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Constrained Egalitarian Allocations*

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This paper proposes a constrained egalitarian solution concept for TU games which combines commitment for egalitarianism and promotion of individual interests in a consistent manner. The paper shows that the set of constrained egalitarian allocations is nonempty for weakly superadditive games. The solution is "almost" unique if the desirability relation between players is complete. *Journal of Economic Literature* Classification Number 026. © 1991 Academic Press, Inc.

1. INTRODUCTION

Consider a society, represented as a coalition of n people. Each person has subjective preferences which define his or her *personal* utility function. The n-tuple of utility functions, in conjunction with the opportunities open to the grand coalition and intermediate coalitions, gives rise to a transferable-utility cooperative game in characteristic function form. The standard task of cooperative game theory is to construct a *solution concept* which will predict for every cooperative game the payoff vector(s) representing reasonable outcome(s), that is, outcomes that "rational" utility-maximizing agents may agree upon.

Suppose, however, that the society consists of individuals who subscribe to equality as a desirable *social* end. Thus, agents are not *rational fools*,[1] and do not necessarily choose actions to maximize individual util-

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[1] See Sen (1977). Prior to Sen, Harsanyi (1953) had also made the important distinction between a person's "ethical" and "subjective" preferences, the former being what an individual would prefer on the basis of impersonal social considerations alone, while the latter expresses what he or she actually prefers.

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ity, since moral reasoning (such as commitment[2] to equality) also influences individual actions. A natural question is: given a tension between social values and individual interests, how will society choose rules or solution concepts?

Note that if primacy is given to either commitment or individual interest, then the problem has an easy solution. For suppose individuals place commitment to equality ahead of individual interest. Then, the extreme egalitarian rule that prescribes equal division of the aggregate worth of the society, *irrespective* of the worths of intermediate coalitions, is the natural candidate. However, such a rule will inevitably run into problems if individuals are not willing to accord primacy to egalitarian principles over individual interest since some coalition may receive an allocation whose aggregate value falls short of its worth. The coalition may then "block" the equal division rule.

Conversely, if primacy is given to individual interests, then one may suggest an allocation rule which yields the "most equal" allocation (according to an agreed measure of inequality) from within the *core* of the cooperative game. However, this rule treats coalitions asymmetrically. For, suppose an allocation x *not* in the core is proposed. Then, a specific coalition, say S, blocks it since it can "arrange" a feasible allocation, say y, for itself so as to make everyone in S better off. But S itself is a potential subsociety, and individuals in S also have a commitment to egalitarian principles. In particular, if S forms, then it has to choose the most equal allocation within the core of the game restricted to S. In other words, y may not be achievable, so that the threat to block x is not "credible."[3]

In Dutta and Ray (1989), we proposed an egalitarian solution concept for transferable utility games which married commitment for egalitarianism and promotion of individual interests in a consistent manner. Broadly the social ethic is captured in the design of social *rules,* while individual behavior is expected to be "selfish" within the context of these rules. More specifically, the objective was to identify allocations for the grand coalition that are "egalitarian" and "unblocked." By the first of these two terms, we meant that the grand coalition must choose the *Lorenz-maximal* elements of the set of allocations that are feasible for it.[4] By the

[2] Indeed, one way in which Sen (1977) defines commitment "is in terms of a person choosing an act that he believes will yield a lower level of personal welfare to him than an alternative that is also available to him."

[3] Thus, the problem is similar to that of subgame perfection in noncooperative games; one must know what a coalition can *credibly* do once it deviates.

[4] We chose the Lorenz criterion as our (partial) ordering of unequal allocations because this criterion is widely accepted as embodying a set of minimal ethical judgements that "should" be made in carrying out inequality comparisons. Additional ethical judgements needed to complete the ordering are not so widely agreed upon. On this, see Atkinson (1970), Dasgupta *et al.* (1972), and Sen (1973).

second, we meant a notion of blocking that is different from the standard concept, in that two restrictions are imposed on a blocking coalition. First, a coalition can only block using allocations that are in turn "unblocked" by any subcoalition. Second, among these allocations, a coalition can only use the Lorenz-maximal ones, because each member of the coalition subscribes to the social ethic of egalitarianism. Therefore, the commitment to egalitarianism was applied in a *consistent* manner across coalitions. The construction of the egalitarian solution has, consequently, a recursive structure.

The following result was proved in Dutta and Ray (1989). Suppose that the concept of blocking is "weak," in the sense that every member of the blocking coalition is at least as well off as before, and at least one member is strictly better off. Then, for each transferable utility game, there can exist *at most one* allocation for the grand coalition satisfying the above properties. This allocation, which we called the *egalitarian allocation,* is not necessarily a core allocation, even though it has the stability properties we have described above. However, for *convex* games, the egalitarian allocation exists, lies in the core, and Lorenz-dominates every other core allocation. In this paper, we refer to the egalitarian allocation as the *W-constrained egalitarian allocation*, or WCEA for short.[5] The purpose of this paper is to examine a parallel concept, called *S-constrained egalitarian allocations* (SCEA). The construction is identical, except that the concept of blocking we use here requires *every* member of the blocking coalition to be *strictly* better off.

At first blush, this slight modification does not appear to have any serious implications for our solution concept. However, this is not the case. Indeed, we show that the new set of allocations that are generated are markedly different from the old. The reader who wishes to obtain some feeling for this right away can turn to Examples 1 and 2 in Section 3 of the paper.

Indeed, the two sets of allocations "rarely" coincide. We provide a complete characterization of when they do in fact coincide (see Theorem 7). The condition imposed on the game for coincidence to occur implies a very small class of possible games. Moreover, not only do the two sets of allocations fail to coincide, they have very different qualitative properties.

To begin with, S-constrained egalitarian allocations exist under an extremely mild condition on the game, which we call *weak superadditivity*.[6] In contrast, the W-constrained egalitarian allocation may not exist even for *balanced* games, although as we have mentioned earlier, existence is

[5] We use a different terminology in this paper, as the name "egalitarian allocation" has been used before for a different solution concept (see Kalai and Samet, 1985).

[6] This simply states that the worth of the grand coalition is at least as great as the sum of the worths of any set of subcoalitions that partition the grand coalition.

guaranteed for convex games. The fact that S-constrained egalitarian allocations exist under very mild assumptions significantly widens the potential class of applications.

On the other hand, S-constrained egalitarian allocations do not possess the extremely appealing uniqueness property that was established for their "weak" counterpart. However, we identify classes of games in which all S-constrained egalitarian allocations have the same *Lorenz curve*. These classes include the class of the three-player games and games in which the *desirability relation*[7] between players is complete. However, that the concept is in general multivalued is demonstrated by means of a four-player game in Section 3.

Of course, the two concepts are not completely unrelated, and a discussion of the relationship may be found in Section 5 of the paper. In general, there is always an S-constrained egalitarian allocation which Lorenz-dominates the W-constrained egalitarian allocation. For convex games and all four-player games, *every* S-constrained egalitarian allocation Lorenz-dominates the weak egalitarian allocation. But this property is not true in general, and an example of a five-player game is provided to illustrate this.

Another property of S-constrained egalitarian allocations is that whenever they are different from their weak counterpart, they also fail to lie in the core. Yet they are stable, in the sense that no coalition, using an S-constrained egalitarian allocation of its own, can deviate profitably from the S-constrained egalitarian allocation for the grand coalition.

2. Notation

For any nonempty subset S of $\{1, 2, \ldots, n\}$, denote by $|S|$ the cardinality of S. We write $\mathfrak{R}^S$ for $\mathfrak{R}^{|S|}$($|S|$-dimensional Euclidean space), where the coordinates are numbered according to the indices present in S. For two vectors x and y in $\mathfrak{R}^S$, we write $x = y$ if $x_i = y_i$ for all $i \in S$, $x > y$ if $x \neq y$ and $x_i \geq y_i$ for all $i \in S$, and $x \gg y$ if $x_i > y_i$ for all $i \in S$. For any $x \in \mathfrak{R}^S$, denote by $\hat{x}$ the vector obtained by permuting the indices of x such that $\hat{x}_1 \geq \hat{x}_2 \geq \ldots \geq \hat{x}_{|S|}$. Also, suppose some $x \in \mathfrak{R}^S$ is given. The projection of x on T, a subset of S, is given by x_T.

3. Constrained Egalitarian Allocations

We consider a transferable utility game in characteristic function form. $N = \{1, 2, \ldots, n\}$ is the player set. A *coalition* is a nonempty subset of N.

[7] See Maschler and Peleg (1966).

To every coalition S is attached a number $v(S)$, called the *worth* of that coalition. A feasible allocation for S is any $|S|$-dimensional vector x with $\sum_{i \in S} x_i = v(S)$.

The notion of a constrained egalitarian allocation is now developed. First, we introduce the concept of egalitarianism that we use. Consider two k-person allocations x and y in $\mathfrak{R}^k$ such that $\sum_{i=1}^{j} \tilde{y}_i \leq \sum_{i=1}^{j} \tilde{x}_i$ for all $j = 1, \ldots, k$, with strict inequality for some j. This partial ordering has a well-known characterization as an ordering that agrees with basic ethical notions of egalitarianism (see, e.g., Kolm, 1969; Dasgupta *et al.*, 1973; Fields and Fei, 1978). To put it loosely, y Lorenz-dominates x iff x can be reached from y by a sequence of "income transfers" from "poor" to "rich." Naturally, *all* pairs (x, y) cannot be compared in this way, so it is no surprise that the Lorenz ordering is partial.

Two allocations x and y have the *same Lorenz curve* if $\tilde{x} = \tilde{y}$.

We assume that our society seeks Lorenz improvements whenever these are "achievable," where the meaning of the word in quotes will be made precise below. Thus for each set A of k-person allocations of a given total, let EA be the set of all allocations that are Lorenz-undominated within A. That is

$$EA = \{x \in A | \quad \text{there is no} \quad y \in A \quad \text{such that} \quad y \quad \text{Lorenz-dominates} \quad x\}. \tag{1}$$

We are now ready to define constrained egalitarian allocations. For two vectors $x, y \in \mathfrak{R}^k$, let $x\, d\, y$ denote a domination relationship. In our paper, d represents $>$ or $\gg$. We start by defining *Lorenz cores* (relative to d).

The Lorenz core of a singleton coalition is $L^d(\{i\}) \equiv v(\{i\})$. Now suppose that the Lorenz core (relative to d) of a coalition of size $(k + 1)$ is defined by

$$L^d(S) = \{x \in \mathfrak{R}^S | x \quad \text{is feasible for} \quad S, \text{and there is no} \quad T \subset S \text{ and } y \in EL^d(T) \text{ such that } y\, d\, x_T\}. \tag{2}$$

The set of *constrained egalitarian allocations* (relative to d) *for* T is given by $EL^d(T)$. And $EL^d(N)$ denotes the set of *constrained egalitarian allocations* (*relative to* d).

Define the set of *W-constrained egalitarian allocations* as the set of constrained egalitarian allocations *relative to* $>$. This is the concept studied in Dutta and Ray (1989), and it was established there that this set contains at most one element. Consequently, we shall refer to this element as *the* W-constrained egalitarian allocation, whenever it exists.

The concept that we study here is the set of *S-constrained egalitarian allocations*, defined as the set of constrained egalitarian allocations relative to the strong domination relation $\gg$. We refer to the Lorenz core (relative to $\gg$) as the S-Lorenz core, written $L^*(\cdot)$, while the Lorenz core relative to $>$ is called the W-Lorenz core, written $L(\cdot)$.

Finally, if x is feasible for S, and there is $T \subset S$ with $y \in EL^*(T)$ such that $y \gg x_T$, we say that y *L^*-blocks* x. We shall also say in this case that T L^*-blocks x. (Write "L-block" for the corresponding notion with weak blocking $>$.)

Remarks. (1) If T L^*-blocks x (or L-blocks x) then it possesses a credibility property. For T can then block x using an allocation y which is *itself a constrained egalitarian allocation* for T. So, if T deviates and forms a subsociety on its own, with the same egalitarian norms being applied to itself, it can still "achieve" the allocation y. And *every* member of T is better off with y than with x. This is what we earlier referred to as "selfish behavior within the context of societal rules."

(2) For any coalition S, the *core* of S is defined by

$$C(S) = \{x \in \mathcal{R}^S | x \quad \text{is feasible for} \quad S, \text{and there is no } T \subset S \text{ such that } v(T) > \textstyle\sum_{i \in T} x_i\}.$$

Clearly, for each S, $C(S) \subseteq L^*(S)$ and $C(S) \subseteq L(S)$. The requirement that coalitions can only block credibly, i.e., with allocations that are constrained egalitarian allocations for those coalitions, enlarges the set of "unblocked" allocations.

The concept of S-constrained egalitarian allocations differs from our earlier formulation only via the domination relation ($>$ is replaced by $\gg$). At first sight, it might appear that this slight modification in the concept of blocking should not cause any drastic alteration in the structure of Lorenz cores or in the sets of SCEAs and WCEAs. However, this intuition is not correct. In Section 5, we provide a detailed comparison of the sets SCEAs and WCEAs. Here, we merely point out two significant differences emanating from the two different notions of blocking. First, while we show in Dutta and Ray (1989) that the set of WCEAs may sometimes be empty (because, for instance, the W-Lorenz core may be empty), existence is not a problem in so far as SCEAs are concerned. Second, while the set of WCEAs can contain *at most* one element, there may be multiple SCEAs. And there are other differences, which we shall soon explore.

The following examples illustrate some characteristics of the two types of constrained egalitarian allocations, as well as the structure of the corresponding Lorenz cores.

EXAMPLE 1. Let $N = \{1, 2, 3\}$, $v(N) = v(\{1, 2\}) = 2.2$, $v(\{1, 3\}) = v(\{2, 3\}) = 1.4$, $v(\{1\}) = v(\{2\}) = 1$, $v(\{3\}) = 0$. Then, $L(N) = \{x \in \mathcal{R}^3 | x_1 > 1, x_2 > 1, x_1 + x_2 = 2.2\}$ and $EL(N) = \{1.1, 1.1, 0\}$, while $L^*(N) = \{x \in$

$\mathfrak{R}^3 | (x_1 \geq 1.1, x_2 \geq 1)$ or $(x_1 \geq 1, x_2 \geq 1.1)$ and $x_1 + x_2 + x_3 = 2.2\}$, and $EL^*(N) = \{x^1, x^2\}$, where $x^1 = (1.1, 1, 0.1)$ and $x^2 = (1, 1.1, 0.1)$.

Note that the coalition $\{1, 2\}$ L-blocks both x^1 and x^2. But, $\{1, 2\}$ cannot L^*-block either allocation since it can only use $(1.1, 1.1)$ to block with. Similarly, coalitions $\{1, 3\}$ and $\{2, 3\}$ can only use the allocation $(1, 0.4)$ for blocking, and hence cannot L-block $(1.1, 1.1, 0)$ or L^*-block x^1 or x^2. However, $C(N) = \phi$ since in the usual core sense, coalitions are allowed to block with any feasible allocation.

Example 1 illustrates the point that SCEAs may not be unique whereas the WCEA must, of course, be unique. The next two examples show the relative advantage of the "strong" concept over its "weak" counterpart.

EXAMPLE 2. $N = A \cup B$, where $|B| = |A| - 1 > 0$. For any $S \subseteq N$, $v(S) = \min(|S \cap A|, |S \cap B|)$. This example corresponds to the well-known "right and left gloves" situation (see Aumann, 1985). It is well known that the core of this game gives one unit each to members of B, and none to members of A, even if A and B are both very large sets. This "discontinuity" arises from the fact that $|B| < |A|$.

In this game, there is no WCEA. Of course, the W-Lorenz core, $L(N)$, being a superset of the core, is nonempty. However, the WCEA does not exist because $L(N)$ is *not closed*. There exists, however, a unique SCEA, given by x, where

$$x_i = \frac{|B|}{2(|B| + 1)} \quad \text{if} \quad i \in A$$

$$x_i = \frac{1}{2} \quad \text{if} \quad i \in B.$$

EXAMPLE 3. Let $N = \{1, 2, 3\}$, and v be a symmetric game, with $v(S) = v(N) = 1$ if $|S| = 2$, and $v(S) = 0$ if S is a singleton coalition.

Here, the WCEA does not exist because $L(N)$ is empty. The set of SCEAs is given by $\{(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2}), (0, \frac{1}{2}, \frac{1}{2})\}$.

4. EXISTENCE AND CHARACTERIZATION RESULTS

This section contains several results on S-Lorenz cores and S-constrained egalitarian allocations. In particular, we provide a characterization of S-Lorenz cores. We then go on to show that, under a condition weaker than superadditivity, the S-Lorenz core and the set of SCEAs are always nonempty. The next issue taken up is uniqueness. Of course, the fact that S-constrained egalitarian *allocations* are not necessarily unique is evident from Example 1 itself. However, in Example 1, $\bar{x}^1 = \bar{x}^2$, so that x^1 and x^2 both have the same *Lorenz curve*. Is it then true that in any game all SCEAs have the same Lorenz curve? We show that this is indeed true in games where the desirability relation between players is complete, and in all three-person games. However, Example 4 shows that this property does not carry over even to convex games with four or more players. Contrast this with the general uniqueness result for W-constrained egalitarian allocation.

The following definitions are useful. For any coalition S, define

$$a(S, v) \equiv \frac{v(S)}{|S|}.$$

This is the *average worth* of the coalition S. When there is no ambiguity about v, we simply write $a(S)$ instead of $a(S, v)$.

The *equal division allocation* for S is the allocation x for S such that for all $i \in S$, $x_i = a(S)$. Denote it by e_S. Finally, S is an *equity coalition* if $a(S) \geq a(T)$ for all $T \subseteq S$.

We start with a couple of preliminary lemmas.

LEMMA 1. *If S is an equity coalition, then $EL^*(S)$ is a singleton set containing the equal division allocation.*

Proof. It is sufficient to prove that $L^*(S)$ contains e_S. Suppose not. Then, there is $T \subset S$ and $y \in EL^*(T)$ such that $y \gg e_S(T)$. So, $\sum_{i \in T} y_i > a(S)|T|$, implying $a(T) > a(S)$. This contradicts the hypothesis that S is an equity coalition. ■

LEMMA 2. *For some $S \subseteq N$, let $y \in EL^*(S)$. For any $i \in S$, if $y_i > \min_{j \in S} y_j$, then there exists an equity coalition T containing i and satisfying*

(i) $a(T) = y_i$.
(ii) $T \subset \{k \in S | y_k < y_i\} \cup \{i\}$.

Proof. Pick any $S \subseteq N$, and suppose $y \in EL^*(S)$, where $y \neq e_S$. Choose any $i \in S$ such that $y_i > \min_{j \in S} y_j$. Let T be any equity coalition satisfying

(a) $i \in T$ and if $j \in T - \{i\}$, then $y_j < y_i$.
(b) T has an average worth at least as high as that of any equity coalition containing i and satisfying (a).

Clearly, such an equity coalition exists since $\{i\}$ is an equity coalition. Suppose that $a(T) > y_i$. From Lemma 1, $e_T \in EL^*(T)$. Also $e_T \gg y_T$ in view of (a). Hence T L-blocks y, a contradiction.

So, suppose $a(T) < y_i$. Define $\varepsilon = \frac{1}{2} \min_{j \in S}(y_i - a(T), y_i - y_j)$. Choose any $j \in S - \{i\}$, and construct a new allocation y' for S as follows: $y'_i = y_i - \varepsilon$, $y'_j = y_j + \varepsilon$, and for all $k \neq i, j$, $y'_k = y_k$. It is easily verified that $y' \in$

$L^*(S)$. But, y' Lorenz-dominates y, since $y_i > y_i' > y_j' > y_j$. Hence $y \notin EL^*(S)$.

Hence, there must exist an equity coalition T containing i with $a(T) = y_i$ and $T \subset \{j \in S | y_k < y_i\} \cup \{i\}$.

THEOREM 1. $L^*(N) = \{x | x$ *is feasible for* N *and for no* $S \subset N$: $e_S \gg x_S\}$.

Proof. Let $R \equiv \{x | \Sigma_{i\in N} x_i = v(N)$, and for no $S \subset N$: $e_S \gg x_S\}$. We want to show that $R = L^*(N)$.

Suppose $x \in R$, but $x \notin L^*(N)$. Then there exists $S \subset N$, $y \in EL^*(S)$ such that $y \gg x_S$. Clearly, $y \neq e_S$ since $x \in R$. Let $y_i = \max_{j\in S} y_j$. Then by Lemma 2, there is an equity coalition $T \subset S$ containing i with $a(T) = y_i$. But, then $e_T \gg x_T$, contradicting the supposition that $x \in R$. Hence, $R \subseteq L^*(N)$.

Suppose now that $x \in L^*(N)$, but $x \notin R$. Then, there exists S such that $e_S \gg x_S$. Since $x \in L^*(N)$, $e_S \notin L^*(S)$. By Lemma 1, S is not an equity coalition. So, there exists an equity coalition $T \subset S$ with $a(T) > a(S)$. By Lemma 1, $e_T \in EL^*(T)$. But $e_S \gg x_S$ and $a(T) > a(S)$ imply that $e_T \gg x_T$. Hence, $x \notin L^*(N)$. So, L*(N) $\subseteq R$. ■

This completes the proof of the theorem.

Remark. Theorem 1 shows that the S-Lorenz core is precisely the set of allocations which cannot be blocked (in the strong sense) by any coalition using its *equal division allocation*. Selten (1972) had earlier introduced this set for zero-normalized characteristic function games.[8] However, as Selten (1987) makes clear, the desire to conform to social norms was *not* the driving force in introducing equity considerations in his formulation. Selten (1987) remarks that "the main importance of equity considerations seems to lie in their usefulness for establishing baselines in strategic reasoning. One looks at what would be obtained in the absence of payoff differences in order to obtain bounds on power-adequate payoff distributions." Thus, Selten's approach recommends the *entire* set $L^*(N)$ as the set of "reasonable" payoff vectors. In our approach, the set of "reasonable" payoff vectors is derived by a *consistent use of commitment to the social norm*. Hence, our solution set is the set of "most equal" payoff vectors from within $L^*(N)$. Quite clearly, Selten's approach and ours differ both in motivation and in the solution concepts.[9]

[8] Given any game (N, v), the corresponding zero-normalized game v_0 is given by $v_0(S) = v(S) - \Sigma_{i\in S} v(\{i\})$, for all $S \subseteq N$.

[9] Crott and Albers (1981) are closer to our approach. In their model, agents believe in the "symmetry" principle that equal division will result when "persons' inputs do not differ." Symmetry, however, is different from equity. Crott and Albers assume the symmetry principle holds for subcoalitions, but not for the grand coalition, whereas we apply the egalitarian principle consistently across all coalitions.

Theorem 1, of course, is silent on whether $L^*(N)$ is nonempty. One might think that since the S-Lorenz core is similar to the core at least in spirit, and since the core is nonempty only for balanced games, the S-Lorenz core will also be nonempty only under relatively stringent conditions. Recall also (from Example 3) that $L(N)$ can also be empty. Our next theorem shows, however, that $L^*(N)$ is nonempty under a condition even weaker than superadditivity.

Recall that v is *superadditive* if for all disjoint S, T,

$$v(S) + v(T) \le v(S \cup T). \tag{3}$$

Define v to be *weakly superadditive* if for all partitions $\{S_1, \ldots, S_m\}$ of N,

$$v(N) \ge \sum_{i=1}^{m} v(S_i). \tag{4}$$

LEMMA 3. *Let v be weakly superadditive. Then the S-Lorenz core of N is nonempty.*

Proof. We construct a specific allocation x^* which is in $L^*(N)$. The allocation x^* is generated by an algorithm which we describe below.

First, choose

$$S_1^* \in \underset{S \subseteq N}{\operatorname{argmax}}\{a(S)\}. \tag{5}$$

Recursively, having chosen $S_1^*, S_2^*, \ldots, S_k^*$, if $\{S_1^*, S_2^*, \ldots, S_k^*\}$ does not form a partition of N, then choose

$$S_{k+1}^* \in \underset{S \subseteq N - \cup_{j=1}^{k} S_j^*}{\operatorname{argmax}} \{a(S)\}. \tag{6}$$

Let $\{S_1^*, S_2^*, \ldots, S_k^*\}$ be the partition of N generated by this algorithm. Let $x \in \Re^N$ be defined by

$$x_i = a(S_j^*), \qquad \text{where } i \in S_j^*,\ i = 1, \ldots, n. \tag{7}$$

From weak superadditivity,

$$\delta = \frac{v(N) - \sum_{i=1}^{n} x_i}{n} \ge 0. \tag{8}$$

Construct x^* such that $\forall i \in N$, $x_i^* = x_i + \delta$. Suppose $x^* \notin L^*(N)$. Then, from Theorem 1, there exists a coalition T such that $e_T \gg x_T^*$. Let $x_i^* =$

$\max_{j \in T} x_j^*$, and fix j such that $i \in S_j^*$. Then, $T \cap \{\bigcup_{l=1}^{j-1} S_l^*\} = \emptyset$, and $a(T) > a(S_j^*)$. But, this contradicts the construction of the partition $\{S_1^*, S_2^*, \ldots, S_K^*\}$. Hence, $x^* \in L^*(N)$. ∎

Note that $L^*(N)$ is closed, and hence compact. Since $L^*(N)$ is always nonempty, this immediately yields the following.

THEOREM 2. *If v is weakly superadditive, then the set of strong egalitarian allocations is nonempty.*

So, a weak condition on the game guarantees the existence of SCEAs. This result should be contrasted with the results obtained for the W-constrained egalitarian allocation in Dutta and Ray (1989), which showed that convex games always possess a WCEA. However, the WCEA may not exist even in balanced games (see Example 3 in this paper). Obviously, Theorem 2 widens the scope of applicability of constrained egalitarianism quite dramatically.

We turn now to the question of uniqueness. Examples 2 and 3 have already demonstrated that the set of SCEAs is not necessarily unique. However, it may be argued that the nonuniqueness exhibited in these cases is not a cause for much concern, since the set of SCEAs (in each game) has the *same Lorenz curve*. We now examine conditions under which this property holds.

Let (N, v) be any game, and $i, j \in N$. Say that i is *at least as desirable as j in v*, written $i\ d(v) j$ if for all $S \subseteq N - \{i, j\}$, $v(S \cup \{i\}) \geq v(S \cup \{j\})$.

Maschler and Peleg (1966) proved that the desirability relation between players is *always transitive*. There are many games of interest in which $d(v)$ turns out to be complete,[10] but of course, this is not generally true.

THEOREM 3. *Suppose (N, v) is a weakly superadditive game satisfying any one of the following conditions.*

(i) *$d(v)$ is complete.*
(ii) *$|N| \leq 3$.*
(iii) *N is an equity coalition.*

Then, all allocations in $EL^*(N)$ have the same Lorenz curve.

Proof. (i) Suppose $d(v)$ is complete. Since $d(v)$ is transitive, assume w.l.o.g. that $i\ d(v)\ i + 1$ for $i = 1, \ldots, n - 1$. For all $i = 1, \ldots, n$, define $S_i = \operatorname{argmax}_{S \subseteq N_i} a(S)$, where $N_1 = N - \{1, \ldots, i - 1\}$ for $i > 1$. Note that we must have

(a) $S_i = \{i, i + 1, \ldots, i + k(i)\}$, for some nonnegative integer $k(i)$.
(b) $a(S_i) \geq a(S_{i+1})$.

[10] For instance, quota games $[q : w_1, w_2, \ldots, w_n]$, where a coalition S is winning iff $\sum_{i \in S} w_i \geq q$, have complete desirability relations.

Suppose $S_1 = \{1, 2, \ldots, k\}$. Then, from (b), $\sum_{i=1}^{k} a(S_i) \le v(S_1)$. Similarly, if $S_{k+1} = \{k+1, k+2, \ldots, m\}$, then $\sum_{i=k+1}^{m} a(S_i) \le v(S_{k+1})$. Repeating this and using superadditivity, we get $\sum_{i=1}^{n} a(S_i) \le v(N)$.

Consider the allocation z where $z_i = a(S_i)$ for each i. Then $v(N) - \sum_{i=1}^{n} z_i = \delta \ge 0$. Observe that in the set of allocations $\{y \in \mathcal{R}^n \mid y \ge z \text{ and } \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} z_i + \delta\}$, there is a unique Lorenz-maximal element; call it x. Note that $x_i \ge x_{i+1}$ for all i and $x_i > z_i$ implies $x_j = x_{j+1}$ for all $j \ge i$. Let K be the largest index such that $x_K = z_K$.

Using Theorem 1 and the construction of x, it is easy to see that $x \in L^*(N)$.

Now, choose $y \in L^*(N)$. We want to show that if the Lorenz curves of x and y are different, then x Lorenz-dominates y.

Let σ be a permutation of $\{1, \ldots, n\}$ such that $y_{\sigma(i)} \ge y_{\sigma(i+1)}$ for all $i = 1, 2, \ldots, n-1$. For all $i = 1, 2, \ldots, n$, define $S_i' = \operatorname{argmax}_{S \subseteq N_i'} a(S)$, where $N_1' = N$ and $N_i' = N - \{\sigma(1), \ldots, \sigma(i-1)\}$. Now, choose *any* $k \le K$. It will suffice to show that

$$\sum_{i=1}^{k} y_{\sigma(i)} \ge \sum_{i=1}^{k} x_i. \tag{9}$$

For in view of the construction of x, it follows that if $K < n - 1$, then $x_{K+1} = x_n$. So, (9) will show that either x and y have the same Lorenz curves *or* x Lorenz-dominates y.

Since $y \in L^*(N)$, we have for all $i \le k$,

$$y_{\sigma(i)} \ge a(S_i'). \tag{10}$$

Since $\delta_i = 0$ for all $i \le k$,

$$x_i = a(S_i). \tag{11}$$

The proof of (9) is completed by showing that $a(S_i') \ge a(S_i)$ for all i. Choose any i. If $N_i = N_i'$, then $a(S_i) = a(S_i')$. Suppose $N_i \ne N_i'$. Since $N_i = \{i, i+1, \ldots, n\}$, the players in N' but *not* in N_i are at least as desirable as i, the "most" desirable player in N_i. So it is clear, that $a(S_i') \ge a(S_i)$.

Hence, if $\bar{y} \ne x$, then x Lorenz-dominates y. So, the set of SCEA must consist of allocations whose Lorenz curves coincide with that of x.

(ii) Suppose now that $|N| = 3$. Let y^1 and y^2 both be in $EL^*(N)$ with $\bar{y}^1 \ne \bar{y}^2$. Clearly, $\bar{y}_1^1 = \bar{y}_1^2$. Hence, we must have $(\bar{y}_2^1, \bar{y}_3^1) \ne (\bar{y}_2^2, \bar{y}_3^2)$. Since $\sum_{i=1}^{3} y_i^1 = \sum_{i=1}^{3} y_i^2 = v(N)$, either y^1 Lorenz-dominates y^2 or y^2 Lorenz-dominates y^1. But, they y^1 and y^2 cannot both belong to $EL^*(N)$. So, in all three-person games, the set of SCEAs must have the same Lorenz curve. Obviously, this proposition is also true in two-person games.

(iii) Finally, note that if N is an equity condition, then $e_N \in L^*(N)$. Then, e_N must be the *only allocation* in $L^*(N)$.

This completes the proof of Theorem 3. ■

Remark. It should be clear to the reader that if $d(v)$ is complete and asymmetric (that is, $id(v)j$ implies *not* $jd(v)i$), then x is the only *allocation* in $EL^*(N)$. Indeed, the proof shows that x will then Lorenz-dominate every other allocation in $L^*(N)$.

Unfortunately, in Example 4, we show that if none of the conditions of Theorem 3 are satisfied, then even in *convex* games, allocations in $EL^*(N)$ need not have the same Lorenz curve.

Recall that a game is *convex* if for all $S, T \subseteq N$,

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T). \tag{12}$$

EXAMPLE 4. Let $N = (1, 2, 3, 4)$. Let v be described by the following schedule:

S	$v(S)$
{1}	23
{2}	40
{3}	39
{1, 2}	100
{1, 3}	82
{2, 3}	79
{1, 2, 3}	159
N	167

and for all $S \neq \{1, 2, 3\}$, $v(S \cup \{4\}) = v(S)$.

It is easy to check that v is *convex* and that $y^1 = (53, 40, 39, 35)$ and $y^2 = (36.5, 53, 41, 36.5)$ both belong to $EL^*(N)$. The Lorenz curves corresponding to y^1 and y^2 intersect each other.

The nonuniqueness of Lorenz curves associated with SCEAs causes a problem. Consider, for instance, Example 4. Should society "choose" y^1 or y^2? Since individuals believe in equality as a desirable end, it makes sense to recommend that society choose the allocation which corresponds to a more equal distribution. However, since y^1 and y^2 have intersecting Lorenz curves, it is not possible to get an unambiguous ranking (in terms of equality) over y^1 and y^2. Thus, in all such cases, the prescription to choose "the most equal" distribution must be supplemented by the advocacy of some *particular* measure of equality. Mere reliance on Lorenz domination will not suffice. In this sense, the W-consistent egalitarian allocation provides a sharper picture by virtue of its uniqueness. However, we recall again that the WCEA lacks general existence properties.

5. The SCEA, the WCEA, and the Core

Example 4 in the previous section also illustrates two other features of the set of SCEAs. First, the WCEA for that game is $x = (53, 53, 53, 8)$, and *both* SCEAs y^1 and y^2 Lorenz-dominate x. Second, neither y^1 nor y^2 belongs to the core since the coalition $\{1, 2, 3\}$ can block either allocation. There is some intuitive reason to expect both features to hold in general.

Suppose, for example, that S_1^* is a coalition maximizing average worth among all coalitions. If x is the WCEA, then it is clear that $x_i = a(S_1^*)$ for all $i \in S_1^*$. In contrast, if y is a SCEA, then some individual, say $j \in S_1^*$, must be allocated $a(S_1^*)$. It is not necessary to allot individuals in S_1^* other than j the amount $a(S_1^*)$ because so long as $y_j = a(S_1^*)$, S_1^* cannot L^*-block y.

In constructing y, one now has to choose $S_2^* \in \text{argmax}_{S \subseteq N-\{j\}}\{a(S)\}$ and again only *one* individual in S_2^* need be given $a(S_2^*)$. However, in constructing the WCEA, one cannot ignore coalitions intersecting with S_1^* even at the second stage. Of course, the process of constructing SCEAs is considerably more complicated since the choice of individuals at each stage cannot be made arbitrarily. Indeed, the reason why all SCEAs do not have the same Lorenz curve is *precisely* that the choice of individuals at each stage is not unambiguous, *unless* $d(v)$ is complete.

In this section, we first analyze the cases under which the first feature of Example 4, namely all SCEAs Lorenz-dominating the WCEA, hold. It turns out that whenever the WCEA is not SCEA, this is true in all games with $n \leq 4$. This phenomenon is also true in all convex games. It is also the case that for *all* games, there is at least one SCEA which Lorenz-dominates the WCEA, if the WCEA is not an SCEA. However, we construct in Example 5 a five-player game in which one SCEA and the WCEA are Lorenz-noncomparable. We then go on to examine the conditions under which the second feature of Example 4 obtains: namely, all SCEAs belong to the core. It turns out that this issue is related to the WCEA also being a SCEA. We show that the class of games under which the WCEA is a SCEA coincides with the class of games under which the SCEA is unique *and* belongs to the core. Typical examples of this class are games in which N is an equity coalition, *inessential* games, and two-player games. We provide a complete description of this class of games. A "nontrivial" game will generally *not* fall into this class.

We start by showing that in *all* games, if the WCEA is not an SCEA, then at least one SCEA Lorenz-dominates the WCEA.

Theorem 4. *If $EL(N) \neq \emptyset$ and $EL(N) \cap EL^*(N) = \emptyset$, then there exists an SCEA which Lorenz-dominates the WCEA.*

Proof. We prove this result by showing that $L(N) \subseteq L^*(N)$. Suppose $x \notin L^*(N)$. Then by Theorem 1, there is $T \subseteq N$ such that $e_T \gg x_T$. If T is an equity coalition, then clearly $\{e_T\} = EL(T)$, and T L − blocks x. If T is not an equity coalition, then there exists an equity coalition $S \subset T$ with $e_S \gg x_S$. Then, S L-blocks x.

Hence, $L(N) \subseteq L^*(N)$. It now follows, using the fact that $L^*(N)$ is compact, that if the WCEA exists, then it must be Lorenz-dominated by *some* SCEA. ■

A stronger result can be obtained for all three- and four-player games, and convex games. We prove these in turn.

THEOREM 5. *Let $|N| \leq 4$. Suppose that $EL(N) \neq \emptyset$ but that $EL(N) \cap EL^*(N) = \emptyset$. Then every SCEA Lorenz-dominates the WCEA.*

Proof. Suppose $|N| = 3$. Then, the result follows from Theorems 3 and 4. (If $|N| = 2$, then the WCEA is also the SCEA.) So, let $|N| = 4$. If N is an equity coalition, then e_N is the only element of $EL^*(N)$ and is also a WCEA. Hence, assume that N is not an equity coalition. Let S^* be (one of) the largest coalition(s) with the highest average worth, i.e., $a(S^*) \geq a(T)$ for all $T \subseteq N$ and if $S^* \subset T$ then $a(S^*) > a(T)$. Let x denote the WCEA. Suppose $|S^*| = 3$. Then, $x_i = a(S^*)$ for all $i \in S^*$. It is then easy to check that the theorem is true.

Suppose $|S^*| = 1$. Renumber players so that $S^* = \{1\}$. Let y^1 be the SCEA which Lorenz-dominates x. Such a y^1 exists, by Theorem 4. Let y^2 be *any other* SCEA. Then $y_1^1 = y_1^2 = x_1 = v(\{1\})$. Moreover, from Theorem 4, *and* the fact that $|S^*| = 1$, it follows that $\bar{y}^1 = \bar{y}^2$. But, then y^1 Lorenz-dominates x implies that y^2 Lorenz-dominates x.

It remains to consider the case: $|S^*| = 2$. W.l.o.g. assume $S^* = \{1, 2\}$. Let y be an SCEA which does not Lorenz-dominate x. We must have $x_1 = x_2 > \max(x_3, x_4)$. Assume $y_1 = x_1$, and w.l.o.g. let $y_3 = \max(y_3, y_4)$. Either (i) $y_2 \geq y_3$ or (ii) $y_3 > y_2$.

Suppose (i) holds. Then, $y_2 < x_2$ and since y does not Lorenz-dominate x,

$$y_1 + y_2 + y_3 > x_1 + x_2 + \max(x_3, x_4). \tag{13}$$

Hence,

$$y_3 > \max(x_3, x_4) \tag{14}$$

and

$$y_4 < \min(x_3, x_4). \tag{15}$$

Since $y_3 \geq y_4$ either (i) $a(\{3, 4\}) = y_3$ or (ii) $a(\{3\}) = y_3$. If $a(\{3, 4\}) = y_3$, then (14) implies that $\{3, 4\}$ L-blocks x. If $a(\{3\}) = y_3$, then again (14) implies that $\{3\}$ L-blocks x.

So, we must have $y_3 > y_2$. Since y does not Lorenz-dominate x, we must have $y_3 > x_3$. From Lemma 2, there exists T containing 3 such that (i) $a(T) = y_3$ and (ii) $T \subseteq \{2, 3, 4\}$. Now, if $2 \notin T$, then T L-blocks x. If $2 \in T$, then construct y' such that $y'_1 = y_1 = a(S^*)$, $y'_2 = y_3$, and $y'_3 = y'_4 = (v(N) - y'_1 - y'_2)/2$. It is tedious but easy to check that

(i) y' Lorenz-dominates y, *and*
(ii) $y' \in L^*(N)$ *or* $x \notin L(N)$.

Hence, either $y \notin EL^*(N)$ *or* $x \notin L(N)$. This contradiction completes the proof of the theorem.

Before proving the corresponding result for convex games, we need to define the structure of the WCEA for convex games. As shown in Dutta and Ray (1989), the WCEA for convex games is obtained by means of the following algorithm.

Let (v, N) be a convex game. Define $v_1 = v$.

Step 1. Let S_1^* be the largest coalition having the highest average worth in the game (v_1, N). Convexity of v_1 ensures that S_1^* is unique. Let

$$x_1^* = a(S_1^*, v_1) \qquad \text{for all } i \in S_1^*. \tag{16}$$

Step k. Suppose that $S_1^*, \ldots, S_{k-1}^*$ have been defined recursively and $\cup_{i=1}^{k-1} S_i^* = Z_{k-1} \neq N$. Define a new game (v_k, N_k) with player set $N_k = N - Z_{k-1}$, and for all $S \subseteq N_k$, $v_k(S) = v_{k-1}(S \cup S_{k-1}^*) - v_{k-1}(S_{k-1}^*)$. The reader can check that v_k will also be convex. Just as in Step 1, define S_k^* to be the largest coalition having the highest average worth in (v_k, N_k). Define

$$x_i^* = a(S_k^*, v_k) \qquad \text{for all } i \in S_k^*. \tag{17}$$

Clearly, in m of these steps ($m \leq n$) there will be a partition of N into sets $\{S_1^*, S_2^*, \ldots, S_m^*\}$. Let x^* be the allocation defined by equations of the form (16) and (17). Note that if $i, j \in S_k^*$ for any $k = 1, \ldots, m$, then $x_i^* = x_j^*$. Also, if $i \in S_k^*$ and $j \in S_l^*$ and $k < l$, then $x_i^* > x_j^*$.

The following was proved in Dutta and Ray (1989).

PROPOSITION 1. *Suppose (v, N) is a convex game. Then, x^* is the WCEA of (v, N). Moreover, $x^* \in C(N)$.*

Now we may state

THEOREM 6. *Suppose v is a convex game. Then either the WCEA is the unique SCEA or every SCEA Lorenz-dominates the WCEA.*

Proof. Let x^* be the WCEA, and let y be some SCEA such that $y \neq x^*$. We have to prove that y Lorenz-dominates x^*. It will suffice to show that if there is $i \in N$ such that $y_i > x_i^*$ then there is no $j \in N$ with $y_j < y_i$. Suppose, on the contrary, that there are $i, j \in N$ with $y_i > x_i^*$ and $y_i > y_j$.

Then, from Lemma 2, there is an equity coalition T containing i such that (i) $a(T) = y_i$ and (ii) $T \subset \{k \in N | y_k < y_i\} \cup \{i\}$.

Let $i \in S_k^*$ where $\{S_1^*, \ldots, S_m^*\}$ is the partition of N induced by the algorithm generating x^*. Define $A \equiv T \cup Z_{k-1}$, and $B \equiv T \cap Z_{k-1}$, where $Z_{k-1} \equiv \cup_{j=1}^{k-1} S_j^*$. Consider two cases.

Case 1. $B = \emptyset$. Then, $a(T) = y_i > x_i^* \geq x_j^*$ for all $j \in T$. So, $\sum_{j \in T} x_j^* < v(T)$. Then, $x^* \notin C(N)$, contradicting Proposition 1.

Case 2. $B \neq \emptyset$. Using convexity,

$$v(A) = v(T \cup Z_{k-1}) \geq v(T) + v(Z_{k-1}) - v(B). \tag{18}$$

Now

$$v(T) - v(B) = a(T)|T| - a(B)|B| \geq a(T)(|T| - |B|) \tag{19}$$

using the fact that T is an equity coalition and $B \subset T$. Combining (18) and (19).

$$v(A) \geq a(T)(|T| - |B|) + v(Z_{k-1}). \tag{20}$$

But

$$\begin{aligned}\sum_{i \in A} x_i^* = \sum_{i \in T-B} x_i^* + \sum_{i \in Z_{k-1}} x_i^* &< y_i(|T| - |B|) + v(Z_{k-1}) \\ &= a(T)(|T| - |B|) + v(Z_{k-1}) \leq v(A).\end{aligned}$$

Again, $x^* \notin C(N)$, contradicting Proposition 1. This completes the proof of the theorem. ■

Although we have provided some intuitive argument to support the contention of Theorems 6 and 7, this argument breaks down in general. In Example 5, we describe a five-person game in which the WCEA and *one* SCEA are Lorenz noncomparable. Needless to say, this game is not convex.

EXAMPLE 5. Let $N = \{1, 2, 3, 4, 5\}$. Let v be described by the following schedule:

S	$v(S)$	S	$v(S)$
{1}	.96	{24}	1.81
{2}	.70	{34}	1.80
{3}	.70	{123}	2.36
{4}	.00	{134}	2.70
{12}	1.66	{124}	2.70
{13}	1.66	{234}	2.85
{14}	2.00	{1234}	3.81
{23}	1.40		

Moreover, 5 is a *dummy player*, so that $v(S \cup \{5\}) = v(S)$, for all $S \subseteq \{1, 2, 3, 4\}$.

Then, the WCEA is $x = (1, .7, .7, 1, .41)$. The SCEAs are $y^1 = (1, .95, .9, .48, .48)$, $y^2\,(1, .9, .95, .48, .48)$, $y^3 = (1, .7, .7, .95, .46)$.

Although y^3 Lorenz-dominates x, x is not Lorenz comparable with y^1 or y^2.

We end this section by taking up the second feature thrown up by all the examples so far—namely the failure of the SCEAs to be in the core, *even when the latter is nonempty*. In our next result, we characterize the class of superadditive games in which the SCEAs belong to the core. We also show that this is precisely the same class under which the WCEA happens to be SCEA.

This class of games is defined by the following condition.

Condition α. N can be partitioned into sets $\{S, T\}$ such that the following are satisfied:

$$\text{(a)}\quad v(S) = \sum_{i \in S} v(\{i\}). \tag{21}$$

(b) For all $S' \subseteq S$, for all nonempty $T' \subseteq T$,

$$\frac{v(N) - v(S)}{|T|} \geq \frac{v(S' \cup T') - v(S')}{|T'|}. \tag{22}$$

$$\text{(c)}\quad \min_{i \in S} v(\{i\}) > \max_{T' \subseteq T} a(T') \qquad \text{if } S \neq \emptyset. \tag{23}$$

Typical examples of games satisfying condition α are *inessential* games, in which case $S = N$, or games in which N itself is an equity coalition, so that $T = N$. Condition α is a stringent condition and the class of games satisfying α is "small" relative to the universe of all possible characteristic function games.

THEOREM 7: *If v is superadditive, then the following statements are equivalent*:

(i) *v satisfies condition α.*
(*ii*) *The WCEA is the unique SCEA.*
(*iii*) *The unique SCEA belongs to the core of v.*

Proof. We first prove that (i) and (ii) are equivalent. Suppose v satisfies condition α. Define x^* by

$$\text{for all } i \in S, \qquad x_i^* = v(\{i\}) \tag{24}$$

$$\text{for all } i \in T, \qquad x_i^* = \frac{v(N) - v(S)}{|T|} \tag{25}$$

Suppose some coalition R L-blocks x^*. If $R \subseteq S$ then (a) guarantees that R cannot L-block x^*. Similarly, if $R \subseteq T$, then (b) guarantees that R cannot L-block x^*. So, let $R = S' \cup T'$, where $\emptyset \neq S' \subseteq S$ and $\emptyset \neq T' \subseteq T$. Let $y \in EL(R)$. Then,

$$\sum_{i \in R} y_i > \sum_{i \in R} x_i^*$$

or

$$v(S' \cup T') > v(S) + \frac{v(N) - v(S)}{|T|} |T'|.$$

But, this contradicts (b).

So, $x^* \in L(N)$. Given (c), if x is to Lorenz-dominate x^*, then there exists $i \in S$ such that $x_i < x_i^*$. But, then $x_i < v(\{i\})$, and $\{i\}$ L-blocks x. So, x^* is the WCEA of the game.

Since $x^* \in L(N)$ and $L(N) \subseteq L^*(N)$, $x^* \in L^*(N)$. It is easy to check, using (a) and (c), that x^* is the unique SCEA.

Suppose now that x is both the WCEA and the SCEA of (v, N). W.l.o.g. assume that $x_1 \geq x_2 \geq \cdots \geq x_n$. Now, if $x_1 = x_n$, then N is an equity coalition. So, the required partition is $S = \emptyset$, $T = N$. Condition α is satisfied since only (b) has to be checked in this case for $S' = \emptyset$.

So, let $x_1 > x_n$. Define $T = \{j | x_j = x_n\}$, and let $S = N - T$. We show that $\{S, T\}$ satisfies the required conditions.

Let i^* be such that $x_i^* = \sum_{i \in S} x_i$. By construction, $x_i^* > x_j$ for all $j \in T$. Since x is an SCEA, from Lemma 2, we know that there is R containing i^* such that (i) $R \subseteq T \cup \{i^*\}$ and (ii) $a(R) = x_i^*$. If $R \cap T \neq \emptyset$, then R L-blocks x^*, a contradiction. Hence, either $R = \{i^*\}$ or $x_i^* = v(\{i^*\}) > x_n \geq \max_{T' \subseteq T} a(T')$. Hence (c) is satisfied.

Similarly, pick any $i \in S$. Again, there will exist $R \subseteq \{j \in N | x_j < x_i\} \cup \{i\}$ with $a(R) = x_i$. If $R \cap \{j | x_j < x_i\} \neq \emptyset$ then R will L-block x, a contradiction. So $R = \{i\}$ and $x_i = v(\{i\})$. If $v(S') > \sum_{i \in S'} x_i$ for any $S' \subseteq S$, then S' will L-block x. So (a) is satisfied.

Suppose now that for some nonempty $T' \subseteq T$, and some $S' \subseteq S$, (b) is violated. Construct the following allocation y for $(S' \cup T')$.

$$\forall\, i \in S', \qquad y_i = v(\{i\}) = x_i$$

$$\forall\, i \in T', \qquad y_i = \frac{v(S' \cup T') - v(S')}{|T'|}.$$

If x is the WCEA for (v, N), then y must be the WCEA for $(S' \cup T')$. But, then $(S' \cup T')$ L-blocks x, a contradiction.

This shows the equivalence of (i) and (ii). The proof of the theorem is completed by showing the equivalence of (iii) and (i).

It is clear from (a) and (b) that x^* belongs to the core. Hence, (i) implies (iii).

Now, let x, the SCEA, belong to the core. W.o.l.g. suppose $x_1 \geq x_2 \geq \cdots \geq x_n$. If x is the equal division allocation e_N, then the trivial partition $\{N\}$ satisfies all the requirements of condition α.

So, suppose $x \neq e_N$. Then, just as in the previous step, let $T = \{j | x_j = x_n\}$ and $S = \{j | x_j > x_n\}$. The proof that $\{S, T\}$ constitutes the required partition is exactly the same as in the previous step. So. (iii) implies (i). ■

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