Credible Coalitions and the Core[1]

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Abstract: A problem with the concept of the core is that it does not explicitly capture the credibility of blocking coalitions. This notion is defined, and the concept of a modified core introduced, consisting of allocations not blocked by any credible coalition. The core and modified core are then shown to be identical. The concept of credibility is thus implicit in the definition of the core.

Consider a cooperative game in characteristic function form (see, e.g. Aumann 1967). $N = \{1, \ldots, n\}$ is the player set. With each $S \subseteq N$, the characteristic function $v(\cdot)$ associates a nonempty subset $v(S)$ of $\mathbb{R}^S$.[3] Let $H \subseteq v(N)$ be the set of outcomes that can actually occur; it has, at least, the property that

$$v(N) = \{x \in \mathbb{R}^N / \text{there is } y \in H \text{ with } y \geqslant x\} \tag{1}$$

Cooperative games with side payments and transferable utility form a special case, with

$$v(S) = \{x \in \mathbb{R}^S / \sum_{i \in S} x^i \leqslant f(S)\} \tag{2}$$

where $f(\cdot)$ is the *NM* characteristic function.

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[3] For each $S \subseteq N$, $\mathbb{R}^S$ is the subspace of $\mathbb{R}^n$ spanned by the axes belonging to the players of S.

Let $x \in v(S)$. It is said to be *blocked* by $T \subseteq S$ if there is $y \in v(T)$ such that

$$y > x^T \tag{3}$$

where x^T denotes the projection of x on $\mathbb{R}^T$.[4]

An outcome $h \in H$ is a *core allocation* if it is not blocked by any coalition $S \subseteq N$. The *core* $C \subseteq H$ is the set of all core allocations.

There is an obvious problem with the concept of the core, somewhat akin to that of credible threats in noncooperative games (see, e.g., Selten 1965, 1975). Suppose S blocks $h \in H$. What guarantees that this can be "backed up" by an arrangement *within* the coalition S that is *not* blocked, in turn, by some subcoalition of S? Clearly, coalitions incapable of reaching an agreement within *its* members are not "credible". But the credibility of coalitions also depends on similar considerations. Hence we require a recursive definition of the core, where only *credible* coalitions (defined below) are permitted to block proposed allocations. This is done in the following way.

Singleton coalitions are, of course, capable of "credibly" blocking allocations. Therefore define all singleton coalitions to be credible. Now, recursively, a coalition S of size k, $1 \leqslant k \leqslant n$, is *credible* if there is $x \in v(S)$ that is not blocked by any credible subcoalition T of S.

An outcome $h^* \in H$ is a *modified core allocation* if no credible coalition blocks it. Given (1), such an allocation exists if and only if N is credible. The *modified core* $MC \subseteq H$ is the set of all modified core allocations.

The intuition that goes with the credibility concept makes this definition more reasonable than the usual one. A *smaller* number of coalitions (the credible ones) are permitted to block. It is, therefore, interesting that the two definitions actually generate the *same* set of allocations. More precisely, *the core and the modified care are identical*. This is simple enough to demonstrate. Clearly, $h \in C$ implies $h \in MC$, so consider the converse. Suppose, on the contrary, that there is $h \in MC$ such that $h \notin C$. Then there is $S \subseteq N$ such that S blocks h. So S is not credible. Therefore, for each $x \in v(S)$, there is $T \subseteq S$, T credible, such that T blocks x. Now take $s \in v(S)$ such that $s > h^S$, and $t \in v(T)$ such that $t > s^T$. Then $t > h^T$, i.e., T blocks h. But this contradicts our premise that $h \in MC$.

To summarize, the core and modified core are the same. However, this should not prevent us from using the concept of the modified core as a primary one, since it explicitly captures our intuitive notion of what constitutes a credible block by a coalition.

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[4] The reader may interpret $>$ as strict domination in all coordinates, or "semi-strict" domination in at least one coordinate. The analysis is unaffected.

Postscript

The above draft was written in 1983. Similar results, or results similar in spirit, have appeared, since then, In particular, I have in mind Greenberg (1987a), who makes a similar observation on the core in the context of "stable standards of behaviour".

This paper illustrates a general notion: the imposition of the same requirements on deviating subcoalitions as those on the grand coalition. In this context, the work of Bernheim, Peleg and Whinston (1987), Dutta and Ray (1987), Dutta, Ray, Sengupta and Vohra (1987) and Greenberg (1987b) are also relevant.

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