y_t^-]$$

is well defined. Then, under (A.1) and (A.4), C_t satisfies the Keynesian property.

This theorem is used extensively throughout the current paper. Its proof is omitted--the reader is referred to Bernheim and Ray [1983].

Note that Theorem 2.1 does not rule out the possibility that the marginal propensity to consume out of endowment exactly equals unity. Due to severe technical problems associated with this case, several of our results apply only to equilibrium programs which are 'strictly Keynesian', in the following sense. Define, for each $C_t \in C_t$, $y \ge 0$, and $\varepsilon > 0$,

$$\lambda(C_{t},y,\varepsilon) \equiv \sup\{\frac{C_{t}(y^{2}) - C_{t}(y^{1})}{y^{2} - y^{1}} | (y^{1},y^{2}) \geq C_{t}(y^{2}) | (y^{2},y^{2}) \leq C_{t}(y^{2}) | (y^{2},y^{2}) | (y^{2},y^{2})$$

and
$$y - \varepsilon < y^1 < y^2 < y + \varepsilon$$

Clearly, if C_t is an equilibrium consumption function, $\lambda(C_t, y, \varepsilon) \leq 1$ for all $y \leq 0$ and $\varepsilon > 0$. We wish to rule out the case of equality.

<u>Definition</u>: Suppose that for some $C_t \in C_t$, $y \in [0, \overline{y}_t]$, there exists ε such that $\lambda(C_t, y, \varepsilon) < 1$. Then we say that C_t is <u>strictly Keynesian</u> at y. Now suppose that $\langle C^*_t \rangle_0^{\infty}$ is a bequest equilibrium, and $\langle y_t, c_t, k_t \rangle_0^{\infty}$ some equilibrium program originating from $y_0 > 0$. Then this program is <u>strictly Keynesian</u> if C_t^* is strictly Keynesian at y_t for all t > 0.

Results which pertain only to strictly Keynesian equilibrium programs are, of course, limited in scope. However, in cases where we have been able to solve for interesting bequest equilibria explicitly, this condition has been satisfied.

3. Positive Behavior

In intertemporal optimal planning models, an important characteristic of optimum capital stocks and consumption levels is that these converge, over time, to some stationary input-output-consumption configuration. In this section, we establish some analogous results for the limiting behavior of capital stocks under a bequest equilibrium.

In stationary models, <u>stationary</u> equilibria always exist (Bernheim and Ray [1983]). Of course, this does not preclude the existence of nonstationary equilibria in such models. Of particular interest for asymptotic stock behavior are periodic nonstationary equilibria. Definition. An equilibrium $\langle C^*_t \rangle_0^{\infty}$ is periodic if there exists an integer T and T functions $(\bar{c}_1, \dots, \bar{c}_T)$ such that $C^*_{t+Tn} = \bar{c}_t$, $n = 0, 1, 2, \dots, t = 1, \dots, T$. The integer T is the period of the equilibrium.

When equilibria are nonstationary, the intertemporal behavior of stocks is governed by a nonstationary process, even though the underlying model is stationary. In these situations, while limiting stocks may not exhibit convergence, a bound on their oscillatory behavior may be obtained.

Theorem 3.1 Suppose that $\langle C_t^* \rangle_0^\infty$ is a periodic equilibrium (with period T). Then under (A.1) and (A.4) the sequence of equilibrium stocks has at most T limit points in $\mathbb{R} \cup \{+\infty\}$.

Theorem 3.1 is interesting because it tells us that an upper limit on the oscillation of <u>stocks</u> is given by the number of different <u>functions</u> which constitute the periodic equilibrium. It also immediately yields a steady state result for stationary equilibria.

<u>Corollary 3.1</u> (Steady State Theorem for Stationary Bequest Equilibria): Suppose that $\langle C_t^* \rangle_0^{\infty}$ is a stationary equilibrium with equilibrium stocks $\langle k_t^* \rangle_0^{\infty}$. Then under (A.1) and (A.4) $\lim_{t \to \infty} k_t^* \equiv k_t^*$ exists in $\mathbb{R} \cup \{+\infty\}$.

Remarks:

(i) Corollary 3.1 does not preclude the possibility that $\lim_{t \to \infty} k_t^* = \infty$, which is not, strictly speaking, a steady state property. However. $k^* < \infty$ in a large class of situations (at least, in all situations where the corresponding optimal growth 'turnpike' is finite - see below).

(ii) By continuity of $f(\cdot)$, it is clear that equilibrium consumptions and outputs also converge to some limiting values whenever equilibrium stocks do.

(iii) The steady state theorem has been obtained <u>without</u> assuming separability of the utility functions, or convexity of the technology. In these respects, compare the result to that obtained by Mitra and Ray [1983] for planning models.

We now turn to a comparison of limiting capital stocks for bequest equilibrium with turnpike levels obtained in aggregative planning models. An omniscient planner who takes into account the infinite stream of utilities of all generations is clearly acting more farsighted than a single generation which only cares about the consumption of its successor. On that score, one would expect a larger stock to be generated in the long-run, under planning. However, while each generation cares only about its successor, it <u>recognizes</u> that its successor, will compensate by bequeathing a larger amount. This tends to <u>increase</u> the limiting capital stock under a bequest equilibrium. The question of which steady state is larger, is, therefore, nontrivial.

To facilitate comparison, assume that (A.5) holds. In the corresponding planned economy version of the model, a 'planner' seeks a feasible consumption program $\langle c_t^* \rangle_0^\infty$ such that for all feasible consumption

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programs $\langle c_t \rangle_0^{\infty}$,

(3.1)
$$\liminf_{T \to \infty} \sum_{t=0}^{T} \delta^{t} \left[v(c_{t}^{*}) - v(c_{t}) \right] \geq 0,$$

or, if all feasible utility sums converge, the planner maximizes, subject to feasibility constraints,

(3.2)
$$\sum_{t=0}^{\infty} \delta^{t} v(c_{t})$$

Call such a maximizing program an optimal program.

That this maximization process adequately represents the corresponding planned economy may be rationalized in two ways. First, we may simply envisage a <u>formal</u> comparision between two economies, identical in technology and one-period utilities; the one governed by two-period bequest motives, the other by an omniscient planner whose social welfare function is expressible as (3.2), or the form implicit in (3.1). Secondly, we can imagine all consumption choices in the altruistic growth economy being left to the planner who has the same discount factor δ as each generation. In this case, the planner replaces the maximization of (3.2) by $\frac{5}{2}$

(3.3)
$$\max_{\substack{c_t > 0 \\ c_t > 0 \\ c_t$$

(the inclusion of $v(c_0)$ separately signifies that the planner also cares for the utility of generation -1). But this is simply a scalar multiple of (3.3).

We now state a well-known turnpike theorem for the planning problem ((3.1) or (3.2)). $\frac{6}{}$.

Theorem 3.2. (Turnpike Theorem under Optimal Planning): Under (A.1), (A.2), (A.5), (A.5.1), and (A.6), an optimal program with stocks $\langle \hat{k}_t \rangle_0^{\infty}$ exists. The sequence of stocks $\langle \hat{k}_t \rangle_0^{\infty}$ converges, as $t \neq \infty$, to a limit stock $\hat{k} \in [0,\infty)$. If $\hat{k} > 0$, it solves the equation $\delta f'(\hat{k}) = 1$. If $\delta \lim_{k \to 0} f'(k) > 1$, then $\hat{k} > 0$.

Theorem 3.3 establishes a general result on the relative asymptotic behavior of $\langle k_t^* \rangle_0^{\infty}$ and $\langle \hat{k}_t \rangle_0^{\infty}$. For stationary bequest equilibria, the comparison between \hat{k} and k^* is then obtained as an immediate corollary.

<u>Theorem 3.3</u>: Under (A.1), (A.2), (A.5), (A.5.1), and (A.6), <u>suppose that</u> $\langle C_t^* \rangle_0^{\infty}$ is a bequest equilibrium with stocks $\langle k_t^* \rangle_0^{\infty}$. <u>Then</u> lim sup $k_t^* \leq \hat{k}$, the planning turnpike. $t \rightarrow \infty$

<u>Corollary 3.2</u>: <u>Under</u> (A.1), (A.2), (A.5), (A.5.1), <u>and</u> (A.6), <u>a</u> stationary bequest equilibrium with limiting capital stock k^* has the property $k^* < \hat{k}$.

Remark: Whether or not the strict inequality $k^* < \hat{k}$ holds, when $\hat{k} > 0$, remains an open question.

Theorem 3.3 establishes that, in the limit, a planned economy must accumulate at least as much capital as an altruistic growth economy. We reiterate that a recognition of the fact of a longer planning horizon does <u>not</u> allow us to conclude that $\hat{k} \ge k^*$. This is because the relative 'myopia' in a bequest economy is offset by the fact that bequests by successors to future generations may induce a larger bequest (capital stock) by the present generation.

If we add to our list of assumptions the relatively technical condition (A.3), and if we assume strict discounting, then we obtain a much stronger result: bequest equilibrium capital stocks do not exceed optimal planning stocks in any period, given the same initial output. $\underline{T}/$

<u>Theorem 3.4</u>: <u>Under</u> (A.1), (A.2), (A.3), (A.5), (A.5.2) <u>and</u> (A.6), <u>let</u> $\langle y_t^*, k_t^*, c_t^* \rangle_0^{\infty}$ <u>be a program originating from</u> $y_0 \in (0, Y)$, <u>generated by the bequest equilibrium</u> $\langle C_t^* \rangle_0^{\infty}$. <u>Then, for all</u> $t \ge 0, k_t^* \le \hat{k}_t, where \langle \hat{k}_t \rangle_0^{\infty}$ <u>is the sequence of optimal planning</u> <u>stocks</u>.

4. Normative Behavior

Although the literature on altruistic growth economies has ignored the positive aspects of bequest equilibria discussed in Section 3, much attention has been directed towards understanding normative issues. Dasgupta [1974a,b] observed that, for a particular class of models, bequest equilibria are never Pareto optimal. Lane and Mitra [1981] corroborate this result for (possibly non-perfect) equilibria in a somewhat more general class of models. However, they introduce a concept of Pareto-optimality, modified in an interesting way (see below). For particular forms of the utility and production functions, they provide sufficient conditions for the existence of equilibria which are modified Pareto optimal.

Unfortunately, these existing results are not entirely satisfactory. Dasgupta considered only a particular parametrization of the general model. Lane and Mitra restricted attention to Nash equilibria consisting of linear consumption functions. As Goldman [1980] has pointed out, in such an equilibrium agents need not act in their own best interests off the equilibrium path. When an agent contemplates deviations from his equilibrium strategy, he envisions later generations selecting actions which do not maximize their utility--the Nash Equilibrium is not dynamically consistent, or 'perfect' in the sense of Selten [1975]. Aside from the particular parametrization analyzed by Dasgupta, the sets of Linear Nash Equilibria and perfect Nash Equilibria ('bequest equilibria') are, in general, entirely disjoint.

In this section, we obtain stronger versions of the results presented in Lane and Mitra and Dasgupta for perfect equilibria (bequest equilibria) in a more general model.

Following the existing literature, we consider three normative notions: efficiency, Pareto optimality, and modified Pareto optimality. Formal definitions follow:

<u>Definition</u>: A feasible program $\langle y_t, k_t, c_t \rangle_0^{\infty}$ from $y_0 \in (0, Y)$ is <u>efficient</u> if there does not exist a feasible program $\langle y'_t, k'_t, c'_t \rangle_0^{\infty}$ with $c'_t \geq c_t$ for all $t \geq 0$, and $c'_s > c_s$ for some $s \geq 0$.

<u>Definition</u>: A feasible program $\langle y_t, k_t, c_t \rangle_0^{\infty}$ from $y_0 \in (0, Y)$ is <u>Pareto-optimal</u> if there does not exist a feasible program $\langle y'_t, k'_t, c'_t \rangle_0^{\infty}$

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with $u(c'_t, c'_{t+1}) \ge u(c_t, c_{t+1})$ for all $t \ge 0$, and $u(c'_s, c'_{s+1}) > u(c_s, c_{s+1})$ for some $s \ge 0$.

<u>Definition</u>: A feasible program $\langle y_t, k_t, c_t \rangle_0^{\infty}$ from $y_0 \in (0, Y)$ is <u>modified-Pareto-optimal</u> if there does not exist a feasible program $\langle y'_t, k'_t, c'_t \rangle_0^{\infty}$ with $u(c'_t, c'_{t+1}) \ge u(c_t, c_{t+1})$ for all $t \ge 0, u(c'_s, c'_{s+1}) > u(c_s, c_{s+1})$ for some $s \ge 0$, and $c'_0 = c_0$.

The definitions of efficiency and Pareto optimality are standard. The notion of modified Pareto optimality is due to Lane and Mitra [1981]. The restriction that $c_0 \approx c'_0$ for any comparison program $\langle y'_t, k'_t, c'_t \rangle_0^{\infty}$ reflects the recognition that time 0 is <u>not</u> the beginning of all mankind, and therefore, in considering Pareto dominance, the utility of generation -1 (which depends on c_0) must not be tampered with.

Given Theorem 3.3, it is possible to establish the efficiency of equilibrium programs by applying known results.

<u>Theorem 4.1</u>: <u>Under</u> (A.1), (A.2), (A.3), (A.5), (A.5.2), <u>and</u> (A.6), <u>if</u> $\langle y_t, c_t, k_t \rangle_0^{\infty}$ <u>is a feasible program from</u> y_0 <u>generated by some</u> <u>bequest equilibrium</u> $\langle c_t^* \rangle_0^{\infty}$, <u>then it is efficient</u>.

Since the utility of each generation depends on its own consumption as well as that of its successor, efficiency in consumption does not guarantee Pareto optimality. In fact, as long as the marginal propensity to consume of generation 1 is less than unity, a transfer of consumption from generation 0 to generation 1 always yields a Pareto dominating allocation. In this way, we establish

<u>Theorem 4.2</u>: Under (A.1), (A.2), and (A.5), assume $\langle y_t, k_t, c_t \rangle_0^{\infty}$ is a feasible program from y_0 generated by a bequest <u>equilibrium</u> $\langle C_t^* \rangle_0^{\infty}$. Then if C_1^* is strictly Keynesian at y_1 , and if $k_0 > 0, c_0 > 0, \langle y_t, k_t, c_t \rangle_0^{\infty}$ is not Pareto optimal.

Of course, a scheme for dominating the equilibrium program by lowering c_0 leaves generation -1 strictly worse off. If we rule out alternatives which are damaging to this pre-historic generation, it becomes impossible to dominate efficient equilibrium programs. The efficiency of these programs alone is sufficient to guarantee modified Pareto optimality. This is stated in

<u>Theorem 4.3:</u> Let $\langle y_t, k_t, c_t \rangle_0^{\infty}$ <u>be a feasible, strictly Keynesian</u> program from y_0 associated with some bequest equilibrium $\langle c_t^* \rangle_0^{\infty}$. <u>Under</u> (A.1), (A.2), and (A.5), $\langle y_t, k_t, c_t \rangle_0^{\infty}$ is modified Pareto optimal if and only if it is efficient.

Notice that the conditions used to guarantee the equivalence of efficiency and modified Pareto optimality are weaker than those used to establish the efficiency of equilibrium programs (Theorem 4.1). Coupling Theorems 4.1 and 4.3, we obtain as an immediate corollary:

<u>Corollary 4.1</u>: Let $\langle y_t, k_t, c_t \rangle_0^{\infty}$ be a feasible, strictly <u>Keynesian program from</u> y_0 associated with some bequest equilibrium $\langle C_t^* \rangle_0^{\infty}$. <u>Under</u> (A.1), (A.2), (A.3), (A.5), (A.5.2), and (A.6), $\langle y_t, k_t, c_t \rangle_0^{\infty}$ is <u>modified Pareto optimal</u>. Notice that every result in this section aside from Theorem 4.1 applies only to equilibria satisfying a strict Keynesian property. Although the property is somewhat stronger than that actually needed to establish the results, we have not yet discovered an interesting way to weaken it. Although experience suggests that equilibrium programs are characteristically strictly Keynesian, violations of this property cannot be dismissed lightly. Normative behavior in such cases remains an important open question for further research.

To illustrate the complexities involved when the strict Keynesian property is violated, we present the following hypothetical situation. Suppose equilibrium consumption functions $\langle C_t^* \rangle_0^{\infty}$ are strictly Keynsian along the equilibrium program for all t, <u>except</u> for t = 1. Due to the concavity of the parametric functions, it is then impossible to Pareto dominate (in the traditional sense) the equilibrium plan by transferring resources from generation 0 to generation 1, as in the proof of Theorem 4.2. This effectively leaves us with only the class of alternative programs admitted under the definition of modified-Pareto optimality. By Corollary 4.1, it is impossible to arrange a dominating allocation belonging to this class. Consequently, the hypothetical equilibrium is Pareto optimal in both the traditional and modified senses.

5. <u>Proofs</u>: In proving Theorem 4.1, we require the following result.

Lemma 5.1: Suppose that $\langle C_t^* \rangle_0^{\infty}$ is an equilibrium, and let y_0, y_0' be two initial output levels. Let $\langle k_t \rangle_0^{\infty}, \langle k_t' \rangle_0^{\infty}$ be the corresponding sequence of capital stocks. Then, if

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 $y_0 \leq y_0'$ (or $k_0 \leq k_0'$), $k_t \leq k_t'$ for all t > 0.

<u>Proof:</u> Since $y_0 \leq y'_0$, $k_0 = y_0 - C''_0(y_0) \leq y'_0 - C''_0(y'_0) = k'_0$, by Theorem 2.1. Now proceed by induction. Let $k_T \leq k'_T$ for some $T \geq 0$. Then, since f_T is increasing, $y_{T+1} = f_T(k_T) \leq f_T(k'_T) = y'_{T+1}$. Using Theorem 2.1 again, $k_{T+1} = y_{T+1} - C''_{T+1}(y_{T+1}) \leq y'_{T+1} - C''_{T+1}(y'_{T+1}) = k'_{T+1}$. This establishes the lemma. Q.E.D.

<u>Remark</u>: Lemma 5.15 establishes an analogue of the Brock 'monotonicity' result (Brock, [1971]) when initial stocks are changed. Note, moreover, that it holds for a nonstationary model.

<u>Remark</u>: We have established a stronger result: that the T subsequences $\langle k_{t+nT}^{*} \rangle_{n=0}^{\infty}$, t = 0,..., T - 1, are either <u>all</u> monotone nonincreasing, or <u>all</u> monotone nondecreasing.

<u>Proof of Corollary 3.1</u>: Specialize to T = 1 in Theorem 4.1. Q.E.D.

In proving Theorem 3.3, we consider two cases. In the first case, $\lim_{t} \sup k_{t}^{*} \equiv k^{*} < \infty.$ We will take the pure accumulation program $\int_{t}^{t} \int_{0}^{\infty} in \text{ this case to be unbounded (the analysis of } \lim_{t} \overline{y}_{t} < \infty \text{ is}$ similar and easier to handle, and is omitted). In case 1, $\langle k_t^* \rangle_0^\infty$ is a bounded sequence. Choose $\varepsilon > 0$, and define $\hat{y} \equiv f(\limsup y_t^*) + \varepsilon$.

At this point, we require some terminology and notation to describe correspondences and their properties. Let $h:[0, \hat{y}] + [0, \hat{y}]$ be some correspondence. G(h) denotes the graph of h. We say that h satisfies the <u>Keynesian property</u> if, for all (y,c), $(y'c') \in G(h)$, with y' > y, we have $c' - c \leq y' - y$. We say that h is <u>filled</u> if it is convex valued, and if $0 \in h(\hat{y})$. Define H as the set of all upper hemicontinuous, filled correspondences $h:[0,\hat{y}] + [0,\hat{y}]$ with $0 \leq c \leq y$ for all $c \in h(y)$, where h satisfies the Keynesian property.

We recall two results from Bernheim and Ray [1983]. First, Hendowed with the Hausdorff topology is compact. (We induce this topology on correspondences by placing the Hausdorff topology on their graphs). Second, there is a unique upper semicontinuous selection C satisfying the Keynesian property from any $h \in H$.

For any upper hemi-continuous correspondence h, we define the filled correspondence as follows:

{ $c \in [0,y]$ | there exists c', c" $\in h(y)$

such that $c' \leq c \leq c''$ for y = [0,y]Fil(h)(y) =

> {c \in [0,y] | there exists c" \in h(y) such that 0 \leq c \leq c"} for y = y

For any consumption function $C:[0,y] \rightarrow [0,y]$ satisfying the Keynesian property, let h(C) be the unc correspondence whose graph is

the filled closure of that of C. Then $h \in H$.

Let T be an integer such that $\overline{y}_t \ge \hat{y}$ for all $t \ge T$. For $t \ge T$, define $\hat{C}_t: [0, \hat{y}] \Rightarrow [0, \hat{y}]$ by $\hat{C}_t(y) = C_t^*(y), y \in [0, \hat{y}]$.

We now establish

(iv)
$$h(\hat{C}_{t_{a}+1}) \rightarrow h^{*} \underline{as} q \rightarrow \infty$$
.

<u>Proof</u>: It is easy to obtain a subsequence $\langle t_{m} \rangle_{m=0}^{\infty}$ such that $k_{t_{m}}^{*} + k^{*}$ as $m + \infty$. Since we are in Case 1, $k_{t_{m}-1}^{*}$ and $k_{t_{m}+1}^{*}$ are bounded sequences. Hence there is a subsequence $\langle t_{n} \rangle_{n=0}^{\infty}$ of $\langle t_{m} \rangle_{m=0}^{\infty}$ such that $k_{t_{n}}^{*} + k^{*}$, $k_{t_{n}-1}^{*} + k$, $k_{t_{n}+1}^{*} + k$ as $n + \infty$, with $k \leq k^{*} \geq k$. The sequence $\{h(\hat{C}_{t_{n}+1})\}_{n=0}^{\infty}$ is in H for all $t_{n} \geq T$. By the compactness of H, there is a subsequence $\langle t_{q} \rangle_{q=0}^{\infty}$ with $t_{o} \geq T$, and $h(\hat{C}_{t_{q}+1}) + h^{*} \in H$. The subsequence t_{q} clearly has all the required properties of the lemma. Q.E.D.

Define $\overline{y} \equiv f(\overline{k})$ and let C^* be the unique usc selection from $h^* \in H$ with the Keynesian property.

Lemma 5.3: k^* maximizes $V(\bar{y},k) \equiv v(\bar{y}-k) + \delta v(C^*(f(k))).$

<u>Proof</u>: Note that there exist $T' \ge T$ such that for all $t_q \ge T', y_{t_q} \le f^{-1}(\hat{y})$. Therefore it is meaningful to write that for all $t_q \ge T, k_{t_q}$ maximizes

$$\mathbf{v}(\mathbf{y}_{q} - \mathbf{k}) + \delta \mathbf{v} (C_{q+1}(\mathbf{f}(\mathbf{k})))$$

over k. As $q \neq \infty$, $h(\hat{C}_{t_q}) \neq h^* \in H$, by construction, and $(y_t, k_t) \neq (\bar{y}, k^*)$. Using continuity arguments along similar lines as those in the proof of Lemma 5.9 of Bernheim and Ray [1983], it is easy to see that the statement of the present lemma must be true. Q.E.D.

Using lemmas 5.2 and 5.3, we can now prove

Lemma 5.4: If
$$\limsup k_t^* \equiv k^* < \infty$$
, then $k^* \leq \hat{k}$.

<u>Proof</u>: Suppose, on the contrary, that $k^* > \hat{k}$. Then we claim that

(5.1)
$$\lim_{k \neq k^*} \frac{\delta [C^*(f(k^*)) - C^*(f(k))]}{k^* - k}$$

exists, and equals unity.

Assume this is <u>not</u> true; then there exists $k^n + k^*$ with

(5.2)
$$\lim_{n \to \infty} \frac{\delta[C^*(f(k^*)) - C^*(f(k^n))]}{k^* - k^n} \equiv \lambda \neq 1$$

where λ is defined in the extended reals.

Case 1: $\lambda > 1$.

In this case, there is $\gamma > 1$ and integer N such that for all

n > N,

(5.3)
$$\delta[C^{*}(f(k^{*})) - C^{*}(f(k^{n}))] \geq \gamma(k^{*}-k^{n})$$

Since $k^* > \hat{k}$, and f is continuously differentiable and concave, we have, by Theorem 3.2, $\varepsilon > 0$ and M > N such that for all n > M,

(5.4)
$$f(k^*) - f(k^n) \le f'(k^n)(k^* - k^n) \le \frac{\gamma - \varepsilon}{\delta} (k^* - k^n)$$
.

Combining (5.3) and (5.4), we have, for n > M,

$$C^{*}(f(k^{*})) - C^{*}(f(k^{n})) \geq \frac{\gamma}{\delta} (k^{*}-k^{n}) \geq \frac{\gamma}{\gamma-\varepsilon} [f(k^{*}) - f(k^{n})] ,$$

which contradicts the Keynesian property of C^{*} .

Case 2: $\lambda < 1$.

In this case, there exists $\mu < 1$ and integer N such that for all $n \ge N$,

(5.5)
$$\delta[C^*(f(k^*)) - C^*(f(k^n))] \leq \mu(k^* - k^n)$$

Now, for all n,

$$(5.6) \ V(\bar{y},k^{n}) - V(\bar{y},k^{*}) = [v(\bar{y} - k^{n}) - v(\bar{y} - k^{*})] + \delta[v(C^{*}(f(k^{n})) - v(C^{*}(f(k^{*})))]$$

Using the mean value theorem, there exists $\alpha^n \in [(\bar{y} - k^*), (\bar{y} - k^n)]$ and $\beta^n \in [\min\{C^*(f(k^n)), C^*(f(k^*))\}]$, max{ $C^*(f(k^n)), C^*(f(k^*))\}$] such that

(5.7)
$$V(\bar{y},k^n) - V(\bar{y},k^*) = v'(\alpha^n)(k^* - k^n) + v'(\beta^n)[\delta\{C^*(f(k^n)) - C^*(f(k^*))\}]$$

Using (5.5), (5.7) yields, for $n > N$,

(5.8)
$$V(\bar{y},k^n) - V(\bar{y},k^*) > [v'(\alpha^n) - \mu v'(\beta^n)](k^* - k^n)$$

As $n \neq \infty$, $k^n \neq k^*$, and so $\alpha^n \neq \overline{y} = k^*$, by continuous differentiability of v. Also, since C^* is continuous from the left (this follows from our usc selection of C^* from h^* , and the Keynesian property), and $f(k^n) < f(k^*)$ for all n, $C^*(f(k^n)) \neq C^*(f(k^*))$, so that $\beta^n \neq C^*(f(k^*))$.

<u>Actual</u> consumption along the sequence $t_q + 1$, $c_{t_q+1}^* \neq f(k^*) - k$ as $q \neq \infty$ (see Lemma 5.2). By the use of C^* , and the fact that $h(\hat{C}_{t_q+1}) \neq h(C^*)$,

(5.9)
$$C^{*}(f(k^{*})) \ge f(k^{*}) - k \ge f(\bar{k}) - k^{*} = \bar{y} - k^{*}$$

Hence,

(5.10)
$$\lim_{n \to \infty} v'(\alpha^n) \ge \lim_{n \to \infty} v'(\beta^n)$$

But using (5.8), (5.10), and $\mu < 1$, it follows that $V(\bar{y}, k^n) - V(\bar{y}, k^*) > 0$ for large n, which contradicts Lemma 5.3.

Therefore our claim, given by (5.1), is indeed true. Denoting by $C^{*-}(f(k^{*}))$ the left-hand derivative of $C^{*}(\cdot)$ at $f(k^{*})$, one has

(5.11)
$$\delta C^{*-}(f(k^{*}))f'(k^{*}) = 1$$

By the Keynesian property of C^* , $C^{*-}(f(k^*)) \leq 1$. Hence

(5.12)
$$\delta f'(k^*) \ge 1.$$

But this, along with our assumption that $k^* > \hat{k}$, contradicts Theorem 3.2. Q.E.D.

In the second case, we have the possibility that $\lim_{t} \sup k_{t}^{*} \equiv k^{*} = \infty$. This is ruled out in t

Lemma 5.5: It is impossible for
$$\lim_{t \to \infty} \sup_{t} k_{t}^{*} \equiv k^{*} \underbrace{to equal}_{t} \infty$$
.

<u>Proof</u>: Suppose, on the contrary, that $k^* = \infty$. Then we claim that there exists T such that $k_T^* > \hat{k}$ and $c_{T+1}^* > c_T^*$. Suppose not. Then for all T with $k_T^* > \hat{k}$ (such T exist since $\hat{k} = 0$ or $\delta f'(\hat{k}) = 1$, and (A.6) holds), $c_{T+1}^* \le c_T^*$. For all t with $k_t^* \le \hat{k}, c_{t+1}^* \le f(\hat{k})$. It follows, therefore, that for all $t \ge 0, c_t^* \le B < \infty$. Consider a sequence $<T_n >$ with $k_T^* \to \infty$. Observe that for each n, $c_{T_r+1}^*$ maximizes

(5.13)
$$W(f(k_{T_n}^*), c) \equiv v(c) + \delta v [C_{T_n+2}^*(f(f(k_{T_n}^*) - c))]$$

But W (f(
$$k_{T_n}^*$$
), $c_{T_n+1}^*$) < v(B)(1 + δ), since $c_t^* \leq B$ for all t.
Since $k_{T_n}^* \neq \infty$, so does f($k_{T_n}^*$). By (A.5), there exists n such

that

(5.14)
$$v(f(k_{T_n}^*)) + \delta v(o) > v(B)(1 + \delta)$$
.

For such n, using (5.13) and (5.14),

$$W(f(k_{T_n}^*), f(k_{T_n}^*)) > W(f(k_{T_n}^*), c_{T_n+1}^*)$$
,

a contradiction. So our claim is true, and there exists T with

(5.15)
$$k_{T}^{*} > \hat{k}$$
, and $c_{T+1}^{*} > c_{T}^{*}$

Further, k_{T}^{*} maximizes

$$V(y_{T}^{*}, k) \equiv v(y_{T}^{*} - k) + \delta v(C_{T+1}^{*}(f(k)))$$

and clearly, $C_{T+1}^{*}(f(k_{T}^{*})) = c_{T+1}^{*}$.

Now we simply retrace the steps in the proof of Lemma 5.4, substituting C_{T+1}^{*} for C^{*} , y_{T}^{*} for \overline{y} , and k^{*} for k_{T}^{*} . We obtain a contradiction by demonstrating that for <u>no</u> sequence $k^{n} + k_{T}^{*}$ does

$$\lim_{n \to \infty} \frac{\delta[C_{T+1}^*(f(k_T^*)) - C_{T+1}^*(f(k^n))]}{k_T^* - k^n} \equiv \lambda$$

exist in the extended reals. The cases $\lambda > 1$ and $\lambda < 1$ are ruled out in exactly the same way. To eliminate $\lambda = 1$, assume, on the contrary that $\lambda = 1$. We first observe that (5.10) holds with strict inequality, since

$$C_{T+1}^{*}(f(k_{T}^{*})) = c_{T+1}^{*} > c_{T}^{*} = y_{T}^{*} - k_{T}^{*}$$
,

and $\alpha^n \rightarrow y_T^* - k_T^*$, $\beta^n \rightarrow C_{T+1}^*(f(k_T^*))$.

Now pick $\mu > 1$ such that $v'(\alpha^n) - \mu v'(\beta^n) > 0$ for sufficiently large n. For this μ , there exists M such that for all $n \ge M$,

$$\delta[C_{T+1}^{*}(f(k_{T}^{*})) - C_{T+1}^{*}(f(k^{n}))] \leq \mu(k_{T}^{*} - k^{n})$$

Following the steps leading up to (5.8), we obtain

$$V(y_{T}^{*}, k^{n}) - V(y_{T}^{*}, k_{T}^{*}) \geq [v'(\alpha^{n}) - \mu v'(\beta^{n})](k_{T}^{*} - k^{n}),$$

and by our choice of μ , this contradicts, for large n, the fact that k_{η}^{*} is a maximizer of $V(y_{\eta}^{*}, k)$. Q.E.D.

Proof of Theorem 3.3: Combine Lemmas 5.4 and 5.5. Q.E.D.

<u>Proof of Theorem 4.1</u>: By Theorem 3.3, $\limsup_{t \to \infty} k_t \leq \hat{k}$, where \hat{k} is the planning-turnpike. If $\hat{k} > 0$, it solves δ f'(\hat{k}) = 1, by Theorem 3.2. In this case, $\liminf_{t \to \infty} f'(\hat{k}_t) \geq f'(\hat{k}) = 1/\delta > 1$, by (A.5.2). If $\hat{k} = 0$, $\lim_{t \to \infty} k_t = 0$, and so again, $\lim_{t \to \infty} \inf_{t \to \infty} f'(\hat{k}_t) = f'(0) > 1$, by (A.3). Define a sequence $\langle p_t \rangle$ by $p_0 = 1/f'(f^{-1}(y))$, and $p_{t+1} = p_t/f'(k_t)$, $t \geq 0$. Then it is easily verified that $\lim_{t \to \infty} p_t k_t = 0$, so by a well-known criterion for efficiency (see, for the term of term of the term of term of the term of term of term of the term of ter

A feasible program is <u>interior</u> if $k_t > 0$ for all $t \ge 0$. One can show that if an equilibrium program is strictly Keynesian, then it is interior.

<u>Lemma 5.6</u>: <u>Under</u> (A.1), (A.2), <u>and</u> (A.5) <u>suppose that</u> $\langle C_t^* \rangle_0^{\infty}$ <u>is a bequest equilibrium.</u> Let $\langle y_t, c_t, k_t \rangle_0^{\infty}$ <u>be an equilibrium path</u> <u>generated by</u> $\langle C_t^* \rangle_0^{\infty}$ <u>from</u> $y \in (0, Y)$. <u>Then, if</u> $k_t > 0$,

(5.16)
$$v'(c_t) \leq \delta f'(k_t) v'(c_{t+1}),$$

If, in addition, for any t > 1, C_t^* is strictly Keynesian at y_t , then,

for such t, if $k_{t-1} > 0$,

(5.17)
$$v'(c_{t-1}) < \delta f'(k_{t-1}) v'(c_t)$$

<u>Proof</u>: Suppose, on the contrary to (5.16), that there is some $t \ge 0$, with $v'(c_t) > \delta f'(k_t)v'(c_{t+1})$. Then there is n > 0 such that (5.18) $v(c_t+n) - v(c_t) > \delta v(c_{t+1}) - \delta v(c_{t+1} - f(k_t) + f(k_t-n))$.

To see this, note that for any n > 0, $v(c_t+n) - v(c_t) = v'(\xi)n$, for some $\xi \in (c_t, c_t+n)$, and $\delta v(c_{t+1}) - \delta v(c_{t+1} - f(k_t) + f(k_t - n))$ $= \delta v'(\alpha) f'(\beta)n$, where $\alpha \in (c_{t+1} - f(k_t) + f(k_t - n), c_{t+1})$, $\beta \in (k_t - n, k_t)$. As n + 0, $v'(\xi) + v'(\alpha) + v'(c_{t+1})$, $f'(\beta) + f'(k_t)$. So, by our hypothesis, there exists n > 0 such that (5.18) holds.

Now, since by hypothesis, $k_t > 0$, pick $\eta \in (0, k_t)$, so that (5.18) holds. Suppose generation t consumes $c_t + \eta$ instead of c_t . Then

$$u(c_{t}, C_{t+1}^{*}(f(k_{t}))) = v(c_{t}) + \delta v(c_{t+1})$$

$$< v(c_{t} + n) + \delta v(c_{t+1} - f(k_{t}) + f(k_{t} - n))$$

$$\leq v(c_{t} + n) + \delta v(C_{t+1}^{*}(f(k_{t} - n)))$$

$$= u(c_{t} + n, C_{t+1}^{*}(f(k_{t} - n)))$$

which contradicts the construction of C_t^* (the weak inequality above follows from the fact that $C_{t+1}^*(\cdot)$ satisfies the Keynesian property).

This establishes (5.16).

To establish (5.17), proceed as above. Suppose, on the contrary, that the equilibrium program is strictly Keynesian, and $v'(c_{t-1}) \ge \delta f'(k_{t-1})v'(c_t)$. Then, defining $\lambda_t \equiv \lambda(C_t^*, y_t, \varepsilon_t)$ (with ε_t given by an interval in which C_t^* is strictly Keynesian), we claim that there is $\eta \in (0, \min(k_{t-1}, \varepsilon_t))$ such that

(5.19)
$$v(c_{t-1} + n) - v(c_{t-1}) > \delta v(c_t) - \delta v(c_t - \lambda_t [f(k_{t-1}) - f(k_{t-1} - n)])$$

By an argument exactly analogous to that following (iv), and an analogous choice of α , β and ξ , $v(c_{t-1} + \eta) - v(c_{t-1}) = v'(\xi)\eta$, and $\delta v(c_t) - \delta v(c_t - f(k_{t-1}) + f(k_{t-1} + \eta)) = \delta v'(\alpha)f'(\beta)\eta$. Given that $v'(\xi) + v'(c_{t-1})$ as $\eta \neq 0$, and $(v'(\alpha), f'(\beta)) \neq (v'(c_t), f'(k_{t-1}))$ as $\eta \neq 0$, it follows, using $\lambda_t < 1$ and $v(\cdot)$ increasing, that (5.19) must hold for η small enough; in particular, for some $\eta \in (0, \min(k_{t-1}, \varepsilon_t))$.

Now suppose that generation t-l consumes $c_t + \eta$ instead of c_t . Then, using the strict Keynesian property,

$$u(c_{t-1}, C_{t}^{*}(f(k_{t-1}))) = v(c_{t-1}) + \delta v(c_{t})$$

$$< v(c_{t-1} + n) + \delta v(c_{t} - \lambda_{t}[f(k_{t-1}) - f(k_{t-1} - n)])$$

$$\leq v(c_{t-1} + n) + \delta v(C_{t}^{*}(f(k_{t-1} - n)))$$

$$= u(c_{t-1} + n, C_{t}^{*}(f(k_{t-1} - n)))$$

which contradicts the construction of C_{t-1}^* . This establishes (5.17). Q. E. D. Lemma 5.7. Under (A.1), (A.2), and (A.5) let $\langle y_t, k_t, c_t \rangle_0^{\infty}$ be the optimal planning program from $y \in (0, Y)$. Then if $c_t > 0$ for any t > 0,

(5.20)
$$v'(\hat{c}_{t}) \ge \delta f'(\hat{k}_{t}) v'(\hat{c}_{t+1})$$

<u>Proof</u>. Suppose, on the contrary, that $v'(\hat{c}_s) < \delta f'(\hat{k}_s)v'(\hat{c}_{s+1})$, and $\hat{c}_s > 0$, for some $s \ge 0$. Then, by an argument similar to that in Lemma 5.6, there is $\eta \in (0, \hat{c}_s)$, such that

(5.21)
$$v(\hat{c}_{s}) - v(\hat{c}_{s} - \eta) < \delta v(\hat{c}_{s+1} + f(\hat{k}_{s} + \eta) - f(\hat{k}_{s})) - \delta v(\hat{c}_{s+1})$$

Now define $\langle y'_{t}, c'_{t}, k'_{t} \rangle_{0}^{\infty}$ from $y \in (0, Y)$ by $y' = \hat{y}_{t}, t \neq s + 1, y'_{s+1} = \hat{f}(k_{s} + n), k'_{t} = \hat{k}_{t}, t \neq s, k'_{s} = \hat{k}_{s} + n$, and $c'_{t} = \hat{c}_{t}, t \neq s, s + 1, c'_{s} = \hat{c}_{s} - n \cdot c'_{s+1} = \hat{c}_{s} + f(\hat{k}_{s} + n) - f(\hat{k}_{s}).$ Clearly this is feasible. Moreover $v(c'_{t}) = v(\hat{c}_{t}), t \neq s, s + 1$, and $v(c'_{t}) + \delta v(c'_{t+1}) = v(\hat{c}_{s} - n) + \delta v(\hat{c}_{s+1} + f(\hat{k}_{s} + n) - f(\hat{k}_{s})) > v(\hat{c}_{s}) + \delta v(\hat{c}_{s+1}).$ So $\langle \hat{y}_{t}, \hat{k}_{t}, \hat{c}_{t} \rangle_{0}^{\infty}$ is not optimal. a contradiction.

<u>Proof of Theorem 3.4</u> Suppose. on the contrary, that $k_s^* > \hat{k}_s$ for some <u>first</u> time period $s \ge 0$. Then $y_s^* \le \hat{y}_s$, where $\langle \hat{y}_t \rangle_0^{\infty}$ represents optimal output levels under planning. Clearly, $\langle C_t^* \rangle_s^{\infty}$ is a bequest equilibrium, and $\langle y_t^*, k_t^*, c_t^* \rangle_s^{\infty}$ is a program generated by this equilibrium, from y_s^* . Let $\langle y_t', k_t', c_t' \rangle_s^{\infty}$ be the program generated by $\langle C_t^* \rangle_s^{\infty}$, from \hat{y}_s . Then, by Lemma 5.1, $k_t' > k_t^*$, $t \ge s$. So $k_s' > \hat{k}_s$. Hence $c_s' < \hat{c}_s$.

Now proceed by induction. Suppose that for some $t \ge s$, $c'_t < c'_t$, and $\hat{k}_t < k'_t$. Then, using the strict concavity of v (and/or f), Lemma 5.6 (noting that $k_t^i > 0$), and Lemma 5.7 (noting that $\hat{c}_t > 0$),

(5.22)
$$\delta f'(k_t) v'(c_{t+1}) \ge v'(c_t) > v'(c_t)$$

$$\geq \delta f'(\hat{k}_t) v'(\hat{c}_{t+1}) > \delta f'(k'_t) v'(\hat{c}_{t+1}) .$$

Using (5.22), it follows that $v'(c_{t+1}') > v'(\hat{c}_{t+1})$, so $c_{t+1}' < \hat{c}_{t+1}$. Since $k_t' > \hat{k}_t$ (by hypothesis) and f is increasing, $k_{t+1}' = f(k_t') - c_{t+1}' > f(\hat{k}_t) - \hat{c}_{t+1} = \hat{k}_{t+1}$. Hence, $c_t' < \hat{c}_t$, for all $t \ge s$. This establishes the inefficiency of $\langle y_t', k_t', c_t' \rangle_s^{\infty}$ from y_s , which contradicts Theorem 4.1.

Hence
$$k_t \leq k_t$$
 for all $t \geq 0$. Q. E. D.

<u>Proof of Theorem 4.2</u>: Since C_1^* is strictly Keynesian at y_1 , and $k_0 > 0$, we have, by Lemma 5.6

(5.23)
$$v'(c_0) < \delta f'(k_0) v'(c_1)$$
.

So, by a standard argument, (see, e.g., proof of Lemma 5.6), there is η (0, c₀) such that

(5.24)
$$v(c_0) - v(c_0 - n) < \delta v(c_1 + f(k_0 + n) - f(k_0)) - \delta v(c_1)$$

Now define $\langle y'_{t}, k'_{t}, c'_{t} \rangle_{0}^{\infty}$ by $y'_{t} = y_{t}, t \neq 1, k'_{t} = k_{t}, t \neq 0, c'_{t} = c_{t}, t \neq 0, 1, and <math>y'_{1} = f(k_{0} + n), k'_{0} = k_{0} + n, c'_{0} = c_{0} - n, c'_{1} = c_{1} + f(k_{0} + n) - f(k_{0}).$ Clearly, $\langle y_{t}, k_{t}, c_{t} \rangle_{0}^{\infty}$ is feasible. For $t \geq 2, u_{t}(c'_{t}, c'_{t+1}) = u_{t}(c_{t}, c_{t+1}).$ For $t = 1, u_{t}(c'_{t}, c'_{t+1}) = u_{t}(c_{t}, c_{t+1}).$ For $t = 0, using (5.24), u_{t}(c'_{t}, c'_{t+1}) = u_{t}(c_{t}, c_{t+1}).$ For $t = 0, using (5.24), u_{t}(c'_{t}, c'_{t+1}) = u_{t}(c'_{t}, c'_{t+1}) = u_{t}(c'_{t}, c'_{t+1}).$

$$u_0(c'_0, c'_1) = v(c_0 - n) + \delta v(c_1 + f(k_0 + n) - f(k_0))$$

> $v(c_0) + \delta v(c_1) = u_0(c_0, c_1)$.

This proves the theorem.

<u>Proof of Theorem 4.3</u>: It is completely straightforward to show that efficiency is a necessary condition for Pareto optimality. For sufficiency, it suffices to exhibit a strictly positive sequence of numbers $\langle \alpha_t \rangle_0^{\infty}$ such that for any feasible program $\langle y'_t, k'_t, c'_t \rangle_0^{\infty}$ with $c'_0 = c_0$,

$$(5.25) \quad \lim_{T \to \infty} \inf \{\alpha_0 \delta[v(c_1') - v(c_1)] + \sum_{t=1}^{T} \alpha_t [u(c_t', c_{t+1}') - u(c_t, c_{t+1}))] \leq 0$$

This clearly would establish the modified Pareto-optimality of $\langle y_t, k_t, c_t \rangle_0^{\infty}$

To this end, we first establish the existence of a strictly positive sequence $\langle \beta_t \rangle_0^{\infty}$, with $\beta_0 > \beta_1 > \beta_2 > \dots$, such that for all feasible programs $\langle y_t^{\prime}, k_t^{\prime}, c_t^{\prime} \rangle_0^{\infty}$,

(5.26)
$$\liminf_{T \to \infty} \sum_{t=0}^{T} \beta_t \delta^t [v(c_t') - v(c_t)] \leq 0$$

Since C_t^* is strictly Keynesian at y_t for all $t \ge 1$, and so $\langle y_t, k_t, c_t \rangle_0^{\infty}$ is interior, we have, by (5.17)

(5.27)
$$0 < \frac{v'(c_t)}{\delta f'(k_t)v'(c_{t+1})} < 1, t \ge 0.$$

Q. E. D.

Now define $\beta_0 = 1$, and, recursively,

(5.28)
$$\beta_{t+1} = \frac{v'(c_t)\beta_t}{\delta f'(k_t)v'(c_{t+1})}$$

Then, by virtue of (5.27), $\beta_0 > \beta_1 > \beta_2 > \dots > 0$. Now define

(5.29)
$$p_t = \delta^t \beta_t v'(c_t) , t \ge 0.$$

Then, using (5.28),

(5.30)
$$p_{t+1} = \frac{p_t}{f'(k_t)}$$
, $t \ge 0$, with $p_0 = v'(c_0)$

Now, by Theorem 4.1, $\langle y_t, k_t, c_t \rangle_0^{\infty}$ is efficient. Hence, by a result of Cass and Yaari [1971],

(5.31)
$$\liminf_{T \to \infty} \sum_{t=0}^{T} p_t (c_t' - c_t) \leq 0$$

for all feasible programs $\langle y'_t, k'_t, c'_t \rangle_0^{\infty}$.

But for $t \ge 0$, $v(c_t') - v(c_t) \le v'(c_t)(c_t' - c_t)$, by concavity of $v(\cdot)$. Multiplying both sides by $\beta_t \delta^t$, summing over 0 to T, and using (5.29), one has (5.26).

Note that by (5.26), we also have that for all feasible programs $\langle y'_t, k'_t, c'_t \rangle_0^{\infty}$ with $c'_0 = c_0$,

(5.32)
$$\lim_{T \to \infty} \inf \sum_{t=1}^{T} \beta_t \delta^t [v(c_t') - v(c_t)] \leq 0$$

Now we show that it is possible to select a strictly positive

sequence $\langle \alpha_t \rangle_0^{\infty}$ such that (5.25) and (5.32) are identical. Since the equilibrium program satisfies (5.32) it will therefore also satisfy (5.25) (for these particular utility weights). This establishes that the program is modified Pareto optimal.

Observe that (5.25) and (5.32) are identical if, for all $t \ge 0$

$$\alpha_{t+1} + \delta \alpha_t = \delta^t \beta_{t+1}$$

(This can be verified by rearranging terms). Consequently, if we choose α_0 arbitrarily and let α_+ be defined by

(5.32)
$$\alpha_t = \delta^t \beta_t - \delta \alpha_{t-1} ,$$

we will have constructed a sequence $\langle \alpha_t \rangle_0^{\infty}$ for which (5.25) and (5.26) are identical. Unfortunately, the sequence may not be strictly positive. Our task is to select α_0 appropriately.

Let $\psi_t \equiv (\beta_{2t+1} - \beta_{2t+2})$. Note that, by (5.27) and (5.28), $\psi_t > 0$. Define

$$\alpha_0 = \sum_{t=0}^{\infty} \psi_t$$

We know that $\alpha_0 > 0$. Since $\langle \beta_t \rangle_0^{\infty}$ is a strictly positive, strictly decreasing sequence, the series defining α_0 must converge, and $\alpha_0 < \beta_1$.

Now we solve the difference equation (5.32) . For t odd, we get

$$\alpha_{t} = \delta^{t} [(\beta_{1} - \beta_{2}) + (\beta_{3} - \beta_{4}) + \dots + (\beta_{t-2} - \beta_{t-1}) + \beta_{t} - \alpha_{0}]$$

= $\delta^{t} [\beta_{t} - \sum_{\tau=(t-1)/2}^{\infty} \psi_{\tau}]$

Since $\langle \beta_t \rangle_0^{\infty}$ is strictly positive and strictly decreasing,

$$\sum_{\tau=(t-1)/2}^{\infty} \psi_{\tau} < \beta_{t}, \text{ so } \alpha_{t} > 0.$$

For t even, we get

$$\alpha_{t} = \delta^{t} [\alpha_{0} - (\beta_{1} - \beta_{2}) - (\beta_{3} - \beta_{4}) - \dots - (\beta_{t-1} - \beta_{t})]$$

= $\delta^{t} [\sum_{\tau=t/2}^{\infty} \psi_{\tau}] > 0$

since $\psi_t > 0$ for all t.

Footnotes

- 1/ This model is adapted from Kohlberg [1976], and can be traced to Dasgupta [1974a,b].
- $\frac{2}{c_t} \quad \text{The validity of our results when there is substitutability between } c_t \quad \text{and } c_{t+1} \quad \text{remains an open question.}$
- 3/ The borderline case $\lim f'(k) = \delta^{-1}$ presents more subtle technical problems which we have not explored.
- 4/ The definition originally appears in Arrow [1973].
- 5/ One can, of course, use a similar argument when feasible utility streams diverge.
- 6/ The proof is omitted. For a more general version of this result in an aggregative context, see Mitra and Ray [1983].
- Since this result implies Theorem 3.3, and is obtained under only slightly more restrictive conditions, Theorem 3.3 may appear redundant. However, Theorem 3.3 is used in the proof of Theorem 4.1. Consequently, it is necessary to state these results separately.

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