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Intertemporal Borrowing to Sustain Exogenous Consumption Standards under Uncertainty*

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Consider an agent who is attempting to maintain a given consumption level over time, in the face of a stochastic technology. He is permitted to borrow and lend at given rates of interest. The main results are: (i) if the borrowing rate of interest exceeds the lending rate, the expected net indebtedness of the agent must grow unboundedly large, unless the consumption target is attainable with at most one loan, and (ii) the probabilities of the two events: becoming increasingly indebted, and accumulating unbounded wealth, sum to unity. *Journal of Economic Literature* Classification Numbers: 026, 111.

I. INTRODUCTION

In this paper, I analyze a model where borrowing and lending activities are undertaken to meet some "target consumption plan." The discussion takes place in the context of an aggregative model of intertemporal accumulation, with production subject to random perturbations.

Two main objectives of this paper are (a) a characterization of the set of consumption targets which are *unattainable*, in a sense made precise below, and (b) an examination of the nature of the particular stochastic process that arises here. In particular, its *transience* is established, implying that the attempt to maintain a consumption target results in either infinite indebtedness, or infinite wealth, with probability one.

The exogenous stipulation of a consumption plan, and the consequent analysis of its "attainability," is, of course, only one particular approach to the problem. Borrowing models dealing with rather different questions, and adopting a different (perhaps more conventional) approach, do exist in the literature. A representative example is the work of MacLean *et al.* [9].[1] Here

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[1] See also the paper by Yaari [15].

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certain rules for payment of outstanding debt are laid down (see the "optimistic" and "pessimistic" models in their paper), and while this delineates the feasible set, the latter is of no direct interest, the major focus of analysis being the relationship between "optimal" and "competitive" programs.

The two approaches are, in a sense, complementary. The stance adopted here is an emphasis on a direct examination of the feasible set, by taking a particular plan as exogenously given, and studying the path of indebtedness (or wealth) that it generates.[2]

The methodology of this exercise has close connections with the literature on what may be termed the "economics of survival," arising out of an explicit analysis of the role of exhaustible resources in the dynamic behavior of an economy.[3] Given the finiteness of an essential exhaustive resource, and an infinite horizon to contend with, models displaying both these features often have the property that there may not exist a consumption program bounded away from zero. In that case, an economy may be said to be *incapable of survival.* A number of studies characterizing survival, in the sense of there being a consumption program bounded away from zero, have been carried out. A more stringent definition of survival may require, in addition, that the economy be capable of generating a consumption program at least as great as some exogenously given level at every date. This is precisely the sort of question addressed in this paper.

II. Summary of Results

Section III sets up an infinite-horizon borrowing model to study the following issues.

(1) Call a consumption target *unattainable* if the sequence of expected debts associated with it (net of the capital stock) grows "too large" over time (see below for a precise definition). If credit markets are *imperfect*, in the sense that the borrowing rate exceeds the lending rate, which targets are unattainable?

(2) Consider a number of independent, identical units, of the sort described above. Some fraction of this group will fail to achieve a given target, while the remainder will (this remaining fraction may be positive even if the target is unattainable). By "failure," I mean that the actual indeb-

[2] By indebtedness, I shall mean the accumulated debt *net* of wealth (or capital stock).

[3] The classic exercise in this areas for the autarkic case is due to Solow [14]. Extensions of the Solow result are to be found in Mitra [10] and Cass and Mitra [6]. The concept of survival has been applied in the context of international trade with steadily deteriorating terms of trade by Mitra *et al.* [1]. See also Ray [13].

tedness (net of capital) of the former grows to infinity with time. The remainder will exhibit bounded indebtedness (again net of capital). Is it true that members of this latter group become "almost always," infinitely *wealthy* with time? Phrased differently, is the framework described here such that the two nonnegative probabilities of becoming "infinitely rich" and "infinitely poor" sum to one? In more general terms, is there an inherent tendency for "unbounded inequality" to arise in this model?

When credit markets are imperfect, the answer to the first question is quite surprising. It is demonstrated (Theorem 3.1) that either (a) a particular consumption target is so modest as to necessitate at most one loan, or (b) it is unattainable. To interpret this result in a simple context (see the example in Section IV), suppose that outputs are exogenously given, independent and identical random variables. In this case, Theorem 3.1 asserts the following. As long as there is some positive probability (however small) that the *realised* output may fall below the consumption target, and *some* positive interest rate differential, expected debt grows infinitely large with time. *This occurs even if expected output exceeds the consumption target, the extent of the surplus being irrelevant.*

In other words, the result is not confined to "poor" agents attempting to maintain exorbitant consumption standards. On the contrary, it is established that the long-run effects of credit market imperfections outweigh the advantages of having expected income (or output) exceed planned consumption goals.

The second question is addressed in a context of "consumption–borrowing." The economic agent is restricted to borrow if and only if the current output plus accumulated wealth falls short of the consumption target. The example of Section IV continues to be a special case.

It is shown (Theorem 3.2) that the two probabilities of becoming "infinitely wealthy" and "infinitely indebted" indeed sum to one (irrespective of credit market imperfections). Given that both these probabilities are strictly positive in a large number of cases, the presence of uncertainty, coupled with the phenomenon of credit, appears to generate (in this model) unbounded inequality among independent units, *identical* in the nature of uncertainty that they face.

It is perhaps of interest that the analysis in this paper leads naturally to a stochastic process whose behavior exhibits a marked difference from those in the literature arising out of intertemporal optimization under uncertainty.[4] In particular, the convergence of relevant economic variables to some invariant distribution is not obtained in this exercise.[5] Therefore, the standard

[4] See Futia [7] or Bhattacharya and Majumdar [4] for a comprehensive survey.

[5] See, for example, Brock and Mirman [5].

techniques (in particular, theorems on convergence to some distribution) cannot be applied, and a different method is used to study the long-run, possibly "explosive" behavior of the dynamic process obtained.

III. The Model

Consider an economic agent whose objective is to maintain an exogenously specified consumption target over time. He has at his disposal a positive stock of a single good, to be divided between current consumption, and investment in a stochastic productive activity. At each date, the agent may draw on an external source of the good, to be repaid at some positive rate of interest. He is also free to lend part or all of his endowment at any date at some rate of interest. These rates are given exogenously.

Formally, consider an interval of reals $I=[a,b]$, $0\leqslant a<b<\infty$, and let $\mathscr{B}$ be its Borel σ-algebra. Define $(\Omega,\mathscr{F})\equiv(I^{\infty},\mathscr{B}^{\infty})$, i.e., as the infinite product measurable space obtained from $(I,\mathscr{B})$. Let P be a probability on $(\Omega,\mathscr{F})$, and $\langle Z_t\rangle_1^{\infty}$ the coordinate stochastic process on $(\Omega,\mathscr{F},P)$. Define the increasing sequence of sub σ-algebras of $\mathscr{F}$ by $\mathscr{F}_t=\sigma(Z_1,\dots,Z_t)$, $t\geqslant 1$. Clearly, $\mathscr{F}_t=\mathscr{B}^t$ for all $t\geqslant 1$. Set $\mathscr{F}_0$ be the σ-algebra (ϕ,Ω).

I shall make the following assumption on P.

(A.0) *There exists a strictly positive sequence of reals* $\langle\beta_t\rangle_1^{\infty}$ *such that for each* $t\geqslant 0$ *and* $\delta_{t+1}\in[0,\beta_{t+1}]$, $P[Z_{t+1}\in[a,a+\delta_{t+1}]/\mathscr{F}_t>0$ a.s.

Remarks. Interpret the process $\langle Z_t\rangle_1^{\infty}$ as a sequence of exogenous random shocks affecting the feasible set at each date. Note that assumption (A.0) is satisfied in a wide variety of cases. Consider two simple examples: (a) the coordinate random variables are i.i.d. with support $[a,b]$, and (b) the coordinate random variables form a Markov chain on some finite subset of $[a,b]$, which includes a, and all stationary transition probabilities are positive. One can easily check that (A.0) is met in both (a) and (b).

The technology is represented by a *production function* $g\colon\mathbb{R}^+\times I\to\mathbb{R}^+$, describing, for each capital stock $k\in\mathbb{R}^+$ and "state of nature" $z\in I$, the output $g(k,z)\in\mathbb{R}^+$. The following assumption is made on g.

(A.1) *For each* $z\in I$, $g(\cdot,z)$ *is continuous on* $\mathbb{R}^+$. *For each* $k\in\mathbb{R}^+$, $g(k,\cdot)$ *is nondecreasing (hence measurable) on* $(I,\mathscr{B})$, *and is continuous at* a.

Denote by q (resp. r) the nonnegative rate of interest on lending (resp. borrowing). By the first part of (A.1), the *transformation function* $f\colon\mathbb{R}^+\times I\to\mathbb{R}^+$, given by

$$f(k,z)\equiv\max\{[g(k_1,z)+k_2(1+q)]/(k_1,k_2)\geqslant 0,\ k_1+k_2\leqslant k\}$$

is well defined for all $(k, z) \in \mathbb{R}^+ \times I$. This function represents the method of maximizing "tomorrow's" gross output, given "today's" capital stock, the technology, and lending rate q. I shall make the following assumption on f.

(A.2) *When $z = b$, the partial derivative of f with respect to k, denoted by $f'(k, b)$, exists on $\mathbb{R}^{++}$, and* $\limsup_{k \to \infty} f'(k, b) < 1 + r$.

Remarks on the Assumption. (a) (A.2) is the assumption of credit market imperfection, a situation of central interest in this paper. Taken in conjunction with the definition of the transformation function, it implies $r > q$, and in addition imposes a condition on the asymptotic behavior of the production function under the best state.

(b) Clearly, (A.2) is a "derived" assumption, in the sense that it has been made on the transformation function rather than on its "primitives," the production and lending functions. It is satisfied, for example, if the production function under the best state b is increasing, concave, and differentiable, with $\lim_{k \to \infty} g'(k, b) < 1 + r$, and if $q < r$.

The definition of f, together with (A.1), permits it to enjoy the following additional properties, which are easily verified.

(P.1) *For each $k \in \mathbb{R}^+$, $f(k, \cdot)$ is nondecreasing (hence measurable on $(I, \mathscr{B})$), and is continuous at a.*

(P.2) *For each $z \in I$, the function $f(k, z) - k$ is nondecreasing in k, and is continuous in k.*

The initial stock is given by $K > 0$. All transformations (production, credit operations) take one period of time; consumption commences at date 1. Denoting output (resp. capital) in period t by X_t (resp. K_t), one has

$$X_{t+1} = f(K_t, Z_{t+1}), \qquad t \geqslant 0 \tag{1}$$

with K_0 defined in (3) below.

Denote accumulated indebtedness in period t by D_t, and fresh borrowings by L_t. Define a borrowing scheme as a sequence $\langle L_t \rangle_0^\infty$ of $\mathscr{F}_t$-measurable functions on Ω. A negative L_t is a repayment of outstanding debt. For some borrowing scheme, and given consumption sequence $\langle c_t \rangle_1^\infty$, one has

$$0 \leqslant D_0 = L_0; \qquad D_{t+1} = D_t(1 + r) + L_{t+1} \geqslant 0, \qquad t \geqslant 0 \tag{2}$$

$$0 \leqslant K_0 = K + L_0; \qquad K_t = X_t - c_t + L_t \geqslant 0, \qquad t \geqslant 1. \tag{3}$$

The inequalities in (2) and (3), relating to the nonnegativity of debt and capital, impose restrictions which every borrowing scheme must satisfy. In

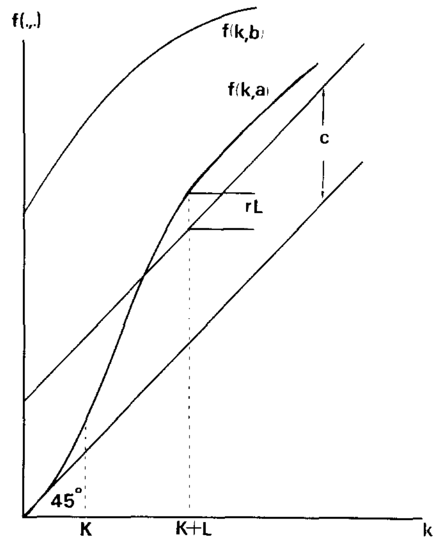

FIGURE 1

particular, (3) dictates that a loan must always be taken in the event of a consumption shortfall.

Define for $t \geqslant 0$, $B_t = D_t - K_t$. This is the measure of indebtedness *net* of existing capital stock, the true indicator of "inability to repay," since in this model the capital stock may be used for debt repayments[6] and must therefore be subtracted from outstanding indebtedness.

For each $c \in \mathbb{R}^+$, let $\mathscr{C}(c) = \{\langle c_t \rangle_{t=1}^{\infty} \colon \inf_{t>1} c_t \geqslant c\}$.

This is the set of all consumption paths fulfilling target requirements. A target c is said to be *unattainable* if $\limsup_{t \to \infty} EB_t = +\infty$ for all $\langle c_t \rangle \in \mathscr{C}(c)$. Otherwise, it is *attainable*.

Characterization of Attainable Targets

I now address the question: Which targets are attainable? Clearly, if the target is so modest that it is producible by the transformation function *under the worst state*, while leaving the initial capital stock intact, it is attainable! In other words, the condition $c \leqslant f(K, a) - K$ is a *sufficient* condition for the attainability of c. However, this condition may be relaxed somewhat. Borrowing for investment in production permits the agent to take advantage of regions in the transformation function providing a large surplus even under the worst state. Figure 1 illustrates this.

[6] In general, if there is some "basic" capital which is neither directly consumed nor used for debt repayments (infrastructure), this may be allowed for (note that the technology need *not* satisfy $g(0) = 0$). K_t may then be interpreted as the excess of total capital over the "basic" stock, at time t.

In this example, the condition $f(K, a) - K \geqslant c$ is *not* met, yet the target is attainable. A production loan of L in the very first period yields an output under the "worst" state (a), which is sufficient to meet c, replenish the new capital stock of $K + L$, and pay at least the interest on the loan. In this case, it is immediate that $D_t \leqslant L$ for $t \geqslant 0$, in particular, that EB_t is bounded above. If, in addition, the stochastic process $\langle Z_t \rangle_1^\infty$ admits of $\delta > 0$ such that $P[Z_t \geqslant a + \delta \text{ i.o.}] = 1$ (satisfied in all but some trivial cases), the debt is repayable in finite time with probability one. Therefore, the problem of escalating expected indebtedness no longer arises in this situation.

We formalize the condition under which this example works as

CONDITION P (Attainability of Targets under Production/Consumption Borrowing): *The target c satisfies*

$$c \leqslant f(k, a) - k - r(k - K) \qquad \textit{for some } k \geqslant K.$$

Remark. When this condition is met, there is *some* capital stock (not less than K), such that enough output is producible from it *in the worst state*, to meet the given target, repay the interest on the loan incurred, and maintain this capital stock.

A natural question that arises is: What happens if the condition outlined above is *not* met? After all, the condition represents an assumption on the *worst* possible production function, and is therefore unlikely to be met in a large number of cases (see, for example, Section IV).

Theorem 3.1 addresses this issue.

THEOREM 3.1. *Under a general borrowing scheme, and* (A.0)–(A.2):

(i) *If condition P is met, the target c is attainable.*

(ii) *If condition P is violated, the target c is unattainable, and* $\lim_{t \to \infty} EB_t = \infty$ *for all* $\langle c_t \rangle \in \mathscr{C}(c)$.

Proof. Part (i) is easily verified by choosing $c_t = c$ for all $t \geqslant 1$, and adapting the discussion above to a formal proof. I establish part (ii). First, I show that there exists $(n, \gamma) \gg 0$ such that $c \geqslant f(k, a + \gamma) - k - r(k - K) + \eta$ for all $k \geqslant K$. Denote $f(K, a) - (1 + r)K$ by N. It is easy to verify that $(1 + r)k - f(k, b) \to \infty$ as $k \to \infty$. So there exists $\underset{\sim}{K} > K$ such that $f(k, b) - (1 + r)k \leqslant N$ for all $k \geqslant \underset{\sim}{K}$. Using this, (P.1), and the continuity of f in k, it follows that the maximum of $f(k, z) - (1 + r)k$, on $[K, \infty)$, always exists, and is attained by some $k \in [K, \underset{\sim}{K}]$, *for each* $z \in I$. Since Condition P is violated,

$$c > \max_{k \in [K, \underset{\sim}{K}]} \{f(k, a) - (1 + r)k\} + rK.$$

Since $f(k, \cdot)$ is continuous at a, it follows from the continuity of the maximum (on the compact set $[K, \underset{\sim}{K}]$) that there exists $\gamma > 0$ such that

$$c > \max_{k \in [K, \underset{\sim}{K}]} \{f(k, a+\gamma) - (1+r)k\} + rK.$$

Finally, there exists $\eta > 0$ such that

$$c \geqslant \max_{k \in [K, \underset{\sim}{K}]} \{f(k, a+\gamma) - (1+r)k\} + rK + \eta.$$

Rewriting this, one obtains $c \geqslant f(k, a+\gamma) - k - r(k-K) + \eta$ for all $k \geqslant K$, where $(\gamma, \eta) \gg 0$.

Define $A \equiv [a, a+\gamma]^T \prod_{T+1}^{\infty} I$. Then $A \in \mathscr{F}$. Let $A_t \equiv \{Z_t \in [a, a+\gamma]\}$. Then $P(A) = P(\bigcap_1^T A_t)$. Note that $P(A_1 \cap A_2) = \int_{A_1} P(A_2/\mathscr{F}_1)\, dP > 0$, since $P(A_1) > 0$ and (A.0) holds. Using (A.0) repeatedly, it follows now from a simple induction argument that $P(A) \equiv \theta > 0$.

Now I claim that there exists $S < \infty$ with $f(k, b) - (1+r)k \leqslant S$ for all $k \geqslant 0$. This is shown by establishing that $\limsup_{k \to \infty} f(k, b)/k < 1 + r$. Consider any convergent sequence $f(k_n, b)/k_n$, as $k_n \to \infty$. Then $f(k_n, b) \to \infty$ by (P.2). The corresponding sequence $f'(k_n, b)$ is bounded, hence has a convergent subsequence $f'(k_{n_r}, b)$. It suffices to study the limit of $f(k_{n_r}, b)/k_{n_r}$. By L'Hospital's Rule, this is simply equal to $\lim_{r \to \infty} f'(k_{n_r}, b) < 1 + r$. Thus the limits of all convergent subsequences of $f(k, b)/k$ are less than $(1+r)$. This establishes the claim.

Pick $s \in (0, 1)$ and $m > 0$ such that $\limsup_{k \to \infty} f'(k, b) < 1 + m < 1 + rs$, and define for some $\hat{M} > 0$, $M \equiv \max[\hat{M}, (S-c)/(1-s)r]$. Finally, let T be the smallest positive integer not exceeding $(M+K)/\eta$.

LEMMA 3.1. *For all* $\omega \in A$, $B_T(\omega) \geqslant M$.

Proof. Let $\omega \in A$ be given. Suppose that $B_t(\omega) \geqslant B_0(\omega) + t\eta$ for some $t \geqslant 0$. Distinguish between two cases.

Case (i). $K_t(\omega) \leqslant K$.

Then $B_{t+1}(\omega) = D_{t+1}(\omega) - K_{t+1}(\omega) = (1+r)\, D_t(\omega) - f(K_t(\omega), Z_{t+1}(\omega)) + c_{t+1} \geqslant [D_t(\omega) - K_t(\omega)] + [c - \{f(K_t(\omega), a+\gamma) - K_t(\omega)\}] \geqslant B_t(\omega) + [c - \{f(K, a+\gamma) - K\}] \geqslant B_t(\omega) + \eta \geqslant B_0(\omega) + (t+1)\eta$.

Case (ii). $K_t(\omega) > K$.

Then $B_{t+1}(\omega) = D_{t+1}(\omega) - K_{t+1}(\omega) = (1+r)\, D_t(\omega) - f(K_t(\omega), Z_{t+1}(\omega)) + c_{t+1} \geqslant (1+r)\, D_t(\omega) - f(K_t(\omega), a+\gamma) + c \geqslant (1+r)\, D_t(\omega) - K_t(\omega) - r[K_t(\omega) - K] + \eta = (1+r)\, D_t(\omega) - (1+r)\, K_t(\omega) + rK + \eta \geqslant (1+r)\, D_t(\omega) - (1+r)[D_t(\omega) - B_0(\omega) - t\eta] + rK + \eta = B_0(\omega) + (t+1)\eta + r[B_0(\omega) + K] \geqslant B_0(\omega) + (t+1)\eta$, since $B_0(\omega) + K = D_0(\omega) - K_0(\omega) + K = L_0(\omega) - K_0(\omega) + K \geqslant 0$.

Using Cases (i) and (ii) repeatedly, and noting that $B_t(\omega) \geqslant B_0(\omega) + t\eta$ holds for $t = 0$, one obtains $B_T(\omega) \geqslant B_0(\omega) + T\eta \geqslant T\eta - K \geqslant M$. Q.E.D.

LEMMA 3.2. *Suppose that $B_{\hat{T}}(\omega) \geqslant M$ for some $\omega \in \Omega$, and positive integer $\hat{T}$. Then $B_t(\omega) \geqslant M(1 + rs)^{t-\hat{T}}$ for all $t \geqslant \hat{T}$.*

Proof. For the given $\omega \in \Omega$, suppose that $B_t(\omega) \geqslant M$ for some $t \geqslant \hat{T}$. Then $B_{t+1}(\omega) = (1 + r)\, D_t(\omega) - f(K_t(\omega), Z_{t+1}(\omega)) + c_{t+1} \geqslant (1 + r)(D_t(\omega) - K_t(\omega)) + c - S \geqslant (1 + rs)\, B_t(\omega) + (1 - s)\, rM + c - S \geqslant (1 + rs)\, B_t(\omega)$. Using this last inequality repeatedly, one obtains, for $t \geqslant \hat{T}$, $B_t(\omega) \geqslant M(1 + rs)^{t-\hat{T}}$. Q.E.D.

LEMMA 3.3. *There exists $B \in \mathbb{R}$ such that $B_t(\omega) \geqslant B(1 + m)^t$ for all $t \geqslant 0$ and $\omega \in \Omega$.*

Proof. Clearly, there is $\bar{K}$ such that $f(K, b) \leqslant (1 + m)K$ for all $K \geqslant \bar{K}$. If $K_t(\omega) \leqslant \bar{K}$, $B_{t+1}(\omega) \geqslant -f(K_t(\omega), b) \geqslant -f(\bar{K}, b)$. If $K_t(\omega) \geqslant \bar{K}$, $f(K_t(\omega), b)/K_t(\omega) \leqslant 1 + m < 1 + r$, so that $B_{t+1}(\omega) = D_{t+1}(\omega) - K_{t+1}(\omega) \geqslant (1 + r)\ D_t(\omega) - f(K_t(\omega), b) + c \geqslant (1 + r)\ D_t(\omega) - [f(K_t(\omega), b)/K_t(\omega)]\ K_t(\omega) \geqslant (1 + m)(D_t(\omega) - K_t(\omega)) = (1 + m)\ B_t(\omega)$. Thus defining $B = \min(-K, -f(\bar{K}, b))$ and noting that $B_0(\omega) \geqslant -K$, the result follows. Q.E.D.

By Lemma 3.3, the random variables B_t are bounded below for each t, so that EB_t exists for all $t \geqslant 0$. Combining Lemmas (3.1)–(3.3), one obtains, for $t \geqslant T$,

$$EB_t = \int_A B_t\, dP + \int_{\Omega \setminus A} B_t\, dP \geqslant \theta M(1 + rs)^{t-T} + (1 - \theta)\, B(1 + m)^t.$$

Since $rs > m$, this proves the theorem. Q.E.D.

Implications of Theorem 3.1. The theorem states that if unattainability of a proposed consumption target is defined by growing expected indebtedness associated with that target, the attainable targets are in fact surprisingly modest. Either the target must be so small as to necessitate at most *one* loan, made in the first period, then repaid, or it is unattainable in the sense of exploding expected indebtedness.

Transience of the Indebtedness Process

I now turn to the second question: Does a particular economic agent become either infinitely rich, or infinitely indebted, with probability one? This question is of some interest in its own right, because an affirmative answer could imply an inherent tendency to growing, unbounded inequality among *identical* units, within the framework of the present model.

I shall address this question in a simpler context than the one studied above. In this framework, which I shall term the *consumption borrowing model*, loans are taken if and only if the output in any period falls short of the consumption target. I shall, therefore, work under the assumption.

(A.3) *As an accounting device, past debt must be paid off, to the extent permitted by the current stock of the good. Fresh borrowing then occurs if and only if the stock net of debt repayments falls short of the given consumption for that period.*

Remark. Under (A.3), a result analogous to that of Theorem 3.1 may be established. In this case, a necessary and sufficient condition for attainability is $c \leqslant f(K, a) - K$, i.e., attainable consumption targets are precisely those necessitating *no* borrowing at all.

In the consumption borrowing model, $D_t > 0$ if and only if $K_t = 0$. Thus, for any exogenous $\langle c_t \rangle_1^\infty$, using (1)–(3), and (A.3),

$$D_t = \max(B_t, 0), \qquad t \geqslant 0 \tag{4}$$

$$K_t = -\min(B_t, 0), \qquad t \geqslant 0 \tag{5}$$

$$B_t + X_t = c_t + (1+r) D_{t-1}, \qquad t \geqslant 1. \tag{6}$$

I shall concentrate on consumption paths with $c_t = c$, for $t \geqslant 0$.

First, I strengthen (A.0) to

(A.0′) *There exists $\beta > 0$ such that for each $\gamma \in (0, \beta]$, there is $\theta > 0$ with $P[Z_{t+1} \in [a, a+\gamma] / \mathscr{F}_t] \geqslant \theta$ a.s. for all $t \geqslant 0$.*

Assumptions (A.4) and (A.5) below put more structure on the stochastic process $\langle Z_t \rangle_1^\infty$ and the transformation function.

(A.4) *There exists $\eta > 0$ such that $P[Z_t \geqslant a + \eta$ i.o.$] = 1$.*

(A.5) *For each $k \in \mathbb{R}^+$, $f(k, \cdot)$ is strictly increasing at a.*[7]

Remarks on the Assumptions. Like (A.0), (A.0′) asserts (loosely speaking) that bad states can occur at every date, irrespective of the past. It assumes, in addition to (A.0), that the probability of such occurrences is bounded away from zero. This assumption holds for a wide variety of cases; both examples in the remarks following (A.0) satisfy (A.0′). Assumption (A.4) is relatively harmless, satisfied in most interesting cases. Assumption (A.5) simply asserts that a change in the "state of nature" in the neighborhood of the worst state does indeed have some effect on production.

In this framework, one can establish

THEOREM 3.2. *Suppose that the borrowing rate of interest r is positive. Under* (A.0′), (A.1), *and* (A.3)–(A.5), *the process $\langle B_t \rangle_0^\infty$ satisfies*

$$P[\lim_{t \to \infty} B_t = +\infty] + P[\lim_{t \to \infty} B_t = -\infty] = 1.$$

[7] In other words there exists $\delta > 0$ such that $f(k, z)$ is increasing on $[a, a+\delta]$ for each $k \in \mathbb{R}^+$.

The proof of Theorem 3.2 depends on two lemmas, the proofs of which I omit for lack of space.[8]

LEMMA 3.4. *Under* (A.1), *and* (A.3)–(A.5), *if there exists* $\bar{K} \geqslant 0$ *such that* $c \leqslant f(\bar{K}, a) - \bar{K}$, *then* $\{K_t \geqslant \bar{K}$ *for some* $t \geqslant 0\} = \{\lim_{t\to\infty} K_t = \infty\}$ *a.s.*

LEMMA 3.5. *Let* H, N *be two Borel sets of* $\mathbb{R}$, *and suppose that there exists* $\xi > 0$ *such that* $P[B_\tau \in N$ *for at least one* $\tau > t/B_t, ..., B_0] \geqslant \xi$ *a.s. on* $\{B_t \in H\}$. *Then*

$$\{B_t \in H \text{ i.o.}\} \subset \{B_t \in N \text{ i.o.}\} \qquad a.s.$$

Proof of Theorem 3.2. For some $s \in (0, 1)$ and $\hat{M} > 0$, define $M \equiv \max[\{f(0, b) - c\}/(1-s)r, \hat{M}]$. Note that $\{B_t \geqslant M$ for some $t \geqslant 0\} = \{B_t \to \infty$ as $t \to \infty\}$. To see this, suppose that for some $\omega \in \Omega$, and $T \geqslant 0$, $B_t(\omega) \geqslant M$ for some $t \geqslant T$. Then $B_{t+1}(\omega) = c - X_{t+1}(\omega) + (1+r)\, B_t(\omega) \geqslant (1+r)\, B_t(\omega) - [f(0, b) - c] \geqslant (1+r)\, B_t(\omega) - (1-s)r\, M \geqslant (1+r)\, B_t(\omega) - (1-s)r\; B_t(\omega) = (1+s)\; B_t(\omega)$. Thus if $B_T(\omega) \geqslant M$, this argument yields $B_t(\omega) \geqslant M(1+rs)^{t-T}$ for $t \geqslant T$, so that $B_t(\omega) \to \infty$ as $t \to \infty$.

Hence if the theorem is not true, there exists $A \in \mathscr{F}$ with $P(A) > 0$, and on A

$$-K^* \leqslant \liminf_{t\to\infty} B_t \leqslant \limsup_{t\to\infty} B_t \leqslant M \tag{7}$$

where $K^* = \min\{K \mid f(K, a) - K \geqslant c\}$, or defined as some positive number with $B_t \in [-K^* + \mu, M]$ i.o., for some $\mu > 0$, with positive probability, if no such K exists. The first inequality in (7) follows from Lemma 3.4, the last from the argument above.

I claim, now, that in the case where $K^* = \min\{K \mid f(K, a) - K \geqslant c\}$, there exists $\mu > 0$ with $P[B_t \in H \text{ i.o}] > 0$, where $H \equiv [-K^* + \mu, M]$. If not, it is easy to see that $P[\lim B_t = -K^*] = P(A)$. Passing to the limit in (4)–(6), $K^* + c^* = \lim_{t\to\infty} f(K^*, Z_t(\omega))$ for all ω such that $B_t(\omega) \to -K^*$. But since $f(K^*, a) \geqslant K^* + c^*$, $f(K^*, a) \geqslant \lim_{t\to\infty} f(K^*, Z_t(\omega))$. Using (A.5), $\lim_t Z_t = a$ with positive probability, contradicting (A.4). Hence the claim is true. When there is no K with $f(K, a) - K \geqslant c$, the set H can be constructed by definition of K^*.

Thus define $H \equiv [-K^* + \mu, M]$ for some $\mu > 0$ so that $P[B_t \in H \text{ i.o.}] > 0$, and define $N \equiv [M, \infty)$. By definition of K^*, it follows that defining $\underset{\sim}{K} \equiv -\min(-K^* + \mu, 0)$,

$$c > f(\underset{\sim}{K}, a) - \underset{\sim}{K},$$

so that there exists $(\delta, \gamma) \gg 0$ with $\gamma \leqslant \beta$ (given in (A.0′)) and

$$c \geqslant f(\underset{\sim}{K}, a + \gamma) - \underset{\sim}{K} + \delta. \tag{8}$$

[8] See Ray [13].

Define T as the smallest integer not less than $(M+K^*-\mu)/\delta$, and $A_t \equiv \{Z_t \in [a, a+\gamma]\}$. Then it is easy to check that $\{B_t \in H\} \cap A_{t+1} \cap ,..., \cap A_{t+T} \subset \{B_{t+T} \in N\}$. Hence on $\{B_t \in H\}$,

$$\begin{aligned} &P\{B_\tau \in N \text{ for some } \tau > t/B_0,..., B_t\} \\ &\quad \underset{\text{a.s.}}{\geqslant} P\{B_{t+T} \in N/B_0,..., B_t\} \\ &\quad \underset{\text{a.s.}}{\geqslant} P\{B_{t+T} \in N/\mathscr{F}_t\} \underset{\text{a.s.}}{\geqslant} P(A_{t+1} \cap ,..., \cap A_{t+T}/\mathscr{F}_t). \end{aligned} \tag{9}$$

Now let $P_t(\cdot)$ be the regular conditional probability version of $P(\cdot/\mathscr{F}_t)$, for $t \geqslant 0$ (this exists in the framework here; see, for example, Ash [1, Theorem 6.6.5 and remarks following Theorem 6.6.6]). Then

$$\begin{aligned} P\left(\bigcap_{t+1}^{t+T} A_s/\mathscr{F}_t\right) &\underset{\text{a.s.}}{=} P_t\left(\bigcap_{t+1}^{t+T} A_s\right) = \int_{A_{t+1}} P_{t+1}\left(\bigcap_{t+2}^{t+T} A_s\right) P_t(d\omega) \\ &= \cdots = \int_{A_{t+1}} \cdots \int_{A_{t+T-1}} P_{t+T-1}(A_{t+T})\, P_{t+T-2}(d\omega) \cdots P_t(d\omega) \\ &\underset{\text{a.s.}}{\geqslant} \theta \int_{A_{t+1}} \cdots \int_{A_{t+T-2}} P_{t+T-2}(A_{t+T-1})\, P_{t+T-3}(d\omega) \cdots P_t(d\omega) \\ &\underset{\text{a.s.}}{\geqslant} \theta^T = \xi > 0 \qquad \text{(these steps use (A.0') repeatedly).}^9 \end{aligned} \tag{10}$$

Combining (9) and (10), one gets

$$P\{B_\tau \in N \text{ for at least one } \tau > t/B_0,..., B_t\} \geqslant \xi > 0 \qquad \text{a.s.}$$

for all $t \geqslant 0$, on $\{B_t \in H\}$. By Lemma 3.5, $\{B_t \in H \text{ i.o.}\} \subset \{B_t \in N \text{ i.o}\}$ a.s. But $\{B_t \in N \text{ for some } t \geqslant 0\} \subset \{\lim_{t \to \infty} B_t = +\infty\}$ a.s., by the argument in the beginning of this proof. Hence $P\{B_t \in H \text{ i.o}\} = 0$, a contradiction to an earlier derivation that $P\{B_t \in H \text{ i.o.}\} > 0$. Q.E.D.

Remarks on Theorem 3.2. (a) The result is independent of (A.2), which is essentially the assumption of credit market imperfections. All that is required is that r, the borrowing rate of interest, be positive.

(b) In a large class of situations (see the example in Section IV), both $P(\lim_t B_t = \infty)$ and $P(\lim_t B_t = -\infty)$ are positive. This result therefore leads to a situation where a number of independent, initially identical agents experience increasing, unbounded inequality amongst themselves.

[9] These steps are written as if $T \geqslant 3$, but the result clearly holds for all $T \geqslant 1$.

IV. An Example: The Phenomenon of Rural Indebtedness[10]

In agrarian economies such as in India, high and growing peasant indebtedness, apart from creating widespread misery among a large class of people, may give rise to a stagnating "semifeudalist" sector.[11] Although the peasant is not tied to the landlord, or forced to relinquish his produced surplus by the "legal" contract of a feudalist society, the obligation to pay off debt effectively constitutes a semifeudal tie. The situation is exacerbated by the fact the indebtedness may be inherited and hence passes on from generation to generation.[12]

There are two issues here; the first dealing with the economic mechanism generating large-scale peasant indebtedness, the second addressing the effects of such indebtedness. In the main, the literature has employed deterministic models to analyze the second of these issues, attributing the existence and growth of rural indebtedness to a combination of unproductive techniques of agricultural production, high rents and periodic spurts of consumption. The framework developed in Section III may be used to study this particular issue in some detail.

In these rural economies, the phenomenon of credit market imperfection appears to be an empirical fact.[13] I take such imperfection as a primitive assumption of the analysis.[14]

Environmental uncertainty is captured by the product space $(\Omega, \mathscr{F}, P)$ introduced in Section III. The probability measure is taken to satisfy

(A.6) *The coordinate random variables* $\langle Z_t \rangle_1^\infty$ *are independent and identically distributed.*

Remark. Z_t is to be interpreted as the random harvest of the peasant at date t, net of rent payments.

Let Z be a random variable with the common distribution of the Z_t's. By (A.6), $P = \mu^\infty$, where μ is a probability on $[I, \mathscr{B}]$. Assume further

(A.7) μ *is nondegenerate and satisfies*: *there exists* $\delta > 0$ *such that for all* $0 < \gamma \leqslant \delta$, $\mu[a, a+\gamma] > 0$ *and* $\mu[b-\gamma, b] > 0$.

[10] Some additional results in this special case are presented in Ray [13]. For further discussion of the issues involved, see Gangopadhyay and Ray [8]. In this section, I omit all proofs.

[11] See, for example, the discussion in Bhaduri [2, 3].

[12] See, for example, Nagesh [12]. This phenomenon partially justifies the use of a large (infinite!) horizon model to analyze the problem.

[13] Bhaduri [2] reports rates of interest on borrowing ranging from 50% to 200% in some villages in India.

[14] See, however, Bhaduri [3] for an interesting analysis of the formation of various interest rates in rural economies.

Consider the following Markov process on $(\Omega, \mathscr{F}, P)$:

$$B_{t+1} = p_t B_t + c - Z_{t+1}, \qquad B_0 \text{ given}$$
$$p_t = \begin{cases} 1+r & \text{if } B_t > 0 \\ 1+q & \text{if } B_t \leqslant 0 \end{cases}; \qquad (r,q) \geqslant 0. \tag{11}$$

To interpret (11), regard B_t as the indebtedness of the tenant (typically to the landlord–moneylender) if it is positive and as his wealth if it is not positive. At every date, fresh borrowings occur if and only if the harvest (Z_t) plus previous wealth (if any) falls below consumption requirements (c), with the extent of borrowing equalling the shortfall. Typically, the rate of interest on borrowing (r) exceeds that on saving (q). Because of inaccess to organized credit, loans are obtained at an exorbitant rate of interest. On the other hand, wealth is simply stocked in kind or cash, with q typically zero or even negative. The condition $r > q$ is a more widespread feature. One observes this in the most developed credit markets.

Assume $B_0 \leqslant 0$; i.e., *the absence of indebtedness to start with.* Recall that a is the minimum harvest, and consider the condition

$$a \geqslant c + qB_0. \tag{12}$$

When (12) holds, the worst harvest combined with interest from initial wealth suffices to meet consumption needs. In the situation under consideration, (12) does not seem to be realistic, since q is usually small or even zero, and a is typically less than c. In any case, a situation in which (12) holds would never display indebtedness, and is thus not the central object of study here.

Observe that this is a special case of the model developed in Section III. Moreover, condition (12) is precisely the analogue of Condition P.

I now summarize some facts about the process (11) in the form of the following propositions.

PROPOSITION 4.1. *If* (12) *holds,* $\lim_{t\to\infty} B_t = -\infty$ *a.s.*

Remark. In view of the discussion following (12), this is hardly surprising.

PROPOSITION 4.2. *Suppose* $r \geqslant q = 0$ *and* $EZ < c$. *Then* $\lim_{t\to\infty} B_t = +\infty$ *a.s.*

Remarks. While $r \geqslant q = 0$ is of relevance for the situation at hand, it is not applicable to more general situations, where one would expect $q > 0$. Moreover, the "unproductive" technology summarized by $EZ < c$, while an appropriate one for agrarian economies a few decades ago, is perhaps no longer a valid characterization. Rents, while still high, are lower than before, so that net harvests are higher, and some technological advances have been made, so that while one would expect the worst harvest to fall below c, it is perhaps inappropriate to assume that $EZ < c$. The next proposition demonstrates that if either $EZ < c$ or $q = 0$ is relaxed, the "almost sure" statement of Proposition 4.2 no longer holds.

PROPOSITION 4.3. *Suppose that* (12) *is violated.*

(i) *If* $c < b$ *and* $(r, q) \gg 0$, *then* $P(\lim_{t\to\infty} B_t = +\infty) > 0$ *and* $P(\lim_t B_t = -\infty) > 0$.

(ii) *If* $r > q = 0$, *but* $EZ > c$, *then, too,* $P(\lim_{t\to\infty} B_t = +\infty) > 0$ *and* $P(\lim_{t\to\infty} B_t = -\infty) > 0$.

Remark. Proposition 4.3 uses assumptions which appear to be most relevant. While $c > a$, a likely situation, $EZ > c$ is an appropriate recognition of the productiveness of technology "on average." In such situations, however, a definitive result of the "almost sure" type is unlikely to obtain, and is in fact impossible if (12) is violated and $r > 0$. Theorem 4.1 provides a result on the sequence of *expected values* of B_t.

THEOREM 4.1. *Suppose* (12) *is violated, and* $r > q > 0$. *Then* $\lim_{t\to\infty} EB_t = +\infty$.

This follows immediately from Theorem 3.1.

Remarks. (1) The statement in Theorem 4.1 is *independent* of the actual value of the probability of a bad harvest, of the size of the interest differential, and of whether the technology is "productive" or not. Viewed in this light, it is a strong result indicating that whenever there is *some* "reason" for indebtedness, i.e., (12) is violated, expected indebtedness accumulates to plus infinity because of credit market imperfections.

(2) One may question why loans are still advanced when indebtedness is high, and even when there is no possibility of debt repayment. I shall not address this issue here; for a discussion, one is referred to the analysis in Gangopadhyay and Ray [8].

One also has

THEOREM 4.2. *When* $r > 0$, $P[\lim_{t\to\infty} B_t = +\infty] + P[\lim_{t\to\infty} B_t = -\infty] = 1$.

This follows as a special case of Theorem 3.2.

Remarks. The theorem states that a particular peasant (or a peasant family) will (almost surely) become either infinitely rich or infinitely indebted over time. This inherent tendency (quite independent of the phenomenon of credit market imperfection), coupled with Proposition 4.3, suggests that there is a built-in mechanism generating inequality over time in situations where this model is applicable. The actual values of the two probabilities will determine the proportion of peasants who are to come under the semifeudal tie in the long run, and the proportion who become increasingly wealthy.

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