ALTRUISTIC GROWTH ECONOMIES

I. Existence of Bequest Equilibria
II. Properties of Bequest Equilibria

by

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ALTRUISTIC GROWTH ECONOMIES*
PART I. EXISTENCE OF BEQUEST EQUILIBRIA

by

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I. Introduction

In this paper, we study an aggregative growth model with intergenerational altruism. Each generation is active for a single period. At the beginning of this period it receives an endowment of a single homogeneous good which is the output from a 'bequest investment' made by the previous generation. It divides the endowment between consumption and investment. The return from this investment constitutes the endowment of the next generation.1/ Each generation derives utility from its own consumption and that of its immediate successor. However, since altruism is limited, in the sense that no generation cares about later successors, the interests of distinct agents come into conflict.

Models of this type have been used to analyze a number of issues concerning intergenerational resource allocation. One line of research, pursued by Arrow [1973] and Dasgupta [1974a], elucidates the implications of Rawls' principle of just savings. These authors were primarily concerned with the characterizing optimal growth under a particular welfare criterion.

Others have addressed the question of how an 'altruistic growth economy' might actually evolve over time. This literature, initiated by Phelps and Pollak [1968],2/ makes extensive use of the Nash equilibrium concept. Several interesting issues emerge.

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First, is the resulting intertemporal allocation Pareto-efficient? Phelps and Pollak [1968] and Dasgupta [1974a,b] reply in the negative. However, Lane and Mitra [1981] argue that the traditional definition of Pareto-efficiency is inappropriate in this context. They demonstrate that when the set of feasible programs is properly restricted, Nash Equilibrium programs within a certain class are indeed Pareto-efficient (in a model involving special functional forms).

Second, how do capital stock and level of consumption behave over time? In particular, how does the equilibrium program compare to that which would be selected by an omniscient planner? Although this question has received some attention from Phelps and Pollak [1968] and Kohlberg [1976], it remains largely unanswered.

There are important practical issues to be stressed in this context. Barro [1974] has argued that under certain special conditions, intergenerational altruism neutralizes the real effects of Social Security and deficit financing. If these conditions are not met, such government policies could be employed as strategic instruments in instances of intergenerational conflict.

The framework of intergenerational altruism is also useful for analyzing how bequests effect the distribution of wealth in an intertemporal context. For a discussion of such issues, see Loury [1981].

Many of the theoretical issues which arise in this framework are closely related to the literature on 'consistent plans', pioneered by Strotz [1956] and Pollak [1968]. The postulate of a sequence of 'planners' with conflicting goals bears strong formal resemblance to that of a single planner with changing tastes. Consequently, some of the general results obtained in this literature may be applicable to altruistic growth models.

The purpose of this paper and its sequel is to address three important theoretical issues which remain open: the existence of equilibrium, the normative properties of equilibrium programs, and the asymptotic behavior of capital stocks in an altruistic growth economy.

It is perhaps surprising that no satisfactory existence theorem has been exhibited for this important and useful class of models.[3/] Authors studying altruistic growth equilibria (see, for example, Phelps and Pollak [1968], Dasgupta [1974a,b], Kohlberg [1976], Lane and Mitra [1981]), while aware of the existence problem, have typically concerned themselves with the properties of equilibria.

The lack of an existence proof is particularly troubling in the light of a counterexample due to Kohlberg [1976] for a particularly simple model, [4/] which demonstrates that Nash equilibria with certain reasonable properties (stationarity and continuous differentiability of the equilibrium strategies) may not, in general, exist. Lane and Mitra [1981] suggest that a proof of existence (for non-stationary equilibria) appears in the literature on consistent plans (Peleg and Yaari [1973]). However, the notion of equilibrium adopted there restricts all agents to select linear consumption functions. This is clearly unsatisfactory. In particular, when an agent contemplates deviations from his equilibrium strategy, he envisions later generations selecting actions which do not in general, maximize their utility. Thus the equilibrium is not perfect, in the sense of Selten [1965]. Furthermore, Peleg and Yaari do not address the question of whether stationary equilibria exist. Goldman [1980] supplies an existence proof for perfect equilibria, but this is applicable only to models with finite time horizons. Moreover, such truncation prevents him from considering the existence of stationary equilibria.

In this paper, we provide two results on the existence of equilibria in altruistic growth models. First, under very general conditions, perfect Nash equilibria always exist in reasonably well-behaved strategies (equilibrium consumption functions are upper semicontinuous, continuous from the left, with limits on the right). Furthermore, if the model is itself stationary, then a stationary equilibrium will exist as well. Consequently, Kohlberg's counterexample results from the restriction that consumption functions must be continuously differentiable.

In the sequel to this paper, we investigate the positive and normative aspects of equilibrium programs for altruistic growth economies. One question addressed there concerns the asymptotic ior of capital stocks. In particular, will the long run capital stock which arises from intergenerational conflict be higher or lower than the 'turnpike' associated with the solution to the optimal planning problem? On *a priori* grounds, the answer is not clear. Agents who take only a limited interest in the future will tend to bequeath less than those who are far-sighted. However, since each generation views its children's bequest as pure waste, it must bequeath a larger sum to obtain the same consumption value.

In the sequel, we obtain steady-state results for equilibrium capital stocks completely analogous to the well-known optimal planning results. By comparing 'steady-states', we show that no limit point of equilibrium capital stocks can exceed the planning turnpike. Consequently, limited intergenerational altruism may provide the basis for a theory of chronic capital shortages.

A second set of questions addressed in the sequel concern normative issues. In particular, are equilibrium programs efficient? If so, are they Pareto optimal in the traditional sense, or modified Pareto optimal in the sense of Lane and Mitra [1981]? Although previous authors have addressed these questions, their analyses have been confined either to particular parametric specifications of the model, or to the class of Nash Equilibria characterized by linear consumption functions (Dasgupta [1974a,b], Lane and Mitra [1981]). We have already mentioned the shortcomings of adopting the second approach. In the sequel, we extend existing results to the class of perfect equilibria.

The current paper is organized as follows. Section 2 displays the model, basic assumptions, and definitions of equilibria. In Section 3, we show that, regardless of the strategies adopted by future generations, the optimal consumption function for the current generation displays a 'marginal propensity to consume' out of endowment which does not exceed unity. That is, each generation's bequest is a normal good. Besides being of independent interest, this result is used extensively to establish our central theorems. Existence of equilibria is established in Section 4. All proofs are deferred to Section 5. Section 6 discusses additional open questions.

II. The Model

The model is a generalization of Kohlberg's (Kohlberg [1976]). There is one commodity, which may be consumed or invested. The transformation of capital stock into output takes one period, and is represented by a sequence of <u>production functions</u> $\langle f_t \rangle_0^\infty$. We assume, for each $t \geq 0$,

(A.1) $f_t\colon \mathbb{R}_+ \to \mathbb{R}_+$ <u>is continuous and increasing</u>

In each time period, decisions concerning production and consumption are made by a fresh generation. Thus, generation t is endowed with some initial output (y_t), which it divides between consumption (c_t), and investment $(k_t = y_t - c_t)$. Each generation derives utility from its own consumption, and the consumption of the generation immediately succeeding it. Preferences are represented by a sequence of utility functions $\langle u_t \rangle_0^\infty$. We assume, for $t \geq 0$,

(A.2) $u_t\colon \mathbb{R}_+^2 \to \mathbb{R}$ <u>is continuous, increasing and strictly concave.</u> [5/]

(A.3) <u>For all</u> $c_t, c_t', c_{t+1}, c_{t+1}'$ <u>with</u> $c_t \geq c_t' \geq 0$, $c_{t+1} \geq c_{t+1}' \geq 0$,

$$u_t(c_t, c_{t+1}) - u_t(c_t', c_{t+1}) \geq u_t(c_t, c_{t+1}') - u_t(c_t', c_{t+1}') \quad .$$

<u>Remark</u>: (A.3) is simply an assumption of <u>weak complementarity</u>. [6/] For u_t differentiable, it is equivalent to $\partial^2 u_t / \partial c_t \, \partial c_{t+1} \geq 0$. Note that (A.3) subsumes the case analyzed by Kohlberg (1976): $u_t(c_t, c_{t+1}) = v(c_t) + \delta v(c_{t+1})$, where δ is positive and $v(\cdot)$ is continuous, increasing and concave.

Assumptions (A.1)-(A.3) will be maintained throughout the paper. We take the historically given initial output at time zero, y, to lie in some compact interval $[0.Y]$, $Y > 0$. A program $\langle y_t, c_t, k_t \rangle_0^\infty$ is feasible from $y \in [0,Y]$ if

(2.1) $\qquad y_0 = y$

(2.2) $\qquad y_t = c_t + k_t \quad , \quad t \geq 0$

(2.3) $\qquad y_{t+1} = f_t(k_t) \quad , \quad t \geq 0$

(2.4) $\qquad (y_t, c_t, k_t) \geq 0 \quad , \quad t \geq 0$

Denote by $\langle c_t \rangle_0^\infty$ the corresponding feasible consumption program. The pure accumulation program is a sequence $\langle \bar{y}_t, \bar{c}_t, \bar{k}_t \rangle_0^\infty$ with $\bar{c}_t = 0$ for all $t \geq 0$, $\bar{y}_t = \bar{k}_t$ for all $t \geq 0$, $\bar{y}_{t+1} = f_t(\bar{k}_t)$ for all $t \geq 0$, and $\bar{y}_0 = Y$.

Define C_t as the set of functions $C: [0,\bar{y}_t] \to [0,\bar{y}_t]$, with $C(y) \leq y$ for all $y \in [0,\bar{y}_t]$. Define $U_t(c,y, C_{t+1}) = u_t(c, C_{t+1}(f_t(y - c)))$ for all $C_{t+1} \in C_{t+1}$, and $(c,y) \geq 0$ with $c \leq y \leq \bar{y}_t$.

We will impose the behavioral assumption that all generations select perfect Nash stategies (see Selten [1965]). Formally,

Definition: The sequence $\langle C_t^* \rangle$, $C_t^* \in C_t$, $t \geq 0$, is a bequest equilibrium (or simply, equilibrium) if for all $t \geq 0$ and $y \in [0,\bar{y}_t]$,

$$C_t^*(y) \in \arg \max_{0 \leq c \leq y} U_t(c,y, C_{t+1}^*) \quad .$$

Note that we have restricted attention to the class of strategies for which consumption depends only upon initial endowment. In general, it is possible for agents to condition their choices upon the entire history of the game. We will refer to these as 'endowment dependent', and 'history dependent' strategies respectively. Clearly, one cannot rule out the existence of equilibria in history dependent strategies which are not simply endowment dependent (see, for example, Goldman [1980]). However, it is easy to verify in our model that <u>if</u> generation $t + 1$ chooses an endowment dependent strategy, there exists an endowment dependent best response for generation t. It follows that although we have restricted attention to endowment dependent strategies, our bequest equilibria continue to be equilibria when no restrictions on strategic choice are imposed. Furthermore, since C_t^* must maximize the utility of generation t for all initial endowments, any bequest equilibrium must in addition be perfect.

We shall often refer to this model as an <u>altruistic growth economy</u>. An altruistic growth economy is <u>stationary</u> if $u_t = u$ and $f_t = f$ for all $t \geq 0$. Finally, a bequest equilibrium is <u>stationary</u> if the altruistic growth economy is stationary, and the equilibrium consumption functions $\langle C_t^* \rangle$ satisfy $C_t^*(y) = C_{t+1}^*(y)$ for all $y \in [0,\bar{y}_t]$, $t \geq 0$.

III. <u>The Marginal Propensity to Consume</u>

Kohlberg [1976] has shown that any stationary continuously differentiable equilibrium $C(\cdot)$ of a stationary alruistic growth model satisfies $0 < C' \leq 1$. That is, the marginal propensity to consume out of endowment is positive, but does not exceed unity. Equivalently, both consumption and bequests are normal goods. In this section, we establish that regardless of the strategies adopted by succeeding generations, the optimal consumption function for any particular generation exhibits a marginal propensity to consume not greater than unity. Henceforth, we shall (for obvious reasons) refer to this as the 'Keynesian property'. Thus, Theorem 3.1 generalizes half of Kohlberg's result. Although the theorem is interesting in its own right, it is also essential for the analysis which follows.

Theorem 3.1: Suppose that for some consumption function $c_{t+1} \in C_{t+1}$ used by generation $t + 1$, an optimal consumption function for generation t, $c_t \in C_t$, given by

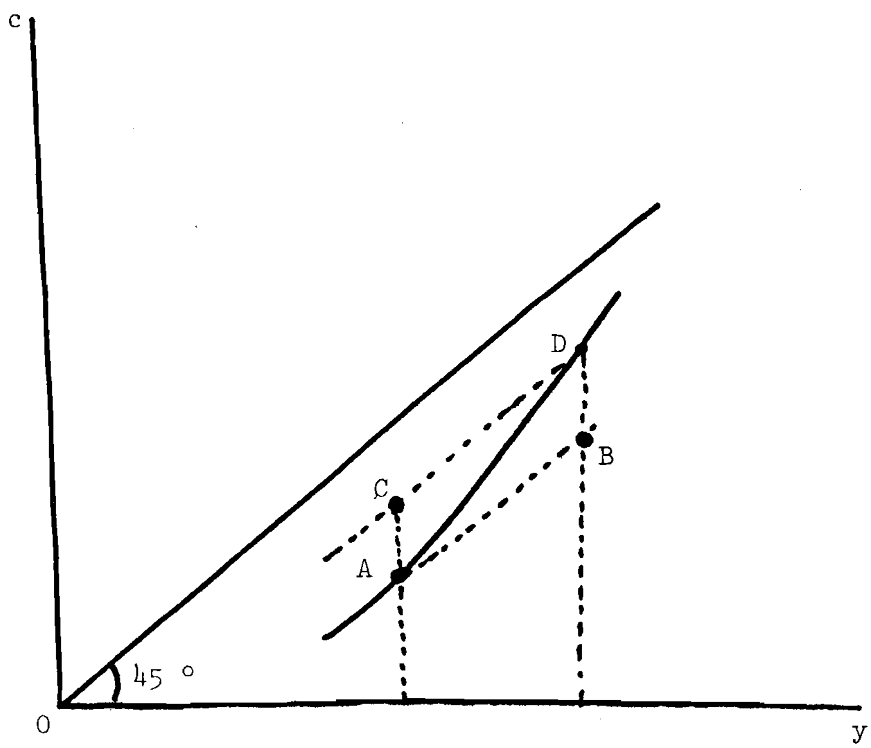
$$c_t(y) \in \arg\max_{0 \leq c \leq y} U_t(c,y;\, c_{t+1}),\ y \in [0,\bar{y}_t]$$

is well defined. Then for all $y_1, y_2 \in [0, \bar{y}_t]$ with $y_1 < y_2$, $c_t(y_2) - c_t(y_1) \leq y_2 - y_1$.

An intuitive understanding of this result can be obtained by consulting Figure 1. We suppose that points A and D lie in the graph of the consumption function, and that the slope between them exceeds 1. Notice that the bequest associated with each point is given by the vertical distance between that point and the 45° line. Let points B and C be defined as follows; at B, agent t has the same endowment as at D, but bequeaths an amount equal to his bequest at A; at C, agent t has the same endowment as at A, but bequeaths an amount equal to his bequest at D. Notice that the lines between A and B and between C and D have slopes of one.

Now we observe that agent t (weakly) prefers moving from B to D. How should he then feel about moving from A to C? The 'future' is identical for A and B (his bequest is the same); similarly for C and D. Thus moving from A to C differs from moving from B to D only in that initial consumption is lower -- the incremental exchange of future consumption for current consumption is the same. If marginal utility of current consumption is decreasing, then C must be strongly preferred to A -- a contradiction. Note that this reasoning is valid only if the reduction in c_t does not raise the marginal utility of c_{t+1} too much (i.e., c_t and c_{t+1} are not substitutes).

<u>Figure 1</u>



Two qualifications are in order. First, this result depends upon weak complementarity (A.3). Second, we doubt that a similar theorem could be obtained in a disaggregated model. Consequently, it may be difficult to generalize the existence theorems proven in the next section to other interesing models by using the techniques employed there. These cases are left as open questions.

IV. Existence Theorems

Although much is now known about the properties of equilibria for models such as that presented in Section 2, previous investigations have failed to produce a completely satisfactory existence theorem. In this section, we present two theorems which establish the existence of perfect equilibria for the altruistic growth model described in section 2. These results may be summarized as follows. For the most general version of our model, non-stationary equilibria in well-behaved strategies always exist (Theorem 4.1). If in addition, the model is stationary, then at least one such equilibrium is stationary as well (Theorem 4.2). Formally,

Theorem 4.1: There exists a bequest equilibrium $\langle C_t^* \rangle$ where for all $t \geq 0$, $C_t^* \in \mathcal{C}_t$ is upper semi-continuous, continuous from the left, with limits on the right.

Theorem 4.2: For stationary models, there exists a stationary bequest equilibrium $\langle C_t^* \rangle$, $C_t^* \in \mathcal{C}_t$, where C_t^* is upper semicontinuous, continuous from the left, with limits on the right, for all $t > 0$.

As the proofs of these theorems are rather intricate, we provide here a sketch of the arguments employed. The behavioral assumption underpinning the perfect Nash concept is that agent t chooses his best strategy (C_t) by maximizing his utility for every possible initial level of endowment, taking C_{t+1} as given. Whether or not the solution to this maximization problem is well defined clearly depends upon the properties of C_{t+1}. We show that, in particular, if C_{t+1} is upper semicontinuous, then t's best responses are well-defined for every initial level of endowment, and form an upper hemicontinuous correspondence. C_t may then be any function selected from this correspondence.

It is, of course, possible to select C_t such that it is <u>not</u> upper-semicontinuous. In this case, C_{t-1} will not necessarily be well-defined. However, it is always possible to select C_t to be upper-semicontinuous, in which case this problem is not encountered. Consequently, we can without loss of generality look for equilibria in upper semicontinuous strategies (notice that, unlike Peleg and Yaari, we have not restricted agents to a subset of strategies, since each generation will always have a globally best response which lies in the desired subset).

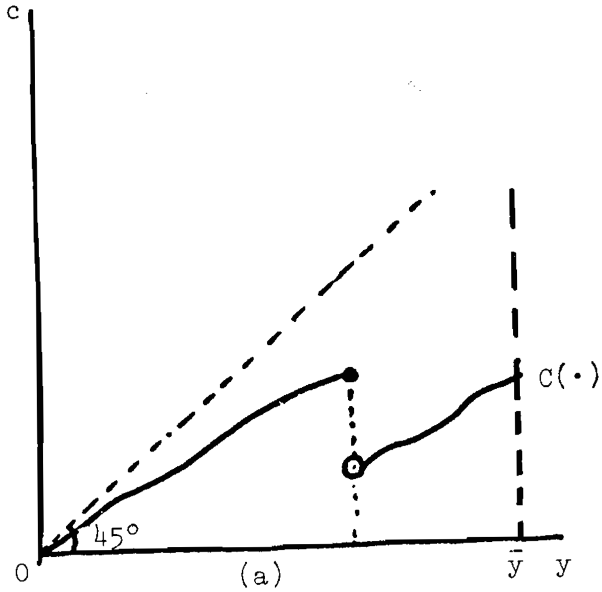
Next, we observe that the upper hemicontinuous correspondence which forms t's best responses must satisfy the Keynesian property (Theorem 3.1). It is easy to see that there is always one and only one upper-semicontinuous selection from such a correspondence, formed by taking the maximum value of consumption for each level of endowment.

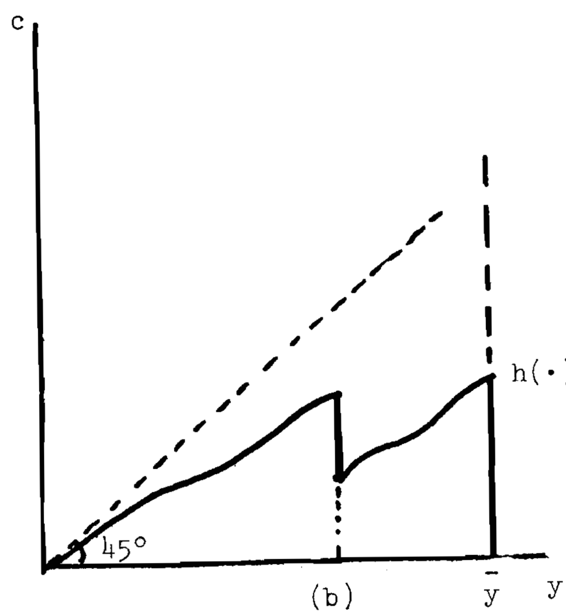
Consequently, for every upper-semicontinuous strategy C_{t+1} chosen by generation $t + 1$, generation t has one and only one upper-semicontinuous best response.

Our next step is to determine the properties of this best response mapping. In particular, we must verify continuity. To do so, we must endow the space of upper semicontinuous consumption functions with an appropriate topology. In practice, it is much easier to identify consumption functions with upper hemicontinuous correspondences from which they are selected, and to work in terms of the latter space. We know that for every upper hemicontinuous correspondence satisfying the Keynesian property, we can select one and only one upper semicontinuous function. Knowing the upper semicontinuous function, can we reconstruct the correspondence from which it is selected? The answer is, in general, no. However, it is true that there is one and only one convex valued upper hemicontinuous correspondence (with a technical restriction on the upper end point) satisfying the Keynesian property from which the function could have been selected. We call the process of going from upper semicontinuous functions to such correspondences 'filling' the function. This is illustrated in Figures 2(a) and (b). Filling the function C in 2(a) yields the correspondence h in 2(b); the only permissible (upper semicontinuous) selection from h is C.

This reasoning allows us to take agents strategy spaces as consisting of convex valued upper-hemicontinuous correspondences satisfying the Keynesian property. The best response mapping then works as follows. For any strategy h_{t+1} chosen by generation $t + 1$, let

Figure 2





C_{t+1} be the unique upper semicontinuous selection. We obtain t's best response by filling the correspondence which takes t's endowments into his optimum consumption levels (call this h_t). This mapping is single valued. Furthermore, the unique upper semicontinuous selection from h_t is a best response to the unique upper semicontinuous selection from h_{t+1}. We endow the space of 'filled' upper-semicontinuous correspondences with the Hausdorff topology; that is, we take the distance between two correspondences to be the Hausdorff distance between their graphs. As long as endowments have an upper bound (clearly, they are bounded by the pure accumulation path), strategy spaces are compact in this topology. Finally, we show that the best response mapping (taking correspondences to correspondences) is continuous for this topology.

The equilibria mentioned in Theorem 4.1 may now be constructed by successive deletion of strategies. Consider generation t. First construct the set of strategies for t which are best responses to some strategy for $t+1$. This set is necessarily compact by the above topological arguments. Next consider the set of strategies for t which are best responses to some strategy for $t + 1$, which is in turn a best response to some strategy for $t + 2$. This set is also compact, and lies within the first set. We continue this process, forming an infinite sequence of compact nested sets; their intersection is non-empty. By constructing these sets for each t, and by appropriately selecting a member from each set, we construct an equilibrium.

If we know in addition that the model is stationary, a stronger result (Theorem 4.2) can be obtained. First consider the case where the production function crosses the 45° line. We can, without loss of generality, choose a common upper bound on the domain (endowments) in every period, and consider only strategies consisting of correspondences defined over this common domain. The best response mapping for each agent will then be identical, and any fixed point of this mapping will be a stationary equilibrium. We know the mapping is continuous, and that it maps a compact space into itself. Unfortunately, the space is not linear, so convexity cannot be verified. However, it is possible to show that the space is both contractible and locally contractible. Figure 3 illustrates the argument for contractibility. We define a homotopy between the identity map and the constant map (with value equal to the horizontal axis) by simply 'shrinking' the vertical axis. A similar argument applies for local contractibility. The existence of a fixed point follows immediately.

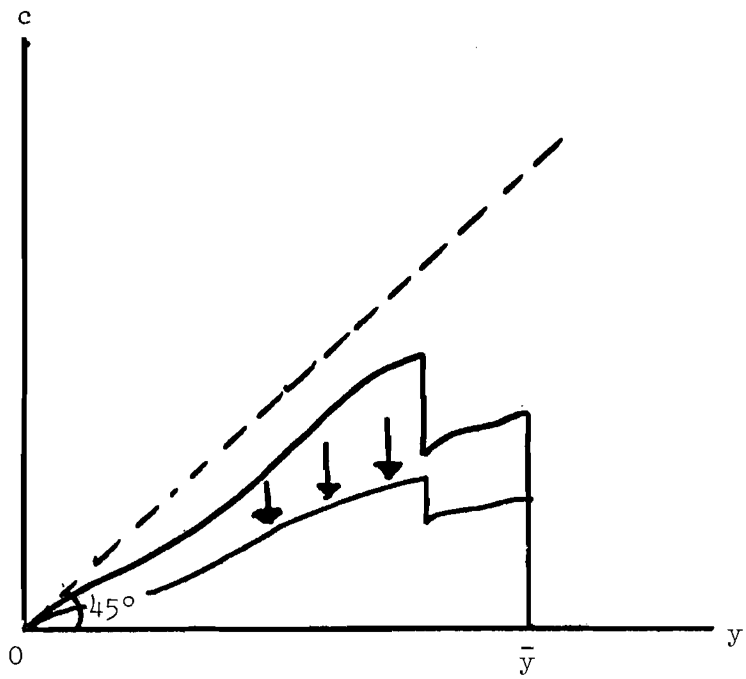
To extend this analysis to cases where the production function need not cross the 45° line, we consider a sequence of economies where we truncate the production function at successively higher levels. An equilibrium for the original economy can be constructed as the limit of equilibria in these artificial economies.

V. Proofs

Proof of Theorem 3.1: Suppose, on the contrary, that there exists $y_1, y_2 \in [0, \bar{y}_t]$, $y_1 < y_2$, with

$$c_t(y_2) - c_t(y_1) > y_2 - y_1. \tag{5.1}$$

Figure 3



Define $\tilde{c}_2 \equiv C_t(y_1) + y_2 - y_1$ $\tilde{c}_1 \equiv C_t(y_2) + y_1 - y_2$. By (5.1), we have $y_2 > \tilde{c}_2 > 0$, $y_1 > \tilde{c}_1 > C_t(y_1) > 0$. Since $U_t(C_t(y_1), y_1; C_{t+1}) > U_t(\tilde{c}_1, y_1, C_{t+1})$, we have, using $\tilde{c}_1 > C_t(y_1)$ and A.2.,

$$(5.2) \qquad C_{t+1}[f_t(y_1 - C_t(y_1))] > C_{t+1}[f_t(y_1 - \tilde{c}_1)]$$

$$= C_{t+1}[f_t(y_2 - C_t(y_2))]$$

Using (5.2) and A.3.,

$$U_t(\tilde{c}_2, y_2; C_{t+1}) - U_t(C_t(y_1), y_1 \; C_{t+1})$$

$$= u_t(C_t(y_1) + y_2 - y_1, C_{t+1}[f_t(y_1 - C_t(y_1))]) - u_t(C_t(y_1), C_{t+1}[f_t(y_1 - C_t(y_1))])$$

$$> u_t(C_t(y_1) + y_2 - y_1, C_{t+1}[f_t(y_2 - C_t(y_2))]) - u_t(C_t(y_1), C_{t+1}[f_t(y_2 - C_t(y_2))])$$

$$> u_t(C_t(y_2), C_{t+1}[f_t(y_2 - C_t(y_2))]) - u_t(C_t(y_2) + y_1 - y_2, C_{t+1}[f_t(y_2 - C_t(y_2))])$$

this last inequality following from A.2 and the fact that $C_t(y_2) + y_1 - y_2 > C_t(y_1)$. Therefore, rearranging terms,

$$(5.3) \qquad U_t(\tilde{c}_2, y_2; C_{t+1}) - U_t(C_t(y_2), y_2; C_{t+1})$$

$$> U_t(C_t(y_1), y_1; C_{t+1}) - U_t(\tilde{c}_1, y_1; C_{t+1})$$

By definition of $C_t(.)$, the left-hand side of (5.3) must be non-positive, while the right hand side must be nonegative. But this contradicts the inequality in (5.3). Q.E.D.

To prove Theorem 4.1, we will need a number of preliminaries. First define, for all $t > 0$, $M_t = \{(x,y) | 0 < x < \bar{y}_t,\ 0 < y < x\}$. For any function or correspondence $g(\cdot)$, define its graph by $G(g) \equiv \{(x,y) | y \in g(x)\}$. For any set G in $\mathbb{R}^2$, denote by $\Pi_y(G)$ its projection onto the y-axis.

For a uhc correspondence $g: [0,a] \Rightarrow [0,a]$, $a > 0$, with $0 < c < y$ for all $c \in g(y)$, define $\mathrm{Fil}(g)$ by $\mathrm{Fil}(g)(y)$ = convex hull of $g(y)$, for all $y \in [0,a)$, and $\mathrm{Fil}(g)(a) = [0, \max g(a)]$. If $a = \infty$, ignore this last requirement. $\mathrm{Fil}(g)$ is the filler of g it is uhc and maps $[0,a]$ to $[0,a]$. A correspondence g is filled if $\mathrm{Fil}(g) = g$.

Next, we attach a name to the property established in Theorem 3.1.

Definition: A set $E \subseteq \mathbb{R}^2$ satisfies the Keynesian property if there does not exist (x',y'), $(x'',y'') \in E$, with $x'' > x'$, , such that $y'' - y' > x'' - x'$. A correspondence with graph in $\mathbb{R}^2$ satisfies the Keynesian property if its graph does.

If a consumption correspondence satisfies the Keynesian property, then the associated 'marginal propensity to consume' can never exceed one.

Define for $t > 0$, $\mathcal{H}_t$ to be the set of graphs of filled, uhc correspondences h_t: $[0, \bar{y}_t] \Rightarrow [0, \bar{y}_t]$, such that $G(h_t) \subseteq M_t$, and such that h_t satisfies the Keynesian property.

Similarly define for $t > 0$, $\tilde{H}_t$ as the set of graphs of uhc correspondences $\tilde{h}_t$: $[0,\bar{y}_t] \rightrightarrows [0,\bar{y}_t]$, with $G(\tilde{h}_t) \subseteq M_t$, and such that $\tilde{h}_t$ has the Keynesian property.

Often we shall refer to $\tilde{h}_t \in \tilde{H}_t$ (or $h_t \in H_t$), at the risk of some harmless notational abuse.

We wish to endow H_t and $\tilde{H}_t$ with a suitable topology. Distance between correspondences will be defined as the Hausdorff distance between their graphs. Formally,

Definition: For every two subsets E and F of a metric space (M,d), let the Hausdorff distance $\delta(E,F)$ (with respect to the metric d on M) be given by

$$\delta(E,F) = \inf\{\varepsilon \in [0.\infty] \mid E \subseteq B_\varepsilon(F),\ F \subseteq B_\varepsilon(E)\}$$

where $B_\varepsilon(X)$, $X \subseteq M$, denotes the ε-neighborhood of X, i.e., $B_\varepsilon(X) \equiv \{x \in M \mid \text{dist }(x,X) < \varepsilon\}$.

Lemma 5.1: Let (M,d) be a compact metric space. Then the set $\mathcal{M}$ of nonempty closed subsets of M together with the Hausdorff distance δ on $\mathcal{M}$ is a compact metric space. Further, for any sequence $\langle E^t\rangle$, $E^t \in \mathcal{M}$, define

$$Li(\langle E^t\rangle) = \{x \in M \mid \text{there is } \langle x^t\rangle,\ x^t \in E^t,\ \lim_t x^t = x\}$$

$$Ls(\langle E^t\rangle) = \{x \in M \mid \text{there is } \langle x^t\rangle,\ x^t \in E^t, \text{ with a convergent subsequence converging to } x\}$$

Then $\langle E^t\rangle$ converges to E in $(\mathcal{M},\delta)$ iff $Li(\langle E^t\rangle) = E =$

$Ls(<E^t>)$.

Proof: See Hildenbrand [1974].

Lemma 5.2: *Suppose that* $h, h' \in H_t$, *with* $G(h) \subseteq G(h')$. *Then* $h = h'$.

Proof: Define $P \equiv G(h') \setminus G(h)$. Suppose P is nonempty. Then there is $(y,c) \in P$, such that either $c > \max h(y)$, or $c < \min h(y)$. Assume the first. Then clearly $y > 0$. Since $G(h)$ is closed and Keynesian, there is $(y^m,c^m) \in G(h)$, with $y^m \uparrow y$ and $\lim(y^m,c^m) = (y,\max(y))$. But then for m sufficiently large, $y - y^m < c - c^m$, so that $P \cup G(h) = G(h')$ violates the Keynesian property, a contradiction. Now assume the second. Then, by definition of a filled graph, $y < \bar{y}_t$. Since $G(h)$ is closed and Keynesian, there is $(y^n,c^n) \in G(h)$ with $y^n \downarrow y$ and $\lim(y^n,c^n) = (y,\min h(y))$. But then for n sufficiently large, $y^n - y < c^n - c$, so that $P \cup G(h) = G(h')$ violates the Keynesian property, a contradiction.

Q.E.D.

Lemma 5.3: *Let* h *be a convex-valued uhc mapping from some compact interval* I *to subsets of a compact interval* $[b_L,b_U]$. *Take* (y,c), $(y',c') \in G(h)$, *with* $y' > y$. *Then for all* $c \in [\min(c,c'), \max(c,c')]$, *there exists* $\tilde{y} \in [y,y']$ *such that* $c \in h(\tilde{y})$.

Proof: Let P^U be the preimage of $[c,b_U]$ and P^L be the preimage of $[b_L,c]$ in $[y,y']$. Both P^L and P^U are closed. Further, $P^L \cup P^U = [y,y']$. Thus, $P^L \cap P^U$ is nonempty. So there exists $\tilde{y}$, $\tilde{c}$, $\tilde{c}'$ such that $\tilde{y} \in P^L \cap P^U$, $\tilde{c}$, $\tilde{c}' \in h(\tilde{y})$, $\tilde{c} > c$, $\tilde{c}' < c$. Since h is convex valued, $c \in h(\tilde{y})$. Q.E.D.

Now we establish

<u>Lemma 5.4: H_t and $\tilde{H}_t$ endowed with the Hausdorff topology are compact metric spaces.</u>

<u>Proof:</u> Let H denote H_t, or $\tilde{H}_t$. It is well-known that each $G(h) \in H$ is closed (by uhc of h). Hence $H \subseteq M_t$, the set of all closed subsets of M_t. To establish that H_t and $\tilde{H}_t$ are closed, pick $\langle h^n \rangle$ in $\tilde{H}_t$ with $G(h^n) \to G(h)$. The projection of $G(h^n)$ onto the first co-ordinate is $[0,\bar{y}_t]$; we check first that this is also true of $G(h)$. Fix $y \in [0,\bar{y}_t]$. We can choose $(y,c^n) \in G(h^n)$ for all n. Since $c^n \in [0,\bar{y}_t]$ for all n, $\langle y, c^n \rangle$ has some convergent subsequence with limit (y,c^*). By Lemma 5.1, $(y,c^*) \in G(h)$, hence $G(h)$ has the required projection property. Similarly, using Lemma 5.1, it is easy to check that $G(h)$ has the Keynesian property. Also $G(h) \in M_t$, since M_t is closed (Lemma 5.2). Thus h is a uhc correspondence with the Keynesian property, with $G(h) \subseteq M_t$. This establishes closedness of $\tilde{H}_t$.

Finally, note that $G(h)$ is filled if $G(h^n) \in H_t$. To see this, pick any (y,c), $(y,c') \in G(h)$ with $c > c'$. Consider any c'' $\in [c',c]$. We will show that there exists $(y''_n, c''_n) \to (y,c'')$, where $(y''_n,c''_n) \in h^n$ for all n. By Lemma 5.1, there exists (y_n,c_n), $(y'_n,c'_n) \in h^n$ for all n, with $(y_n,c_n) \to (y,c)$, $(y'_n,c'_n) \to (y,c')$. Pick N such that for all $n > N$, $c_n > c'' > c'_n$. By Lemma 5.3, there exists, for such n, $(y''_n,c''_n) \in h^n$ with $c''_n = c''$, and

$y_n'' \in [\min(y_n, y_n'), \max(y_n, y_n')]$. Clearly, as $n \to \infty$, $(y_n'', c_n'') \to (y'', c'')$. By Lemma 5.1, $(y'', c'') \in G(h)$. Also, note that $(\bar{y}_t, 0) \in G(h^n)$ for all n. Hence $(\bar{y}_t, 0) \in G(h)$. By the previous argument, $h(\bar{y}_t) = [0, \max h(\bar{y}_t)]$. So h is filled.

This establishes that H is closed. Consequently, since $H \subseteq M_t$, which is compact, H is compact. Q.E.D.

Let S_t be the set of all s: $[0, \bar{y}_t] \to [0, \bar{y}_t]$, s upper-semi-continuous (usc), with $G(s) \subseteq M_t$, and satisfying the Keynesian property.

Lemma 5.5: If $s \in S_t$, then for all $y^* \in [0, \bar{y}_t]$, $\lim_{y \uparrow y^*} s(y) = s(y^*)$. Also, $\lim_{y \downarrow y^*} s(y)$ exists.

Proof: Suppose, on the contrary, that there is y^*, and $y^m \uparrow y^*$ with $\bar{s} \equiv \lim_m s(y^m) < s(y^*)$ (by usc of $s(\cdot)$, it cannot be greater). Then, for m sufficiently large, we have $y^* - y^m < (1/2)[s(y^*) - \bar{s}]$, and $s(y^m) - \bar{s} < (1/2)[s(y^*) - \bar{s}]$. Rearranging, $y^* - y^m < s(y^*) - s(y^m)$, which violates the Keynesian property.

To establish that $\lim_{y \downarrow y^*} s(y)$ exists, assume on the contrary that there are y^*, and two sequences $y_m \downarrow y^*$, $y_n \downarrow y^*$ with $y_m, y_n > y^*$, and $\lim_m s(y^m) > \lim_n s(y^n)$. Then define $\delta = (1/3)[\lim_m s(y^m) - \lim_n s(y^n)]$ and M, N with

(a) $\lim_m s(y^m) - s(y^M) < \delta$

(b) $s(y^N) - \lim_n s(y^n) < \delta$ and

(c) $y^M > y^N$, with $y^M - y^N < \delta$.

It is easily seen that such M, N exist. Then

$$s(y^M) - s(y^N) > [\lim_m s(y^m) - \lim_n s(y^n)] - 2\delta = \delta > y^M - y^N ,$$

which violates the Keynesian property. Q.E.D.

Lemma 5.6 relates elements of H_t to those of S_t.

Lemma 5.6: For $h \in H_t$, let $s(\cdot)$ be defined by $s(y) \equiv \max \{h(y)\}$. Then

(a) $s(\cdot)$ is well defined.

(b) $s(\cdot) \in S_t$

(c) $s(\cdot)$ is the unique selection from h such that $s(\cdot) \in S_t$.

Proof:

(a) This follows from the fact that h is uhc and maps into a compact set.

(b) Pick any sequence $y^m \to y \in [0,\bar{y}_t]$. Then $s \equiv \limsup_m s(y^m) \in h(y)$, by uhc of h. Thus $s \leq \max \{h(y)\} = s(y)$. Verification of the Keynesian property is trivial.

(c) Lemma 5.5, along with part (b), implies, for $y^* > 0$, that $\lim_{y \uparrow y^*} s(y) = s(y^*) = \max\{h(y^*)\}$. Consequently choosing $s(y^*) < \max\{h(y^*)\}$ violates usc. For $y^* = 0$, $\{h(y^*)\} = \{0\}$, and there is nothing to be proved. Q.E.D.

Let $s = S(h)$ be the unique selection (in S_t) from $h \in H_t$. We prove

Lemma 5.7: Consider $\langle h^m \rangle$ in H_t, $h^m \to h \in H_t$. Define $\tilde{G}$ = {limit points of all sequences (y^m,c^m), where $(y^m,c^m) \in G(S(h^m))$ for all m}.

Then $s(y) \equiv \max\{c \mid (y,c) \in \tilde{G}\}$ is well defined for all $y \in [0,\bar{y}_t]$, and $s(\ ) = S(h)$.

Proof: It is easy to verify that $\tilde{G}$ is closed, that its projection into the first co-ordinate is $[0,\bar{y}_t]$, and that $\tilde{G} \subseteq \mathrm{Li}(\{G(h^m)\}) \subseteq G(h)$. Hence $s(y)$ is well defined for all $y \in [0,\bar{y}_t]$. Moreover, it is easy to verify that $\tilde{G}$ has the Keynesian property. Given this, and the fact that $\tilde{G}$ is the graph of some uhc correspondence $\tilde{h} \in H_t$, $s \in S_t$, by Lemma 5.6(b). But s is a selection from $G(h)$, hence by Lemma 5.6(c), $s = S(h)$. Q.E.D.

Next, we establish two lemmas concerning the model itself.

Lemma 5.8: For any $s_{t+1} \in S_{t+1}$, define for $y \in [0,\bar{y}_t]$,

$$h(y) = \arg\max_{0 \leq c \leq y} U_t(c,y;\ s_{t+1}) \tag{5.4}$$

Then $\mathrm{Fil}(h) \in H_t$.

Proof: By definition of H_t, $\mathrm{Fil}(\cdot)$, and Theorem 3.1, it suffices to show that h is well defined and uhc, and that $\mathrm{Fil}(h)$ satisfies the Keynesian property. This last step follows once we show h is well-defined and uhc. For suppose that $\mathrm{Fil}(h)$ does not have the Keynesian property; then there exists (y_1,c_1), $(y_2,c_2) \in G(\mathrm{Fil}(h))$, with $y_1 < y_2$, such that $c_2 - c_1 > y_2 - y_1$. Let $c_1' \equiv \min\{\mathrm{Fil}(h)(y_1)\}$, $c_2' - c_1' \equiv \max\{\mathrm{Fil}(h)(y_2)\}$. Clearly (y_1,c_1'), $(y_2,c_2') \in G(h)$, and $c_2' - c_1' \geq c_2 - c_1 > y_2 - y_1$. But this violates Theorem 3.1, which asserts that h possesses the Keynesian property.

So it remains to verify that h is well-defined and uhc.

Since s_{t+1} is usc and u_t is continuous and increasing, U_t is usc. A usc function reaches a maximum on a compact set; hence $h(y)$ is well-defined for all $y \in [0,\bar{y}_t]$.

To verify uhc of h, consider some sequence $\langle y^m \rangle$ in $[0,\bar{y}_t]$ with $y^m \to y^* \in [0,\bar{y}_t]$, and $\langle c^m \rangle$ with $c^m \in h(y^m)$ for all m, where $c^m \to c^*$. We will show that $c^* \in h(y^*)$. Suppose that this is not true.

Since s_{t+1} is usc, we have

$$\limsup_m U_t(c^m,y^m;\ s_{t+1}) \leq U_t(c^*,y^*;\ s_{t+1}) \tag{5.5}$$

By our assumption that $c^* \notin h(y^*)$, there exists $\tilde{c}$ with $y^* \geq \tilde{c} \geq 0$ and

$$U_t(\tilde{c},y^*;\ s_{t+1}) > U_t(c^*,\ y^*;\ s_{t+1}) \tag{5.6}$$

Define $\langle \tilde{c}^m \rangle$ by $\tilde{c}^m = \max(0,y^m + \tilde{c} - y^*)$ for all $m \geq 0$. Note that $y^m \geq \tilde{c}^m \geq 0$, and that $\tilde{c}^m \to \tilde{c}$. Recall that

$$U_t(\tilde{c}^m,y^m;\ s_{t+1}) = u_t(\tilde{c}^m,s_{t+1}[f_t(y^m - \tilde{c}^m)])$$

Now note that $y^m - \tilde{c}^m \uparrow y^* - \tilde{c}$, and so, since f_t is increasing, $f_t(y^m - \tilde{c}^m) \uparrow f_t(y^* - \tilde{c})$. By Lemma 5.5, $\lim_m s_{t+1}[f_t(y^m - \tilde{c}^m)] = s_{t+1}[f_t(y^* - \tilde{c})]$. Using the continuity of u_t, we have

$$\lim_{m\to\infty} U_t(\tilde{c}^m,y^m;\ s_{t+1}) = U_t(\tilde{c},y^*;\ s_{t+1}) \tag{5.7}$$

But (5.5), (5.6) and (5.7) together imply that for m sufficiently large, $U_t(c^m,y^m;\ s_{t+1}) < U_t(\tilde{c}^m,y^m;\ s_{t+1})$ contradicting, for such m, $c^m \in h(y^m)$. Q.E.D.

Now we introduce some additional notation. For $s_{t+1} \in S_{t+1}$, define $\psi_t(s_{t+1}) \equiv h$ by (5.4) and $\phi_t(s_{t+1}) \equiv \mathrm{Fil}(\psi_t(s_{t+1}))$. Then, by Lemma 5.8, $\phi_t\colon\ S_{t+1} \to H_t$. Next, define $\Phi_t\colon\ H_{t+1} \to H_t$ by $\Phi_t(h_{t+1}) = \phi_t(S(h_{t+1}))$ for each $h_{t+1} \in H_{t+1}$. Finally, for $B \subseteq H_{t+1}$, let $\Phi_t(B) \equiv \bigcup_{h\in B} \Phi_t(h)$.

A central result in the proof of Theorem 4.2 is

Lemma 5.9: $\Phi_t\colon\ H_{t+1} \to H_t$ is continuous for all $t > 0$.

Proof: Consider any sequence $\langle h^m_{t+1}\rangle$ in H_{t+1}, $h^m_{t+1} \to h_{t+1} \in H_{t+1}$. We show that for any limit point h_t of $h^m_t \equiv \Phi_t(h^m_{t+1})$, $h_t = \Phi_t(h_{t+1})$.

Clearly, $h^m_t = \mathrm{Fil}(\tilde{h}^m_t)$, where $\tilde{h}^m_t = \psi_t(s^m_{t+1}) \in \tilde{H}_t$ for $s^m_{t+1} = S(h^m_{t+1})$. Define $\tilde{G} \equiv \{$limit points of all sequences (y^m,c^m), where $(y^m,c^m) \in G(s^m_{t+1})\}$, and $s_{t+1}\colon\ [0,\bar{y}_{t+1}] \to [0,\bar{y}_{t+1}]$ by $s_{t+1}(y) \equiv \max\,\{c|(y,c) \in \tilde{G}\}$ for all $y \in [0,\bar{y}_{t+1}]$. Then by Lemma 5.7, $s_{t+1} = S(h_{t+1})$.

We shall demonstrate that $h_t = \phi_t(s_{t+1})$.

Without loss of generality, assume that $h^m_t \to h_t$, and $\tilde{h}^m_t \to \tilde{h}_t \in \tilde{H}_t$ (this last step is possible, by Lemma 5.4). It is obvious that $G(\tilde{h}_t) \subseteq G(h_t)$, and hence $G(\mathrm{Fil}(\tilde{h}_t)) \subseteq G(h_t)$. But since $\mathrm{Fil}(\tilde{h}_t)$, $h_t \in H_t$, we have $\mathrm{Fil}(\tilde{h}_t) = h_t$, by Lemma 5.2. Therefore it suffices to show that for each $y^* \in [0,\bar{y}_t]$ and $c^* \in \tilde{h}_t(y)$, $U_t(c,y^*; s_{t+1})$ is maximized at c^*. In that case, $G(\tilde{h}_t) \subseteq G(\psi(s_{t+1}))$. so that $G(h_t) = G(\mathrm{Fil}(\tilde{h}_t)) \subseteq G(\mathrm{Fil}(\psi(s_{t+1}))) = G(\phi(s_{t+1}))$. But since $h_t, \phi(s_{t+1}) \in H_t$, we have $h_t = \phi(s_{t+1})$, by Lemma 5.2.

Therefore, pick $y^* \in [0,\bar{y}_t]$ and $c^* \in \tilde{h}_t(y)$. By Lemma 5.1, there exists $(y^m,c^m) \in G(\tilde{h}_t^m)$ with $(y^m,c^m) \to (y^*,c^*)$. Since $s^m_{t+1} = S(h^m_{t+1})$ for all m, and $s_{t+1} = S(h_{t+1})$, we have

$$\limsup_m s^m_{t+1}\,[f_t(y^m - c^m)] \leq s_{t+1}\,[f_t(y^* - c^*)].$$

(This comes from the fact that $s_{t+1}(y^* - c^*) = \max\{h_{t+1}(y^* - c^*)\}$). Consequently, since u_t is increasing in its second argument,

$$\limsup_m u_t(c^m,s^m_{t+1}[f_t(y^m - c^m)]) \leq u_t(c^*,s_{t+1}[f_t(y^* - c^*)]) \tag{5.8}$$

Now suppose, on the contrary, that there exists $\tilde{c} \in [0,y^*]$ with

$$u_t(\tilde{c},s_{t+1}[f_t(y^* - \tilde{c})]) > u_t(c^*,s_{t+1}[f_t(y^* - c^*)]) \tag{5.9}$$

By construction of s_{t+1} there exists a sequence $\langle x^m,s^m_{t+1}(x^m)\rangle$ with $(x^m,s^m_{t+1}(x^m)) \to (f_t(y^* - \tilde{c}),s_{t+1}[f_t(y^* - \tilde{c})])$. Define $\tilde{c}^m = \max(y^m - f_t^{-1}(x^m),0)$ for all m. Then, clearly, $0 \leq \tilde{c}^m \leq y^m$. Note that as $m \to \infty$, $f_t(y^m - \tilde{c}^m) \to f_t(y^* - \tilde{c})$. Now pick $\varepsilon > 0$, and integer M^* such that for all $m \geq M^*$, $|f_t(y^m - \tilde{c}^m) - x^m| < \varepsilon$. Since $y^m - \tilde{c}^m \leq f_t^{-1}(x^m)$, we have, using the Keynesian property of s^m_{t+1}, that

$$s^m_{t+1}\,(f_t(y^m - \tilde{c}^m)) > s^m_{t+1}\,(x^m) - \varepsilon. \tag{5.10}$$

Letting $\varepsilon \to 0$ in (5.10), we have

$$\liminf_m s^m_{t+1}(f_t(y^m - \tilde{c}^m)) \geq \lim_m s^m_{t+1}(x^m) = s_{t+1}(f_t(y^* - \tilde{c})).$$

Consequently, using the fact that u_t is increasing,

$$(5.11)\quad \liminf_m u_t(\tilde{c}^m, s^m_{t+1}[f_t(y^m - \tilde{c}^m)]) \geq u_t(\tilde{c}, s_{t+1}[f_t(y^* - \tilde{c})])$$

But (5.8), (5.9) and (5.11) imply that for sufficiently large m,

$$u_t(\tilde{c}^m, s^m_{t+1}[f_t(y^m - \tilde{c}^m)]) > u_t(c^m, s^m_{t+1}[f_t(y^m - c^m)])$$

which contradicts $(y^m, c^m) \in G(\tilde{h}^m_t)$.

This establishes the lemma. Q.E.D.

Proof of Theorem 4.1

Define, for each $(t,\tau) \geq 0$

$$(5.12)\qquad Q^{t,\tau} \equiv \Phi_t \circ \Phi_{t+1} \circ \dots \circ \Phi_{t+\tau}(H_{t+\tau+1})$$

It is easy to verify that $Q^{t,\tau}$ is compact, using Lemmas 5.4 and 5.9. Further, for all $(t,\tau) \geq 0$, $Q^{t,\tau} \supseteq Q^{t,\tau+1}$. Therefore

$$(5.13)\qquad Q^t \equiv \bigcap_{\tau=0}^{\infty} Q^{t,\tau}$$

is nonempty.

We claim that for each $h_t \in Q^t$, $(\Phi_t)^{-1}(h_t) \cap Q^{t+1}$ is nonempty. Clearly, since $h_t \in \Phi_t(H_{t+1})$, $P \equiv (\Phi_t)^{-1}(h_t)$ is nonempty. Also P is closed, by continuity of Φ_t. Therefore, if $P \cap Q^{t+1}$ is empty, there exists $\tau \geq 0$ such that $P \cap Q^{t+1,\tau}$ is empty. But then

$h_t \in \Phi_t(Q^{t+1,\tau}) = Q^{t,\tau} \supseteq Q^t$, a contradiction.

We now construct the equilibrium. Pick $h_0 \in Q^0$. Generation 0's consumption strategy is $C_0^* \equiv S(h_0)$. Now pick $h_1 \in (\Phi_0)^{-1}(h_0) \cap Q^1$, and define Generation 1's strategy as $C_1^* \equiv S(h_1)$. In general, having picked h_t in this recursive fashion, define Generation t's strategy by $C_t^* \equiv S(h_t)$, and choose $h_{t+1} \in (\Phi_t)^{-1}\ (h_t) \cap Q^{t+1}$.

For any $t \geq 0$, and given C_{t+1}^* according to this contruction, $h_t \equiv \mathrm{Fil}(\tilde{h}_t)$ for $\tilde{h}_t = \psi_t(C_{t+1}^*)$. Therefore, for each $y \in [0,\bar{y}_t]$,

$$S(\tilde{h}_t)\ (y) \in \arg\max_{0 \leq c \leq y}\ U_t(c,y;\ C_{t+1}^*).$$

But the unique construction of $S(\cdot)$ (see Lemma 5.6) easily yields $S(\mathrm{Fil}(\tilde{h}_t)) = S(\tilde{h}_t)$. Hence $C_t^* = S(\tilde{h}_t)$.

Finally, since $C_t^* \in \mathcal{S}_t$ for all $t \geq 0$, Lemma 5.5 assures us that $\langle C_t^* \rangle$ has all the properties claimed in the statement of the theorem. Q.E.D.

Now we turn to the proof of Theorem 4.2. Observe that the pure accumulation program $\langle \bar{y}_t \rangle$ is monotone, by A.1. Hence $\lim_t \bar{y}_t \equiv \tilde{y}$ exists in $\mathbb{R}^+ \cup \{+\infty\}$. Consider two cases:

(1) $\tilde{y} < \infty$

(2) $\tilde{y} = \infty$.

We will first focus on Case 1. Define $\bar{y} \equiv \max(y_0,\tilde{y})$. Let $M = \{(y,c) | y \in [0,\bar{y}],\ c \in [0,y]\}$. Let $\mathcal{H}$ be the space of (graphs of) all uhc, filled, correspondences $h\colon\ [0,\bar{y}] \rightrightarrows [0,\bar{y}]$, with $G(h) \subseteq M$, such that h has the Keynesian property.

Lemma 5.10: *Take any* h'. *There exists a mapping*

$$\chi: \mathcal{H}\times[0,1] \to \mathcal{H} \text{ such that}$$

(5.14) $$\chi(h, 0) = h \quad \text{for all} \quad h \in \mathcal{H}$$

(5.15) $$\chi(h, 1) = h' \quad \text{for all} \quad h \in \mathcal{H}$$

(5.16) $$\chi \text{ is continuous}$$

(5.17) $$\delta(h,h') < \varepsilon \text{ implies } \delta(\chi(h,\lambda), h') < \sqrt{5}\varepsilon \text{ for all } \lambda \in [0,1]$$

Proof We explicitly construct the mapping χ.

$$G[x(h,\lambda)] = \{(y,c) \mid \lambda \max\{h'(y)\} + (1 - \lambda) \max\{h(y)\}$$
$$\geq c \geq \lambda \min\{h'(y)\} + (1 - \lambda) \min\{h(y)\}\}.$$

First we prove that for all $h, h' \in \mathcal{H}$, $\lambda \in [0,1]$, $\chi(h,\lambda) \in \mathcal{H}$ This is accomplished in several steps.

(i) For all $y \in [0,\bar{y}]$, $\chi(h,\lambda)(y)$ is nonempty, convex, and closed. Nonemptiness follows from $h(y)$ and $h'(y)$ being nonempty; convexity and closedness are true by construction. Noting that $h(\bar{y}) = h'(\bar{y})$ we conclude that $\chi(h,\lambda)$ is a filled correspondence.

(ii) $G(\chi(h, \lambda)) \subseteq M$. From (i) and the fact that h, h' are not defined outside of $[0,\bar{y}]$, we know that $\pi_y G(\chi(h,\lambda)) = [0,\bar{y}]$. We need only show that for all $y \in [0,\bar{y}]$, $\chi(h,\lambda)(y) \subseteq [0,y]$. But this follows immediately from the fact that $\min \chi(h,\lambda)(y) \geq \min\{\min\{h(y)\}, \min\{h'(y)\}\}$, and $\max \chi(h,\lambda)(y) \leq \{\max\{h(y)\}, \max\{h'(y)\}\}$ coupled with the observation that $\min\{h(y)\}, \min\{h'(y)\} \geq 0$ and $\max\{h(y)\}$,

$\max\{h'(y)\} \leq y$.

(iii) $G(\chi(h,\lambda))$ is closed. Take any sequence $(y_t, c_t) \in G(\chi(h,\lambda))$ with limit point (y, c). We know

$$\lambda \max\{h'(y_t)\} + (1 - \lambda) \max\{h(y_t)\} \geq c_t \geq \lambda \min\{h'(y_t)\} + (1 - \lambda) \min\{h(y_t)\}$$

Passing to the limit as $t \to \infty$

$$\lambda \lim_t \max\{h'(y_t)\} + (1 - \lambda) \lim_t \max\{h(y_t)\} \geq c$$

$$\geq \lambda \lim_t \min\{h'(y_t)\} + (1 - \lambda) \lim_t \min\{h(y_t)\}$$

But since $y_t \to y$ and h, h' have closed graphs, $\lim_t \max\{h'(y_t)\} \in h'(y)$, $\lim_t \max\{h(y_t)\} \in h(y)$, etc. By definition, $\lim_t \max\{h'(y_t)\} \leq \max\{h'(y)\}$, $\lim_t \max\{h(y_t)\} \leq \max\{h(y_t)\}$, etc. So

$$\lambda \max\{h'(y)\} + (1 - \lambda) \max\{h(y)\} \geq c \geq \lambda \min\{h'(y)\} + (1 - \lambda) \min\{h(y)\} \quad .$$

But then $(y,c) \in G(x(h,\lambda))$. (i) and (iii) together imply that h is a uhc correspondence on $[0,\bar{y}]$.

(iv) $\chi(h,\lambda)$ satisfies the Keynesian property. Take any y', y'' such that $y'' > y'$, and $c'' \in \chi(h,\lambda)(y'')$, $c' \in \chi(h,\lambda)(y')$. Then

$$\lambda \max\{h'(y'')\} + (1 - \lambda) \max\{h(y'')\} \geq c'', \text{ and}$$

$$\lambda \min\{h'(y')\} + (1 - \lambda) \min\{h(y')\} \leq c'$$

Subtracting,

$$\lambda[\max\{h'(y'')\} - \min\{h'(y')\}] + (1 - \lambda)[\max\{h(y'')\} - \min\{h(y')\}] \geq c'' - c'$$

But since h' and h satisfy the Keynesian property,

$$\max\{h'(y'')\} - \min\{h'(y')\} \leq y'' - y', \text{ and } \max\{h(y'')\} - \min\{h'(y')\} \leq y'' - y$$

So $\lambda[y'' - y'] + (1 - \lambda)[y'' - y'] = y'' - y' \geq c'' - c'$.

(i)-(iv) together imply that for all $h, h' \in H$, $\lambda \in (0,1)$ we have $\chi(h,\lambda) \in H$.

Now we turn to the specific properties of χ. For $\lambda = 0$, $G[\chi(h,\lambda)] = \{(y,c) \mid \max\{h(y)\} \geq c \geq \min\{h(y)\}\}$. Since h is filled, this is the definition of h. An identical argument establishes (5.15), so it and (5.14) are verified.

Consider a sequence (h_t,λ_t) converging to (h,λ). We wish to establish that $\lim(\chi(h_t,\lambda_t)) = \chi(h,\lambda)$,i.e., we want to verify (5.16). Consider any point $(y,c) \in \lim(\chi(h_t,\lambda_t))$. By Lemma 5.1, there exists $(y_t,c_t) \in \chi(h_t,\lambda_t)$ converging to (y, c). Then

$\lambda_t \max\{h'(y_t)\} + (1 - \lambda_t) \max\{h_t(y_t)\} \geq c_t \geq \lambda_t \min\{h'(y_t)\} + (1 - \lambda_t) \min h_t($

Passing to limits, and observing the fact that $\lim \max h'(y_t)$, $\lim \min h'(y_t) \in h'(y)$ ($G(h')$ is closed), and $\lim \max h_t(y_t)$, $\lim \min h_t(y_t) \in h(y)$ ($G(h)$ is the collection of limit points of all sequences lying in $G(h_t)$ -- Lemma 5.1), we have

$\lambda \max\{h'(y)\} + (1 - \lambda) \max\{h(y)\} \geq c \geq \lambda \min\{h'(y)\} + (1 - \lambda) \min\{h(y)\}$,

which immediately implies $(y,c) \in G(\chi(h,\lambda))$. That is,

$\lim \chi(h_t,\lambda_t) \subseteq \chi(h,\lambda)$.

Since $x(h_t,\lambda_t) \in H$ and H is compact, $\lim x(h_t,\lambda_t) \in H$. We know $x(h,\lambda) \in H$. By then by Lemma 5.2, $\lim x(h_t,\lambda_t) = x(h,\lambda)$. This establishes (5.16).

Finally, consider property (5.17). We prove this in two steps. First, we show that $G(x(h,\lambda)) \subseteq B_\varepsilon(G(h'))$. We do this by showing that for any $(y,c) \in G(x(h,\lambda))$, there exists $(y^*,c^*) \in G(h')$ such that $d[(y,c), (y^*,c^*)] < \varepsilon$. Note that for any such (y,c), we have $c \in [\min\{\min\{h(y)\}, \min\{h'(y)\}\}, \max\{\max\{h'(y)\}, \max\{h(y)\}\}]$. We know there exist $(y',c'), (y'',c'') \in G(h')$ such that

$$d[(y',c'),(y, \min\{\min\{h(y)\},\min\{h'(y)\}\}] < \varepsilon$$

$$d[(y'',c''),(y, \max\{\max\{h(y)\},\max\{h'(y)\}\})] < \varepsilon \quad .$$

If $c \leq c'$, then $d[(y,c), (y',c')] < \varepsilon$. If $c \geq c''$, $d[(y,c), (y'',c'')] < \varepsilon$. If $c' < c < c''$, then by Lemma 5.3 there exists $y^* \in [\min(y',y''), \max(y',y'')]$ such that $c \in h'(y^*)$. But since $|y^* - y| < \varepsilon$, $d[(y,c), (y^*,c)] < \varepsilon$.

For the second step, we show that $G(h') \subseteq B_{\sqrt{5}\varepsilon}[G(x(h,\lambda))]$. Take any $(y,c) \in G(h')$. We want to show that there exists $(y^*,c^*) \in G(x(h,\lambda))$ such that $d[(y,c), (y^*,c^*)] < \sqrt{5}\varepsilon$. Let $(y',c') \in G(h)$ be the closest point in $G(h)$ to (y,c). Since $\delta(h,h') < \varepsilon$, $d[(y,c), (y',c')] < \varepsilon$. If $y' = y$, then the result is trivial to check, so henceforth we assume $y' \neq y$. Let

$$c'' = \arg\min_{c^0 \in h(y)} |c - c^0| \quad .$$

First we argue that $(c'' - c) \cdot (c' - c) \geq 0$. Suppose this is not true. Then by Lemma 5.3, there exists $y'' \in (\min(y,y'), \max(y,y'))$

such that $c \in h(y)$. But then $d[(y'',c), (y,c)] < d[(y',c'), (y,c)]$ -- a contradiction. Consequently, we know that either

(i) $y' > y$ and $c', c'' < c$

(ii) $y' > y$ and $c', c'' > c$

(iii) $y' < y$ and $c', c'' < c$

(iv) $y' < y$ and $c', c'' > c$

Case (i): by the Keynesian property, since $y' - y < \varepsilon$, $c' - c'' < \varepsilon$. But we also know that $c - c' < \varepsilon$, so $c - c'' < 2\varepsilon$. In this case, $\min\{h(y)\} < c = \max\{h(y)\}$, $\min\{h'(y)\} < c < \max\{h'(y)\}$, so for all λ there exists $c^* \in \chi(h, \lambda)(y)$ with $|c - c^*| < 2\,\varepsilon$. Case (iv): completely analogous to Case (i). Case (ii): consider three subcases.

(a) $\max\{h'(y')\} < c'$. Then for all λ, there exist $\bar{c}'$, $\bar{c}''$ with $\bar{c}'' > c$, $\bar{c}' < c'$ with $(y,\bar{c}'')$, $(y',\bar{c}') \in G(\chi(h,\lambda))$. But then by Lemma 5.3 there exists $(\tilde{y}, \tilde{c}) \in G(x(h, \lambda))$ such that $y' > \tilde{y} > y$, $c' > \tilde{c} > c$. But then $d[(y, c), (\tilde{y}, \tilde{c})] < d[(y, c), (y', c')] < \varepsilon$.

(b) $\min h'(y') > c'$. Then for all λ, there exists $\bar{c}'$ with $c' < \bar{c}' < \min h'(y')$ and $(y',\bar{c}') \in G(X(h,\lambda))$. But by the Keynesian condition $\min\{h'(y')\} < c + (y' - y) < c + \varepsilon$ so $d[(y,c), (y',\bar{c}')] < \sqrt{2}\varepsilon$.

(c) $\min\{h'(y')\} < c' < \max\{h'(y')\}$. Then $c' \in G(\chi(h,\lambda)(y'))$ for all λ.

Case (iii): Completely analagous to case (ii). Q.E.D

<u>Lemma 5.11: H is contractible</u>

Proof: Take any $h \in H$ and construct the corresponding χ. By the properties of χ, the identity map on H is homotopic to the constant map (mapping to h), so H is contractible. Q.E.D.

Lemma 5.12: H is locally contractible. 12/

Proof: Choose any $h \in H$, and an open neighborhood U of h. We must show that there exists a subneighborhood V such that V is contractible over U.

There exists ε such that $B_\varepsilon(h) \subseteq U$. Let $V = B_{\varepsilon\sqrt{5}}(h)$. Construct χ corresponding to h. For all $\lambda \in [0,1]$, $h' \in V$, $\chi(h',\lambda) \in U$. Thus the identity map on V is homotopic over U to a constant map on V, so H is locally contractible. Q.E.D.

Lemma 5.13: If $\tilde{y} < \infty$, a stationary equilibrium exists.

Proof: Define Φ: $H \to H$ by $\Phi(h) = \mathrm{Fil}(\tilde{h})$ for $h \in H$, where $\tilde{h}(y) \equiv \arg\max_{0<c<y} U(c,y;\ S(h))$, $y \in [0,\bar{y}]$. Then, exactly as in Lemma 5.9, Φ is continuous. Moreover, H is compact (as in Lemma 5.4), contractible (Lemma 5.11) and locally contractible (Lemma 5.12). Therefore by Smart ([1974] Corollary 3.1.3) Φ has a fixed point h^*, i.e. there exists $h^* \in H$ with $\Phi(h^*) = h^*$. Define $C^* \equiv S(h^*)$, and $C_t^* \in C_t$ by $C_t^*(y) = C^*(y)$, $y \in [0,\bar{y}_t]$. This is clearly the required stationary equilibrium. Q.E.D.

Using Lemma 5.13, we have established existence of a stationary equilibrium in Case (1). Now we use this result to handle Case (2).

Lemma 5.14: If $\tilde{y} = \infty$, a stationary equilibrium exists.

Proof: Construct a sequence of 'artifical' economies by altering the production functions as follows: for a sequence $k^m \uparrow \infty$, with $\bar{k}^0 > Y$, and $f(k^m) > k^m$, define

$$f^m(k) = \begin{cases} f(k) & , \quad k \leq k^m \\ f(k^m) + \frac{1}{2}(k - k^m) & , \quad k > k^m \end{cases}$$

Let $\bar{k}^m$ be defined by the largest root to the equation $f^m(k) = k$. By Lemma 6.13, for each economy m with production function f^m, there is a stationary equilibrium. Let H^m be the set of all uhc, Keynesian, filled correspondences h from $[0,\bar{k}^m]$ to $[0,\bar{k}^m]$, with $0 \leq c \leq y$ for all $c \in h(y)$. Select $h^m \in H^m$, for each m, such that $s^m = S(h^m)$ produces a stationary equilibrium for the m^{th} economy. Finally, define Φ^m: $H^m \to H^m$ by

$$\Phi^m(h) = \mathrm{Fil}(\tilde{\Phi}^m(h)), \text{ where}$$

$$\tilde{\Phi}^m(h)(y) = \underset{0 \leq c \leq y}{\arg\max}\ U(c,y;\ S(h)) \quad \text{for} \quad y \in [0, \bar{k}^m]$$

In particular, we know that $h^m = \Phi^m(h^m)$.

Now construct a correspondence h_* from $[0,\infty)$ to $[0,\infty)$ in the following way. Define $L_0 = 0$, and $L_{n+1} = \bar{y}_n$, for $n \geq 0$. Then $L_n \uparrow \infty$. For a correspondence h, let $h/[x,y]$ denote its restriction to the interval $[x,y]$. By Lemma 6.4, there is a convergent subsequent h^{m_1}, for which $h^{m_1}/[L_0,L_1]$ has a limit point, h^1_*, which is a correspondence on $[L_0, L_1]$. Recursively, suppose that h_*^n is defined on $[L_{n-1}, L_n]$ as the limit point of some sequence $h^{m_n}/[L_{n-1}, L_n]$. Consider $[L_n, L_{n+1}]$. There is a subsequence of h^{m_n}, $h^{m_{n+1}}$, such that this converges to some correspondence h_*^{n+1} on $[L_n, L_{n+1}]$. Define h_* by $h_*/[L_n, L_{n+1}] = h^{n+1}$. with the provision that at L_n, it is the union of points in h_*^n and h_*^{n+1}. h_* is clearly well-defined on $[0,\infty)$, since $L_n \to \infty$. We show that for all $y \in [0,\infty)$,

$$h_*(y) = \mathrm{Fil}(\tilde{h}_*(y)), \text{ where}$$

$$\tilde{h}_*(y) = \arg\max_{0 \leq c \leq y} U(c, y; S(h_*))$$

If this is so, the selection $S(h_*)$ will induce the required stationary equilibrium in the obvious way.

Choose n such that $L_n > y$, and s such that $L_q > \bar{k}^s > k^s > L_n$ for some $q > n$. Consider the sequence $h^{m_q}/[0, \bar{k}^s]$ (note that $m_q > s$). We know that $h^{m_q}/[0, \bar{k}^s]$ converges to $h^*/[0, \bar{k}^s]$ by the above arguments. Define the following sequence of correspondences on $[0, \bar{k}^s]$:

$$\tilde{h}^{m_q} = \Phi^s(h^{m_q}/[0, \bar{k}^s])$$

Take some limit point of $\tilde{h}^{m_q}$, $\tilde{h}_*^q$. By the continuity of Φ^s (Lemma 5.9), we know that

$$\tilde{h}_*^q = \Phi^s(h_*/[0, \bar{k}^s]) \tag{5.18}$$

Now we argue that $\tilde{h}^{m_q}/[0, L_n] = h^{m_q}/[0, L_n]$. This follows immediately from observing that if $y \leq L_n$,

$$\text{Fil}\{\arg\max_{0\leq c\leq y} u(c, S(h^{m_q})[f^s(y - c)])\} = \text{Fil}\{\arg\max_{0\leq c\leq y} u(c, S(h^{m_q})[f^{m_q}(y - c)])$$

The left hand side is just the definition of $\tilde{h}^{m_q}(y)$; the right hand side is the definition of $h^{m_q}(y)$ (it is a stationary equilibrium). The equality follows because $f^s/[0,L_n] = f^{m_q}/[0,L_n]$, and $y - c \leq L_n$. But then, since $h^{m_q}/[0,L_n]$ converges to $h_*/[0,L_n]$, we know that $\tilde{h}_*^q/[0,L_n] = h_*/[0,L_n]$. So, from (5.18), for $0 \leq y \leq L_n$,

$$h_*(y) = \text{Fil}\{\arg\max_{0\leq c\leq y} u(c,S(h_*/[0,\bar{k}^s])[f^s(y - c)])\}$$

But since f^s and f agree on $[0,L_n]$, this implies that for $0 \leq y \leq L_n$,

$$h_*(y) = \text{Fil}\ [\arg\max_{0\leq c\leq y} u(c,S(h_*)[f(y - c)])]$$

Since this can be established for any y (take L_n large enough), this implies $h_* = \Phi(h_*)$. Define $C_* = S(h_*)$ and $<C_t^*>$ by $C_t^*(y) = C_*(y)$, $y \in [0,\bar{y}_t]$, $t > 0$. This is the required stationary equilibrium. Q.E.D.

Proof of Theorem 4.2: Combine Lemmas 5.13 and 5.14. Q.E.D.

VI. Further Questions

In our opinion, the following questions pose interesting issues for future analysis.

(1) The techniques used here to establish existence of bequest equilibrium does not appear to be applicable in multicommodity models. The question of existence of bequest equilibria in such models remains open.

(2) Kohlberg [1976] established the uniqueness of continuously differentiable, stationary equilibria, whenever these exist. Is this true of stationary equilibria when the strategy space is <u>not</u> restricted (as in the present exercise)?

(3) If the answer to (3) is in the affirmative, the following conjecture is worth exploring: whenever stationary equilibria exist, either (i) nonstationary equilibria do <u>not</u> exist, or (ii) all nonstationary equilibria have the property that the sequence of equilibrium consumption functions converge to that of the stationary equilibrium.

Footnotes

1/ Empirical evidence sugggests that a significant fraction of the U.S. capital stock is transferred to younger generations through bequests (see, for example, Kotlikoff and Summers [1981]). The well-being of a particular generation therefore depends, at least in part, upon the prevalence of altruism amongst its predecessors.

2/ Phelps [1975] extends the original analysis of Phelps and Pollak [1968].

3/ However, there are certainly serious technical problems involved. See the discussion in Section IV, or the actual technique of proof employed (Section V).

4/ The model assumed stationary preferences and a linear, stationary technology.

5/ The assumption that u_t is defined on all pairs of nonnegative real numbers rules out an analysis of such cases as $u_t(c_t, c_{t+1}) = \ln c_t + \delta \ln(t+1)$, $\delta > 0$.

6/ The validity of our results when there is substitutability between c_t and c_{t+1} remains an open question.

7/ H is locally contractible if for each $h \in H$ and neighborhood U of h, there is a neighborhood V with $h \in V \subset U$, contractible to a point over U. (See Dugundji [1958]).

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