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A Consistent Bargaining Set*

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Both the core and the bargaining set fail to satisfy a natural requirement of consistency. In excluding imputations to which there exist objections, the core does not assess the "credibility" of such objections. The bargaining set goes a step further. Only objections which have no counterobjections are considered justified. However, the credibility of counterobjections is not similarly assessed. We formulate a notion of a consistent bargaining set in which each objection in a "chain" of objections is tested in precisely the same way as its predecessor. Various properties of the consistent bargaining set are also analyzed. *Journal of Economic Literature* Classification Numbers: 021, 022, 026. © 1989 Academic Press, Inc.

1. INTRODUCTION

The core and the bargaining set as solution concepts fail to satisfy, at least a priori, a natural requirement of consistency. Consider, for instance, the notion of the core. Let x be an imputation which is not in the core. In

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particular, suppose a coalition S^0 can ensure for its members a payoff vector x^0 which is higher than what these players were getting at the imputation x; i.e., (S^0, x^0) is an objection or a "threat" to x. However, is this threat "credible"? What is there to prevent the existence of a coalition S^1 and a pair (S^1, x^1) such that (S^1, x^1) is an objection to x^0 itself? Ray [8] and Greenberg [4] show that if x does not belong to the core, then there exists an objection (S^0, x^0) to x which is "internally stable" in the sense that no subset of S^0 can have an objection to x^0; i.e., there does exist an objection which is credible in so far as its subsets are concerned. In other words, the core is "internally consistent".

However, the above argument does not apply to the credibility of an objection in relation to coalitions which are not subsets of the original objecting coalition. Indeed, the bargaining set can be seen as a solution concept which attempts to modify the core to take account of this problem. We shall argue that even the bargaining set does not go far enough in this respect; while it does test objections against counterobjections, it does not similarly test the counterobjections or any further objections, and, in this sense, it is not consistent. The purpose of this paper is to analyze the consequences of imposing consistency requirements on the bargaining set. This leads to a new solution concept which we call the consistent bargaining set. It is a set which, in general, is larger than the core but smaller than the bargaining set.

Recall that the bargaining set[1] does not exclude an imputation x simply if it admits an objection. x is excluded from the bargaining set only if there exists an objection (S^0, x^0) which is justified in the sense that it does not admit a counterobjection, i.e., an objection (S^1, x^1) which is also an objection to x^0. While there are many variants of the original notion of the bargaining set introduced by Aumann and Maschler [1], we shall restrict ourselves to the one formulated by Mas-Colell [6], primarily because, for our purposes, it represents a simplification of the earlier notions.

Note that the Mas-Colell bargaining set can be partitioned into the set of core imputations and the set of those imputations to which there is no justified objection. But is an imputation in the latter set a reasonable or allowable allocation? An affirmative answer clearly presumes that the counterobjection is always "justified" so that if it exists, the original objection is *not* justified. But any notion of consistency would demand that the counterobjection be *itself* judged on terms similar to those of the objection! In fact, as we show in the next section, for any imputation which belongs to the bargaining set but not to the core, there must exist a counterobjection to an objection which is itself followed by yet another objection. So the validity of counterobjections should not, in general, be allowed to

[1] A precise definition follows in the next section.

go unchallenged. Mas-Colell [6] does consider a modification in which objections are tested only with respect to counterobjections which are themselves justified in the sense that there exists no further objection to the counterobjection. As he points out, this definition "iterates one more step the objection–counterobjection logic." But why stop at just one more iteration?[2] In the interests of consistency we ought not to disregard a counterobjection simply because it has a further objection; if there is a further objection that, too, should be tested in a consistent manner.

We therefore formulate a solution concept that is consistent in the sense that every objection in a "chain" of objections is tested in precisely the same way as its predecessor.

By scrutinizing counterobjections, and hence making them more difficult, our solution concept yields a set of imputations which is smaller than the bargaining set. Of course, it contains the core. In Section 3 we provide an example where our solution lies *strictly* between the bargaining set and the core. This demonstrates that, in general, both the core and the bargaining set are not consistent; the core is not consistent because it leaves out imputations which have only invalid objections against them and the bargaining set is not consistence because it includes imputations to which every objection has only invalid counterobjections. In the same section we also provide some classes of games in which the bargaining set coincides with our notion of a consistent bargaining set. These are three person superadditive games, ordinal convex games, and games generated from exchange economies with a continuum of agents. These also turn out to be classes of games in which the consistent bagaining set exists.

Since the consistent bargaining set contains the core, existence is certainly guaranteed in all games where the latter exists. One might also expect to be able to establish existence under weaker conditions than those typically used for showing the existence of the core. This seems not to be the case. While the bargaining set is non-empty in transferable utility games (see section VI of Mas-Colell [6]), the same is not true for the consistent bargaining set. In Section 4 we provide a four player example of a superadditive, transferable utility game in which the consistent bargaining set is empty.

In Section 5 we examine some other properties of the consistent bargaining set. We show that it is covariant and symmetric but need not satisfy individual rationality or the reduced game property of Davis and Maschler [2].

[2] In fairness to Mas-Colell, we should emphasize that in his context, namely of an exchange economy with a continuum of agents, his equivalence result implies that the bargaining set is consistent (see also Section 3.3 below).

2. Consistent Objections

2.1. *The Model and Assumptions*

Consider a game with n players where $N = \{1, ..., n\}$ denotes the set of *players* and $\mathcal{N}$ the set of all non-empty subsets of N. A *coalition* S is an element of $\mathcal{N}$. For any coalition $S \in \mathcal{N}$, let R^s denote the $|S|$-dimensional Euclidean space with coordinates indexed by the elements of S. For $x \in R^N$, x_S will denote its projection on R^S. Each coalition S has a feasible set of payoffs or utilities denoted $V(S) \subseteq R^S$. A non-side payment game in characteristic function form can now be denoted (N, V).

For $x, y \in R^S$, we write $x \gg y$ if $x_i > y_i$ for all $i \in S$, $x \geqslant y$ if $x_i \geqslant y_i$ for all $i \in S$, and $x > y$ if $x \geqslant y$ and $x \neq y$. For any set $B \subseteq R^S$, Bdry B denotes the boundary of B and Int B, the interior of B. Let $B^* = \{x \in B \mid \nexists y \in B$ such that $y > x\}$. We say that $x \in R^S$ is an *imputation for* S if $x \in V^*(S)$. $x \in R^N$ is said to be an *imputation* if it is an imputation for N.

A pair (S^0, x^0), where $S^0 \in \mathcal{N}$ and $x^0 \in V^*(S^0)$, is said to be an *objection*[3] to an imputation x if $x^0 > x_{S^0}$.

The *core* of a game V is defined as

$$C(V) = \{x \in V^*(N) \mid \nexists (S^0, x^0) \text{ which is an objection to } x\}.$$

Let (S^0, x^0) be an objection to x. (S^1, x^1), where $S^1 \in \mathcal{N}$ and $x^1 \in V^*(S^1)$, is said to be a *counterobjection* to (S^0, x^0) if $x_i^1 \geqslant x_i^0$ for all $i \in S^0 \cap S^1$, $x_i^1 \geqslant x_i$ for all $i \in S^1 \backslash S^0$, and at least one of these inequalities is strict.

An objection (S^0, x^0) to x is said to be a *justified objection* if there does not exist any counterobjection to (S^0, x^0).

We can now define the *bargaining set* as in Mas-Colell [6].

The bargaining set of a game V is defined as

$$B(V) = \{x \in V^*(N) \mid \nexists (S^0, x^0) \text{ which is a justified objection to } x\}.$$

We shall make use of the following assumptions.

(A1) For all $S \in \mathcal{N}$, $V(S)$ is closed; it is comprehensive in the sense that $V(S) = V(S) - R_+^S$; $0 \in V(S)$; there exists a real number m such that if $u \in V(S)$ and $u \geqslant 0$, then $m > u_j$ for all $j \in S$.

(A2) (Superadditivity) For any two disjoint coalitions S, $T \in \mathcal{N}$, if $x \in V(S)$, $y \in V(T)$, then $(x, y) \in V(S \cup T)$.

[3] Notice that instead of the usual requirement $x^0 \in V(S^0)$, we require that $x^0 \in V^*(S^0)$. The reason for this will become clear in the next section when we define the consistent bargaining set. It should, however, be clear that this modification leaves the core and the bargaining set unchanged.

(A3) For all $S \in \mathcal{N}$ if $x, y \in V(S)$ and $x > y$, then there exists $\bar{x} \in V(S)$ such that $\bar{x} \gg y$.

While (A1) and (A2) are standard assumptions, for some of our results we shall also make use of (A3). Given comprehensiveness, (A3) implies that the boundary of $V(S)$ does not become parallel to any of the coordinate axes. This means that given any feasible utility vector for a coalition, it is possible to increase the utility of any player in the coalition, possibly by reducing the utility of some other player. In exchange economies, it is implied by the strict monotonicity of preferences. It should also be clear that (A3) is satisfied in all transferable utility games.[4]

2.2. *The Consistent Bargaining Set*

We shall need to introduce some additional notation before describing our solution concept. Define a *collection* $\mathcal{A}$ as $\{x; (S^i, x^i)_{i=0}^m\}$, where x is an imputation and, for each $i = 0, ..., m$, x^i is an imputation for S^i. Define $b(\mathcal{A}) \in R^N$ by

$$b_i(\mathcal{A}) = \begin{cases} \max\{x_i, x_i^j \mid i \in S^j, j = 0, ..., m\}, & \text{if } i \in \bigcup_{j=0}^{m} S^j; \\ x_i, & \text{otherwise.} \end{cases}$$

We will sometimes find it more convenient to refer to $b(\mathcal{A})$ as $b(x, x^0, ..., x^m)$. A pair $(\hat{S}, \hat{x})$, where $\hat{S} \in \mathcal{N}$ and $\hat{x} \in V^*(\hat{S})$, is an *objection to the collection* $\mathcal{A}$ if

$$\hat{x} > b_{\hat{S}}(\mathcal{A}).$$

If $(\hat{S}, \hat{x})$ is an objection to the collection $\mathcal{A}$, we shall also say that $\hat{S}$ objects to $\mathcal{A}$.

A collection $\{x; (S^i, x^i)_{i=0}^m\}$ is a *chain* if (S^0, x^0) is an objection to x, and for each $i = 0, 1, ..., m$, (S^i, x^i) is an objection to the collection $\mathcal{A}^{i-1} = \{x; (S^j, x^j)_{j=0}^{i-1}\}$. A *chain of* x refers to a chain with x as the imputation. The *length* of a chain $\mathcal{A}$ is the number of coalitions which appear in $\mathcal{A}$.

A pair $(\hat{S}, \hat{x})$ is a *terminating objection to the chain* $\mathcal{A} = \{x; (S^i, x^i)_{i=0}^m\}$ if it is an objection to $\mathcal{A}$ such that there is no objection to the chain $\{x; (S^i, x^i)_{i=0}^m, (\hat{S}, \hat{x})\}$. $\mathcal{A}$ is a *terminating chain* if there is no objection to $\mathcal{A}$.

Suppose the imputation x is proposed initially and an objection (S^0, x^0) is raised to x. A counterobjection to this objection and further counterobjections can be arranged in the form of a chain. Given that the number of coalitions is finite and given that the objection/counterobjection is drawn from the set of imputations of the dissenting coalition, it follows that

[4] Recall that a transferable utility game is one in which each coalition $S \in \mathcal{N}$ is assigned a real number $v(S)$ such that $V(S) = \{x \in R^S \mid \sum_{i \in S} x_i \leqslant v(S)\}$.

all chains must be of finite length. Ultimately, there must be a terminating objection. We use this fact to assess the validity of the original objection.

Let $\mathscr{A}$ be a chain, and $(\hat{S}, \hat{x})$ an objection to $\mathscr{A}$ (for economy of notation we will also consider the collection $\{x\}$ to be a chain). To the objection $(\hat{S}, \hat{x})$ we shall give a label: *valid* or *invalid.* The labelling must satisfy the following property:

(P) An objection $(\hat{S}, \hat{x})$ to $\mathscr{A}$ is *valid* if there is no valid objection to $\{\mathscr{A}, (\hat{S}, \hat{x})\}$. It is *invalid* if there exists a valid objection to $\{\mathscr{A}, (\hat{S}, \hat{x})\}$.

It is easy to verify the truth of the following:

Fact. Under (A1), there is a unique "labelling" satisfying property (P).

As we mentioned above, all chains of objections must eventually terminate. Suppose $(\hat{S}, \hat{x})$ is a terminating objection to $\mathscr{A}$. Then, by (P) it is valid. We may now work backwards from the valid terminating objections to uniquely determine the "label" of each objection. This is the intuition behind the Fact.

Notice that the validity or invalidity of an objection is to be assessed in the context of the chain that it objects to, since it is the chain that places restrictions on the kinds of counterobjections that can follow upon the objection under consideration.[5]

We are now in a position to define the *consistent bargaining set.* This is the set

$$\mathrm{CB}(V) = \{x \in V^*(N) \mid \nexists \text{ a valid objection to } x\}.$$

An example illustrating this solution concept is provided in Section 3.1 below. Embedded in the concept of a valid objection is the consistency requirement outlined in Section 1. Counterobjections are now being assessed in exactly the same way as the objection itself. In contrast, the bargaining set $B(V)$ and its close cousins ignore the validity of the counterobjection, and hence fail to satisfy the consistency requirement.

An objection is clearly valid if it is justified. Hence

$$C(V) \subseteq \mathrm{CB}(V) \subseteq B(V), \tag{1}$$

so that the consistent bargaining set is sandwiched between the core and the bargaining set.

Recall that if x belongs to $B(V)$ but not to $C(V)$, then there exist chains of x of positive length and all such chains must be at least of length 2. We shall now show that if (A1)–(A3) are satisfied, then there must exist a chain of x which has length greater than 2. In other words, there *must* exist an

[5] The same observation also applies to the criterion by which an objection is judged to be justified or not; it depends on the objection as well as the original imputation, i.e., on the chain.

objection having a counterobjection such that there is a further objection following the counterobjection. Put another way, there must exist a counterobjection which is not terminating, i.e., not justified in the sense of Mas-Colell [6]. While this does not by itself imply that the objection–counterobjection logic is never enough to check the validity of an objection (indeed, in the next section we provide some cases in which it is), it certainly provides an important motivation for studying the consistent bargaining set introduced above.

PROPOSITION 2.2. *Suppose* (N, V) *satisfies* (A1)–(A3) *and* $x \in B(V) \backslash C(V)$. *Then there exists a chain of* x *of length greater than* 2.

Proof. Since $x \in B(V) \backslash C(V)$ there exist chains of x of positive length, and any such chain must be at least of length 2. Suppose the conclusion of the Proposition is false. Then every chain of positive length must be *exactly* of length 2. Let $\{x; (S^0, x^0), (S^1, x^1)\}$ be such a chain. First observe that $S^0 \cap S^1 \neq \varnothing$. Otherwise, from (A1) and (A2) it would follow that $S^0 \cup S^1$ has a justified objection to x.

Let $b^0 = x_{S^0}$. Since S^0 has an objection to x, it follows from (A1) and (A3) that $b^0 \in \operatorname{Int} V(S^0)$. Let $B = S^0 \cap S^1$ and $e_B^0 \in R^{S^0}$ be the vector which has 1 in every coordinate $i \in B$ and 0 elsewhere. By (A1) and (A3) we can now find a real number $t^0 > 0$ such that

$$\bar{x}^0 = b^0 + t^0 e_B^0 \in V^*(S^0).$$

Similarly, by letting $b^1 = x_{S^1}$ and defining e_B^1 to be a vector in R^{S^1}, we can find a real number $t^1 > 0$ such that

$$\bar{x}^1 = b^1 + t^1 e_B^1 \in V^*(S^1).$$

There are three possibilities; $t^1 > t^0$, $t^1 < t^0$, or $t^1 = t^0$.

Suppose $t^1 > t^0$. Then

$$\bar{x}^1 > b_{S^1}(x, \bar{x}^0) \tag{2}$$

and

$$b(x, \bar{x}^1) = b(x, \bar{x}^0, \bar{x}^1). \tag{3}$$

From (2) and the fact that $t^1 > t^0 > 0$ it follows that $\{x; (S^0, \bar{x}^0), (S^1, \bar{x}^1)\}$ is a chain. By hypothesis it must be a terminal chain; i.e., there does not exist $(\hat{S}, \hat{x})$ such that $\hat{x} > b_{\hat{S}}(x, \bar{x}^0, \bar{x}^1) = b_{\hat{S}}(x, \bar{x}^1)$. Since $\{x; (S^1, \bar{x}^1)\}$ is also a chain, this must mean that it is a terminal chain. But this contradicts the hypothesis that $x \in B(V)$.

If $t^1 < t^0$ we can use the same argument as above to show that $\{x; (S^0, \bar{x}^0)\}$ is a terminal chain, which again contradicts the hypothesis that $x \in B(V)$.

Suppose $t^1 = t^0$. We begin by showing that $S^0 \backslash B \neq \emptyset$. Suppose not. Then $S^0 \subseteq S^1$ and it is easy to see that $b(x, x^0, x^1) = b(x, x^1)$. Since $\{x; (S^0, x^0), (S^1, x^1)\}$ is a terminating chain, this must mean that so is $\{x; (S^1, x^1)\}$, which contradicts the hypothesis that $x \in B(V)$.

Since there exists $i \in S^0 \backslash B$ and we know that $b^0 \in \text{Int } V^*(S^0)$, there exists a real number $\bar{\varepsilon} > 0$ such that for all $\varepsilon \leqslant \bar{\varepsilon}$, $b^0 + \varepsilon e_i^0 \in \text{Int } V^*(S^0)$. By (A1) and (A3), for every $\varepsilon \leqslant \bar{\varepsilon}$, there exists $t(\varepsilon) > 0$ such that

$$\bar{x}^0(\varepsilon) = b^0 + \varepsilon e_i^0 + t(\varepsilon)\, e_B^0 \in V^*(S^0).$$

Certainly, for every $\varepsilon > 0$, $t(\varepsilon) < t^0 = t^1$. Thus, for every $0 < \varepsilon < \bar{\varepsilon}$, $\{x; (S^0, \bar{x}^0(\varepsilon)), (S^1, \bar{x}^1)\}$ is a chain, and by hypothesis, a terminal chain. We shall now show that $\{x; (S^1, \bar{x}^1)\}$ is a terminal chain which provides the necessary contradiction for completing the proof. Certainly, $\{x; (S^1, \bar{x}^1)\}$ is a chain. Suppose it is not a terminal chain. Then there exists $(\hat{S}, \hat{x})$ such that $\hat{x} \in V(\hat{S})$ and

$$\hat{x} > b_{\hat{S}}(x, \bar{x}^1).$$

By (A1) and (A3) we can now find $\bar{x} \in V^*(\hat{S})$ such that

$$\bar{x} \gg b_{\hat{S}}(x, \bar{x}^1).$$

By the construction of $\bar{x}^0(\varepsilon)$, this implies that for some $0 < \varepsilon < \bar{\varepsilon}$,

$$\bar{x} \gg b_{\hat{S}}(x, \bar{x}^0(\varepsilon), \bar{x}^1).$$

But this implies that for some $0 < \varepsilon < \bar{\varepsilon}$, $\{x; (S^0, \bar{x}^0(\varepsilon)), (S^1, \bar{x}^1)\}$ is not a terminal chain—a contradiction. ∎

3. Consistency and the Bargaining Set

In this section, we examine the circumstances in which the bargaining set is consistent; i.e., we provide classes of games in which $B(V)$ and $\text{CB}(V)$ coincide. These also turn out to be cases in which $\text{CB}(V)$ is non-empty. We start by showing that there exist transferable utility games where $\text{CB}(V)$ lies *strictly* between $B(V)$ and $C(V)$. This shows that $B(V)$ and $C(V)$ are *not*, in general, consistent. In Sections 3.2 and 3.3 we show that $B(V)$ and $CB(V)$ do coincide in all 3-person superadditive games as well as in the class of ordinally convex games. Section 3.4 contains a discussion of continuum games.

The following lemma will prove useful in this section.

LEMMA 3.0. *Assume that* (N, V) *satisfies* (A1) *and* (A3). *Let* $\mathscr{A}$ *be a chain. Suppose* $\mathscr{S} = \{S^0, ..., S^m\}$ *is the collection of all coalitions which can object to* $\mathscr{A}$ *and* $\bigcap_{i=0}^m S^i \neq \varnothing$. *Then there exists an element of* $\mathscr{S}$ *which has a terminating objection to* $\mathscr{A}$.

Proof. For $i = 0, ..., m$, let $b^i = b_{S^i}(\mathscr{A})$ and let $B = \bigcap_{i=0}^m S^i$. By the same argument as in the proof of Proposition 2.2, for every $S^i \in \mathscr{S}$ we can find a real number $t^i > 0$ such that

$$\bar{x}^i = b^i + t^i e^i_B \in V^*(S^i).$$

Let S^j be a coalition with a maximal t^i. It is now easy to see that $\{\mathscr{A}, (S^j, \bar{x}^j)\}$ is a terminating chain. ∎

3.1. *An Example*

Here we give an example to illustrate our solution concept and to demonstrate that there are cases in which CB(V) lies *strictly* between the core and the bargaining set.

The example is a variant of an example in Mas-Colell [6]. (N, v) is a transferable utility game, with $N = \{1, 2, 3, 4\}$. The characteristic function v is the minimal superadditive function compatible with the values $v(N) = 4$, $v(\{1, 2, 3\}) = 3.03$, $v(\{2, 3, 4\} = 3.06$, $v(\{2, 4\}) = v(\{3, 4\}) = 2.06$ and $v(\{2\}) = 1$. This game has a non-empty core. For instance, the imputation (0, 1.6, 1.6, 0.8) has no objection.

The imputation $x = (1, 1, 1, 1)$ is not in the core. The coalitions $\{1, 2, 3\}$, $\{2, 4\}$, $\{3, 4\}$, and $\{2, 3, 4\}$ can all object to x. However, x is in the bargaining set because none of these coalitions has a justified objection. For example, if $\{1, 2, 3\}$ objects, it has a counterobjection from $\{2, 4\}$ and $\{3, 4\}$. If $\{3, 4\}$ objects, there must be a counterobjection from either $\{2, 4\}$ or $\{1, 2, 3\}$.

Nevertheless, there is a valid objection to this imputation, so that it is not in the consistent bargaining set. This is the objection $(\{2, 3, 4\}, x^0)$, where $x^0 = (1.01, 1.01, 1.04)$. The coalitions which can object to $\{x; (\{2, 3, 4\}, x^0)\}$ are $\{1, 2, 3\}$, $\{2, 4\}$, and $\{3, 4\}$. But no coalition can terminate the chain. Moreover, any two of these have a non-empty intersection and so, by Lemma 3.0, every objection to $\{x; (\{2, 3, 4\}, x^0)\}$ can be terminated. Thus this is a valid objection.

Now consider the imputation (0.94, 1.03, 1.03, 1) which does not belong to the core. In fact, the coalitions $\{1, 2, 3\}$, $\{2, 4\}$, $\{3, 4\}$ can object. And these are the only coalitions which can object. It is also easy to check that none of these can have a justified objection. Moreover, the remainder have a non-empty intersection and, again, by Lemma 3.0, this implies that every objection can be terminated and is, therefore, invalid. Thus this imputation

lies in the consistent bargaining set but not in the core; i.e., the former strictly contains the latter.

We have, therefore, demonstrated that there exist games satisfying

$$C(V) \subset \mathrm{CB}(V) \subset B(V).$$

3.2. *Three-Person Games*

PROPOSITION 3.2.1. *Let* (N, V) *be a three person game satisfying* (A1)–(A3). *Then* $B(V) = \mathrm{CB}(V)$.

Proof. It suffices to show that $B(V) \subseteq \mathrm{CB}(V)$. Suppose not. Then there exists $x \in B(V) \backslash \mathrm{CB}(V)$; x has no justified objection but does have a valid objection. Let (S^0, x^0) be such an objection. Let $\mathscr{S}$ be the collection of all coalitions which have objections to $\mathscr{A} = \{x; (S^0, x^0)\}$. Because x has no justified objection, $\mathscr{S} \neq \varnothing$. Now we claim that $\bigcap_{S \in \mathscr{S}} S \neq \varnothing$. To see this, we first make the following observations:

(1) If S can object to $\mathscr{A}$ it can object to x,

(2) Given (A2), if $S, T \in \mathscr{S}$ and $S \cap T = \varnothing$, then $S \cup T \in \mathscr{S}$, and hence,

(3) For $i \neq j \neq k$ it is impossible that $\{i\}$ and $\{j, k\}$ can both object to x, or that $\{i\}$, $\{j\}$ and $\{k\}$ can all object to x.

To establish our claim, suppose to the contrary that $\bigcap_{S \in \mathscr{S}} S = \varnothing$. Then using observations (1), (2), and (3), it is immediately seen that $\mathscr{S}$ can only take one of the following two forms:

(a) $\mathscr{S} = \{\{i, j\}, \{i\}, \{j\}\}$ or

(b) $\mathscr{S} = \{\{i, j\}, \{j, k\}, \{i, k\}\}$.

In case (a) S^0 can only be $\{i, k\}$, $\{j, k\}$, or $\{k\}$. In either case, using observation (1), we contradict observation (3). In case (b) S^0 must be a singleton, say $\{i\}$. But again using observation (1) we contradict observation (3).

Thus $\bigcap_{S \in \mathscr{S}} S \neq \varnothing$. We can now appeal to Lemma 3.0 to assert that there exists a terminating chain $\{x; (S^0, x^0), (\hat{S}, \hat{x})\}$, where $\hat{S} \in \mathscr{S}$. But this contradicts the supposition that (S^0, x^0) is a valid objection. ∎

Remark 3.2. Assumption (A3) cannot be dispensed with in the above proposition. Consider, for instance, the following three-person game satisfying (A1) and (A2):

$N = \{1, 2, 3\}$; $V(N) = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 \leqslant 1.5\}$; $V(\{i\}) = \{x \mid x \leqslant 0\}$ for all $i = 1, 2, 3$; $V(\{1, i\}) = \{(x_1, x_i) \mid x_1 \leqslant 0, x_i \leqslant 1\}$, for $i = 2, 3$; $V(\{2, 3\}) = \{(x_2, x_3) \mid x_2 \leqslant 0,\ x_3 \leqslant 0\}$. This game satisfies (A1) and (A2)

but not (A3). The reader can also check that the imputation $x = (-\frac{1}{4}, \frac{7}{8}, \frac{7}{8})$ is in $B(V)$. But $(\{1\}, 0)$ is a valid objection to x, so that $x \notin \mathrm{CB}(V)$.

PROPOSITION 3.2.2. *If the conditions of Proposition* 3.2.1 *are satisfied, then* $\mathrm{CB}(V) \neq \varnothing$.

Proof. This follows immediately from Proposition 3.2.1 and the fact that in all three-person games satisfying (A1) and (A3), $B(V) \neq \varnothing$ (see Vohra [11]). ∎

3.3. *Ordinally Convex Games*

In this section we show that the bargaining set is consistent if (N, V) is *ordinally convex*. Indeed we show that in this case $B(V)$, and hence $\mathrm{CB}(V)$, coincides with $C(V)$.

(A4) (*Ordinal Convexity*); For all $S, T \in \mathcal{N}$ and $u \in R^N$ such that $u_S \in V(S)$ and $u_T \in V(T)$, either $u_{S \cup T} \in V(S \cup T)$ or $u_{S \cap T} \in V(S \cap T)$.

Remark 3.3.1. Ordinally convex games have been studied by Vilkov [10], Sharkey [9], and Peleg [7]. It is easy to check that convex, transferable utility games satisfy (A4). The core is non-empty for ordinally convex games (see Greenberg [3]) and, therefore, so is the consistent bargaining set.

The following lemma will be useful in proving the main result of this section.

LEMMA 3.3. *Given* (A1), (A3), *and* (A4), *for all* $S, T \in \mathcal{N}$ *and for all* $u \in R^N$ *such that* $u_S \in V(S)$ *and* $u_T \in V(T)$ *either* $u_{S \cup T} \in V(S \cup T)$ *or* $u_{S \cap T} \in \mathrm{Int}\, V(S \cap T)$.

Proof. Pick any $u \in R^N$ such that $u_S \in V(S)$ and $u_T \in V(T)$. By (A4) either $u_{S \cup T} \in V(S \cup T)$ or $u_{S \cap T} \in V(S \cap T)$. If $u_{S \cup T} \in V(S \cup T)$ or $u_{S \cap T} \in \mathrm{Int}\, V(S \cap T)$ then there is nothing to prove. So suppose $u_{S \cup T} \notin V(S \cup T)$ and $u_{S \cap T} \in \mathrm{Bdry}\, V(S \cap T)$. Construct $u' \in R^N$ such that

$$\text{(i) } u'_{S \setminus T} < u_{S \setminus T} \qquad \text{(ii) } u'_{T \setminus S} < u_{T \setminus S} \qquad \text{(iii) } u'_{S \cap T} > u_{S \cap T}$$

$$\text{(iv) } u'_S \in V(S) \qquad \text{(v) } u'_T \in V(T) \qquad \text{(vi) } u'_{S \cup T} \notin V(S \cup T).$$

Given (A1) and (A3), the reader can check that such a u' exists. Moreover, since $u_{S \cap T} \in \mathrm{Bdry}\, V(S \cap T)$, (iii) implies that $u'_{S \cap T} \notin V(S \cap T)$. But this along with (iv), (v), and (vi) implies a violation of (A4).

PROPOSITION 3.3. *Let* (N, V) *satisfy* (A1), (A3), *and* (A4). *Then* $B(V) = \mathrm{CB}(V) = C(V)$.

Proof. It is clearly sufficient to prove that $B(V) \subseteq C(V)$. Suppose this is not true. Then there exists $x \in B(V) \backslash C(V)$. So there exists a coalition which objects to x. For any $S \in \mathcal{N}$ let $C_S(V)$ denote the core of the game consisting of the players in S with V restricted to S. By considering a minimal coalition which objects to x, it is easy to see that the collection

$$\mathcal{S} = \{S \in \mathcal{N} \mid \exists y \in C_S(V) \text{ such that } y > x_S\}$$

is non-empty. Let S^* be a maximal coalition in $\mathcal{S}$, i.e., $S^* \in \mathcal{S}$ and if $S^* \subset T$, then $T \notin \mathcal{S}$. Pick $y^* \in C_{S^*}(V)$ such that (S^*, y^*) is an objection to x. Because $x \in B(V)$, there must exist some coalition T which has a counterobjection. Let $\mathcal{T}$ be the collection of all coalitions which have a counterobjection to $(x; (S^*, y^*)$. Note that for any $T \in \mathcal{T}$, $T \backslash S^* \neq \emptyset$. Let $e(T)$ be the vector in $R^{T \backslash S^*}$ consisting of 1 in every coordinate. From (A1) and (A3), for every $T \in \mathcal{T}$, there exists a real number $t(T) > 0$ such that

$$(x_{T \backslash S^*} + t(T)\, e(T), y^*_{S^* \cap T}) \in V^*(T).$$

Let $T^* \in \mathcal{T}$ be such that $t(T^*) \geqslant t(T)$ for all $T \in \mathcal{T}$. Because $y^* \in C_{S^*}(V)$, $y^*_{S^* \cap T^*} \notin \operatorname{Int} V(S^* \cap T^*)$. From Lemma 3.3, it must be the case that

$$(x_{T^* \backslash S^*} + t(T^*)\, e(T^*), y^*) \in V(S^* \cup T^*).$$

Pick any $z \in V^*(T^* \cup S^*)$ such that $z \geqslant (x_{T^* \backslash S^*} + t(T^*)\, e(T^*), y^*)$. We shall now prove that $z \in C_{S^* \cup T^*}(V)$. Suppose not. Then there exist $W \subseteq S^* \cup T^*$ and $w \in V(W)$ such that $w > z_W$. Since $y^* \in C_{S^*}(V)$, $W \backslash S^* \neq \emptyset$. Since $w > z_W$, $W \in \mathcal{T}$. But, because $w > z_W$, $t(W) > t(T^*)$, which contradicts the definition of T^*.

Hence, $z \in C_{S^* \cup T^*}(V)$. Moreover, it is clear that $(S^* \cup T^*, z)$ is an objection to x. But this contradicts the definition of S^* since $|S^* \cup T^*| > |S^*|$. ∎

Remark 3.3.2. It is worth pointing out that the equivalence of the Aumann–Maschler bargaining set and the core in convex, transferable utility games was proved by Maschler, Peleg, and Shapley [5]. The proof of Proposition 3.3 also serves as an alternative proof of the Maschler–Peleg–Shapley result.

3.4. *Continuum Games*

An important class of games in economics is the one consisting of games generated from exchange economies with a continuum of agents.

In Section 2 we defined a notion of a consistent bargaining set for games with a finite number of players. A direct extension of that definition to games with a continuum of players does not seem possible; due to the presence of chains of infinite length, it may no longer be possible to

apply our earlier definition to check whether an objection is valid or invalid. Notice, however, that simply the presence of an infinite chain $(x; (S^0, x^0), \ldots)$ by itself does not mean that our earlier criterion cannot be applied to check whether (S^0, x^0) is valid or invalid. For example, it may be the case that there exists a terminating chain $(x; S^0, x^0), (S^1, x^1))$, in which case (S^0, x^0) is invalid. In general, our original classification of objections as valid or invalid may no longer be exhaustive. Clearly, we would include in the consistent bargaining set any imputation to which all possible objections are invalid and exclude any imputation to which there exists a valid objection. In other words,

$$\{x \in V^*(N) \mid \nexists \text{ a valid objection to } x\} \supseteq \mathrm{CB}(V)$$
$$\supseteq \{x \in V^*(N) \mid \text{any objection to } x \text{ is invalid}\}. \tag{4}$$

While it would be of some interest to formulate a definition which classifies objections as valid or invalid in an exhaustive and mutually exclusive manner, given (4), we do know that

$$C(V) \subseteq \mathrm{CB}(V) \subseteq B(V).$$

Mas-Colell [6] has shown that, under the standard assumptions, in exchange economies with a continuum of agents, $B(V) = C(V)$. This clearly implies that in this context, $C(V) = \mathrm{CB}(V) = B(V)$ and, hence, the bargaining set is consistent.

4. An Example where the Consistent Bargaining Set is Empty

In this section, we present an example of a superadditive, transferable utility game with four players in which the consistent bargaining set is empty.

Example 1. Let $N = \{1, 2, 3, 4\}$, $S^1 = \{1, 2, 3\}$, $S^2 = \{2, 3, 4\}$, $S^3 = \{1, 4\}$, $S^4 = \{1, 2, 4\}$, $S^5 = \{1, 3, 4\}$. Also, $v(S^1) = v(S^2) = 66$, $v(S^3) = 46$, $v(S^4) = v(S^5) = 63$, $v(N) = 80$, and for all other coalitions S, $v(S) = 0$. Notice that player 1 is symmetric to player 4, while player 2 is symmetric to player 3.

Throughout this section we shall be concerned only with the game defined in the above example.

Proposition 4.1. $\mathrm{CB}(V) = \varnothing$.

We shall need two lemmas concerning the game in Example 1 before we prove that its consistent bargaining set is empty.

LEMMA 4.1. *If $x \in \mathrm{CB}(V)$ and S^5 does not have an objection to x, then $\{S^1, S^2, S^3, S^4\}$ cannot all have objections to x.*

Proof. Suppose $\{S^1, S^2, S^3, S^4\}$ constitutes the set of objecting coalitions to x. Since S^5 cannot object, $x_2 \leqslant 17$. We first show that if $x \in \mathrm{CB}(V)$, then

$$\max\{x_1, x_4\} < 23. \tag{5}$$

Suppose not. Let $x_1 \leqslant x_4$ and $x_4 \geqslant 23$. Either $x_3 \leqslant 3 + x_4$ or $x_3 > 3 + x_4$. In the former case, (S^1, x^1) is a justified objection[6] where $x^1 = (46 - x_4, 17, 3 + x_4)$. In the latter case (S^4, x^4) is a justified objection where $x^4 = (46 - x_4, 17, x_4)$. Hence, remembering that players 1 and 4 are symmetric, (5) must hold.

Given (5), we must also have $x_3 < 26$. Otherwise, $(S^4, \bar{x})$ is a justified objection where $\bar{x} = (23, 17, 23)$. So suppose $x_3 = 26 - \varepsilon$ for some $\varepsilon > 0$ and $\max(x_1, x_4) = 23 - \delta$ for some $\delta > 0$. Choose γ such that $0 < \gamma < \min\{\varepsilon, 2\delta\}$. Consider the objection $(S^4, \tilde{x})$ where $\tilde{x} = (23 - \gamma/2,\ 17 + \gamma,\ 23 - \gamma/2)$. Then $b(x, \tilde{x}) = (23 - \gamma/2, 17 + \gamma, 26 - \varepsilon, 23 - \gamma/2)$. For $i = 1, 2, 3$, let the *excess* of coalition S^i over the chain $\mathscr{A} = \{x; (S^4, \tilde{x})\}$ be defined as $e^i = v(S^i) - \sum_{j \in S^i} b_j(x, \tilde{x})$. It is now easy to check that

$$e^1 = \varepsilon - \frac{\gamma}{2}, \qquad e^2 = \varepsilon - \frac{\gamma}{2}, \qquad \text{and} \qquad e^3 = \gamma.$$

Thus

$$e^i + e^j > e^k \qquad \text{for any} \quad i, j, k = 1, 2, 3. \tag{6}$$

Let $\mathscr{T} = \{S^1, S^2, S^3\}$. Note that

$$\bigcap_{S \in \mathscr{T}} S = \varnothing \tag{7}$$

while

$$S^i \cap S^j \neq \varnothing \qquad \text{for any } i, j = 1, 2, 3. \tag{8}$$

Since every coalition in $\mathscr{T}$ has a positive excess, it has an objection to the chain $\mathscr{A}$. Of course, any coalition which objects to $\mathscr{A}$ must be in $\mathscr{T}$. However, no objection (S^i, x^i) to the chain $\mathscr{A}$ can be a terminating objection. For if it is a terminating objection, given (7), we would contradict (6). Moreover, given an objection (S^i, x^i), there must be a terminating objection to the chain $\{\mathscr{A}, (S^i, x^i)\}$. This follows from Lemma 3.0 and (8). Thus

[6] Notice that $46 - x_4 > x_1$ since S^3 has an objection to x.

any objection to $\mathscr{A}$ is invalid; i.e., $(S^4, \tilde{x})$ is a valid objection to x, which means that $x \notin \mathrm{CB}(V)$.

LEMMA 4.2. *If* $x \in \mathrm{CB}(V)$, *then* $x \geqslant 0$.

Proof. *Step* 1. $x \in \mathrm{CB}(V)$ implies that $x_1 \geqslant 0$.

Suppose not. Clearly, S^2 cannot have an objection to x. Since $x \notin C(V)$, there must be some coalition with an objection to x. Let S be a coalition which has the maximum excess over x. By superadditivity, if $x_i < 0$, then $i \in S$. Let $S_I = \{i \mid x_i < 0\}$. Now consider $x' \in R^S$ such that $\sum_{i \in S} x'_i = v(S)$ and $x'_i = \max(0, x_i)$ for all $i \in S$, $i \neq 1$. Since S has the maximum excess over x, by superadditivity and the fact that $v\{i\} = 0$ for all i, such an x' must exist. We now claim that (S, x') is a justified objection to x. Suppose not. Then there exists a counterobjection (T, z). Since $b(x, x') \geqslant 0$ by construction, T must contain S^1 or S^3 or S^4 or S^5. Notice that player 1 belongs to all such coalitions. However, if $1 \in T$, then $T \cup S_I$ also has a counterobjection to (S, x'). But this must mean that $T \cup S_I$ has a higher excess than S, a contradiction. Thus $x \notin B(V)$, which contradicts the hypothesis that $x \in \mathrm{CB}(V)$.

Step 2. $x \in \mathrm{CB}(V)$ implies that $x_2 \geqslant 0$.

Suppose not. Clearly, then, S^5 cannot have an objection to x. By Lemma 4.1, we can now claim that $\{S^1, S^2, S^3, S^4\}$ cannot all have objections to x. Therefore, there can be at most three coalitions from $\mathscr{S} = \{S^1, S^2, S^3, S^4, S^5\}$ which have objections to x. The only collection of *three* coalitions in $\mathscr{S}$ which has an empty intersection is $\{S^1, S^2, S^3\}$. But given that $x_2 < 0$, if S^3 has an objection so must S^4. So this cannot be the collection of three coalitions from $\mathscr{S}$ having an objection to x. Thus, if $\{S^j\} \subseteq \mathscr{S}$ is a collection of objections to x, $\bigcap_j S^j \neq \varnothing$. Let $k \in \bigcap_j S^j$.

If there exists no objection to x from $\mathscr{S}$, it is easy to see that $(S_I, 0)$ is a justified objection to x. If there does exist an objection from $\mathscr{S}$ let S be a coalition with the maximum excess over x and let $x' \in R^S$ be such that $\sum_{i \in S} x'_i = v(S)$, and $x'_i = \max(0, x_i)$ for $i \neq k$. By the same argument as in Step 1 above, we can now show that (S, x') is a justified objection to x. We have, therefore, established that if $x_2 < 0$ there exists a justified objection to x, which contradicts the supposition that $x \in \mathrm{CB}(V)$.

Given the symmetry between players 1 and 4 and between players 2 and 3, the same argument as in Steps 1 and 2 can be used to show that if $x \in \mathrm{CB}(V)$ then $x_4 \geqslant 0$ and $x_3 \geqslant 0$. And this completes the proof. ∎

Proof of Proposition 4.1. From Lemma 4.2 we know that if $x \in \mathrm{CB}(V)$, then $x \geqslant 0$. We show that $\mathrm{CB}(V) = \varnothing$ by showing that any imputation $x \geqslant 0$ has a valid objection. Note that since (N, v) is not *balanced*, every imputation must have at least one objection. Let $\mathscr{S} = \{S^1, S^2, S^3, S^4, S^5\}$.

For any imputation $x \geqslant 0$, an objection can only come from a coalition in $\mathscr{S}$. Moreover, one of the following must hold:

(i) Exactly one coalition in $\mathscr{S}$ objects to x.

(ii) Exactly two coalitions in $\mathscr{S}$ object to x.

(iii) Exactly three coalitions in $\mathscr{S}$ object to x.

(iv) Exactly four coalitions in $\mathscr{S}$ object to x.

(v) All five coalitions in $\mathscr{S}$ object to x.

Clearly, in case (i), the objection is justified, so that $x \notin B(V)$ and hence $x \notin \mathrm{CB}(V)$. In case (ii), the two objecting coalitions must have a non-empty intersection (by super-additivity) so that by Lemma 3.0, $x \notin B(V)$. The only collection of *three* coalitions in $\mathscr{S}$ having an empty intersection is $\{S^1, S^2, S^3\}$. But if S^4 and S^5 cannot object, then $x_3 \leqslant 17$ and $x_2 \leqslant 17$. It follows that if S^3 can object so can either S^4 or S^5 (or both)! Therefore, if exactly three coalitions object, they must have a non-empty intersection and Lemma 3.0 can again be applied to show that $x \notin B(V)$. In view of Lemma 3.0, we, therefore, only need to consider collections of four and five coalitions in $\mathscr{S}$, each collection having an empty intersection.

In case (v), the only collections of *four* coalitions in $\mathscr{S}$ having an empty intersection are:

(a) $\{S^1, S^2, S^4, S^5\}$

(b) $\{S^1, S^2, S^3, S^4\}$

(c) $\{S^1, S^2, S^3, S^5\}$.

However, the collection in (iv a) cannot be the set of objecting coalitions, for the fact that S^3 cannot object implies that $x_2 + x_3 \leqslant 34$. On the other hand, the fact that S^4 and S^5 can object implies that $x_3 > 17$ and $x_2 > 17$. Hence $x_2 + x_3 > 34$, a contradiction.

We have already shown in Lemma 4.1 that (iv b) is not possible. Since players 2 and 3 are symmetric, the same argument can also be used to show that case (iv c) is impossible.

Hence, if $x \in \mathrm{CB}(V)$, then all five coalitions in $\mathscr{S}$ must object to x. We begin by showing that in this case

$$\max\{x_1, x_4\} = say\ x_4 < 23. \tag{9}$$

Suppose (9) is not true, so that $x_4 \geqslant 23$. Since S^3 can object to x, $x_1 + x_4 < 46$, which implies that $x_1 < 46 - x_4 \leqslant x_4$. Also, since S^2 can object, $x_2 + x_3 + x_4 < 66$, which in turn implies that $x_3 < 66 - (46 - x_4) - x_2$. Hence, (S^1, x^1) is an objection where $x^1 = (46 - x_4, x_2, 20 + x_4 - x_2)$. We now claim that (S^1, x^1) is a justified objection to x. Note that $b(x, x^1) = (46 - x_4, x_2, 20 - x_2 + x_4, x_4)$. Also, since S^4 and S^5 can object to

x, $x_2 > 17$, $x_3 > 17$. Hence, for any $S \in \mathscr{S}$, $v(S) \leqslant \sum_{i \in S} b_i(x, x^1)$. Thus, there cannot be a counterobjection to (S^1, x^1). This proves that if $x \in \mathrm{CB}(V)$, then (9) must hold.

Next, we prove that

$$x_2 + x_3 < 43. \tag{10}$$

Suppose $x_2 + x_3 \geqslant 43$. Then, in view of (9), (S^3, x^3) is an objection where $x_1^3 = x_4^3 = 23$. Noting that $b(x, x^3) = (23, x_2, x_3, 23)$, $x_2 + x_3 \geqslant 43$ and the fact that $\min\{x_2, x_3\} > 17$, we can check that $v(S) \leqslant \sum_{i \in S} b_i(x, x^3)$ for all $S \in \mathscr{S}$. Hence, (S^3, x^3) must be a justified objection, which contradicts the supposition that $x \in \mathrm{CB}(V)$.

Now, $\min(x_2, x_3) > 17$ and (10) implies that $\max(x_2, x_3) < 26$. Without loss of generality, assume $x_2 = \min(x_2, x_3)$, $x_1 = \min(x_1, x_4)$. Let $x_2 = 17 + \gamma$, $x_3 = 26 - \varepsilon$. By (10), $\varepsilon > \gamma > 0$. Now, consider the objection $(S^4, \tilde{x})$, where $\tilde{x}_1 = 46 - \gamma - \tilde{x}_4$, $\tilde{x}_2 = 17 + \gamma$, and $\tilde{x}_4 = \max((46 - \gamma)/2, x_4)$. Note that S^1, S^2, and S^3 can object to $\{x; (S^4, \tilde{x})\}$. It is also easy to check that the excesses of these coalitions are

$$e^1 = \varepsilon - (23 - \tilde{x}_4) \qquad e^2 = 23 + (\varepsilon - \gamma) - \tilde{x}_4 \qquad e^3 = \gamma.$$

Hence,

$$e^i + e^j > e^k \qquad \text{for all} \quad i, j, k = 1, 2, 3.$$

We can now use the same argument as in the proof of Lemma 4.1 to show that any objection to $\{x; (S^4, \tilde{x}_4)\}$ is invalid; i.e., $(S^4, \tilde{x}_4)$ is a valid objection to x and $x \notin \mathrm{CB}(V)$. This completes the proof that there does not exist any imputation $x \in \mathrm{CB}(V)$. ∎

5. Other Properties of the Consistent Bargaining Set[7]

In this section, we examine whether $\mathrm{CB}(V)$ satisfies some classical properties of cooperative game theory. We focus attention on the following properties: covariance, symmetry, individual rationality, and the reduced game property of Davis and Maschler [2].

Let Γ be a set of games. A *solution* of Γ is mapping σ which assigns to each game $(N, V) \in \Gamma$ a subset $\sigma(N, V)$ of $V(N)$.

For any solution σ, we may now define the properties of interest to us:

Covariance: For all $a \in R^N$, if $(N, V + \{a\})$ is defined by $(V + \{a\})(S) = V(S) + a_S$, $S \in \mathscr{N}$, then $\sigma(N, V + \{a\}) = \sigma(N, V) + a$.

[7] We are grateful to a referee for suggesting the inclusion of a section on the properties discussed here.

Symmetry: For all permutations π of N, $\sigma(N, \pi V) = \pi\sigma(N, V)$, where $\pi V(S) = V(\pi S)$, $S \in \mathcal{N}$.

Individual Rationality: For every $x \in \sigma(N, V)$, $x_i \geqslant \sup\{y \mid y \in V(\{i\})\}$ for all $i \in N$.

In order to define the reduced game property, we need some further notation.

Let $(N, V) \in \Gamma$, let $x \in V(N)$, and let $S \in \mathcal{N}$. The *reduced game* with respect to S and x is the game $(S, V_{x,S})$ where

$$V_{x,S}(S) = \{y_s \mid (y_S, x_{N \setminus S}) \in V(N)\},$$

$$V_{x,S}(T) = \bigcup_{Q \subseteq N \setminus S} \{y_T \mid (y_T, x_Q) \in V(T \cup Q)\} \qquad \text{if} \quad T \subseteq S, T \neq S.$$

The reduced game $(S, V_{x,S})$ has the following interpretation. The players of S are allowed to choose only payoff vectors y_S that are compatible with $x_{N \setminus S}$, the fixed payoff distribution to the members of the complementary coalition $(N \setminus S)$. However, proper subcoalitions T of S may seek the cooperation of subsets Q of $(N \setminus S)$, if each member i of Q is given x_i in the resulting payoff vectors for $(T \cup Q)$. It should be noted that even if (N, V) satisfies (A1), (A2), and (A3), a reduced game $(S, V_{x,S})$ may fail to be superadditive, although it will satisfy (A1) and (A3).

Reduced Game Property: If $S \in \mathcal{N}$ and $x \in \sigma(N, V)$, then $x_S \in \sigma(S, V_{x,S})$.

PROPOSITION 5.1. CB(V) *is symmetric and covariant.*

Proof. Suppose $x \in \mathrm{CB}(V)$. To show that the consistent bargaining set is covariant, we need only show that $x + a \in \mathrm{CB}(V + \{a\})$. Suppose not. Then there is a valid objection (S, y) to $x + a$. It is then trivial to check that $(S, y - a_S)$ is a valid objection to x in the game (N, V). This contradicts the supposition that $x \in \mathrm{CB}(V)$.

The proof of symmetry is also obvious and is left to the reader. ∎

It is easy to construct examples of non-superadditive games in which CB(V) contains imputations which violate individual rationality. In three-person superadditive games it can be shown that all imputations in CB(V) do satisfy individual rationality. We shall now present an example of a five-player, superadditive, transferable utility game in which CB(V) does not satisfy individual rationality. This example also illustrates the fact that CB(V) may contain imputations in which a "dummy" player gets a positive utility.[8] Of course, these properties are also shared by $B(V)$.

[8] This is also the case with the Aumann–Maschler bargaining set.

EXAMPLE 2. (CB(V) violates individual rationality).

We consider a five-player, superadditive, transferable utility game which is a modification of Example 1. Player 5 is a "dummy" player in the sense that *he/she* does not contribute to any other coalition and has 0 individual worth. The payoffs of coalitions not containing player 5 are exactly as in example 1. $N=\{1, 2, 3, 4, 5\}$, $S^1=\{1, 2, 3\}$, $S^2=\{2, 3, 4\}$, $S^3=\{1, 4\}$, $S^4=\{1, 2, 4\}$, $S^5=\{1, 3, 4\}$. Also, $v(S^1)=v(S^2)=66$, $v(S^3)=46$, $v(S^4)=v(S^5)=63$, $v(N)=80$. For all other coalitions $S\subseteq\{1, 2, 3, 4\}$, $v(S)=0$. $v(\{5\})=0$ and for any $S\subseteq\{1, 2, 3, 4\}$, $v(S\cup\{5\})=v(S)$.

We shall show that for any $\varepsilon>0$, $x=(-\varepsilon, 17, 17, -\varepsilon, 46+2\varepsilon)$ belongs to CB(V).

Proof. Notice that player 5 cannot belong to any coalition which has an objection to x. Indeed, the only coalitions which have objections to x are

$$S^1, \quad S^2, \quad S^3, \quad S^4, \quad S^5, \quad \{1\}, \quad \{4\}, \quad \text{and} \quad \{1, 2, 3, 4\}.$$

We shall show that none of them has a valid objection to x.

Suppose (S^1, y) is an objection to x. If $y_1<0$ then $y_2\geqslant 17$, $y_3\geqslant 17$, and $y_2+y_3>66$. Now it is easy to see that $(S^3, (23, 23))$ is a terminating objection to the chain $\mathscr{A}=(x; (S^1, y))$. If $y_1\geqslant 0$, then, clearly, there exist counterobjections to $\mathscr{A}$. Moreover, player 4 belongs to all the coalitions which have further objections. By Lemma 3.0 this implies that there exists a terminating objection to the chain $\mathscr{A}$. Thus, S^1 does not have a valid objection to x.

Given the symmetry between players 1, 4 and 2, 3 a similar argument shows that S^2 does not have a valid objection; either there is a terminating objection to it from S^3 or player 1 belongs to all further objecting coalitions.

Suppose (S^3, y) is an objection to x. Then $(y_1, 17, 17, y_4)$ is an imputation of the game defined in Example 1. As we have already shown in Section 4, there exists a valid objection to any imputation of Example 1. Since the coalitions which can object to the chain $(x; (S^3, y))$ in the present example are exactly those which can object to $(y_1, 17, 17, y_4)$ in Example 1, this means that there exists a valid objection to $(x; (S^3, y))$; i.e., S^3 does not have a valid objection to x.

Since objections from S^4, S^5, and $\{1, 2, 3, 4\}$ also yield imputations for the game in Example 1, the same argument as above implies that none of these coalitions can have a valid objection to x.

Consider the objection to x from coalition $\{1\}$. Now $(\{4\}, 0)$ is a valid objection to the chain $\mathscr{A}=(x; (\{1\}, 0))$. To see this, simply consider objections to $\mathscr{A}$ from coalitions S^1, ..., S^5 and $\{1, 2, 3, 4\}$ and apply in each case

the corresponding argument from above to show that any such objection to $\mathscr{A}$ is invalid.

Given the symmetry between players 1 and 4, it can also be shown that the objection from $\{4\}$ has a valid counterobjection in $(\{1\}, 0)$. This completes the proof that $x \in \mathrm{CB}(V)$. ∎

In the following example we show that $\mathrm{CB}(V)$ need not satisfy the reduced game property.

EXAMPLE 3. $\mathrm{CB}(V)$ does not satisfy the reduced game property).

Consider the game in Section 3.1 and $x = (0.94, 1.03, 1.03, 1) \in \mathrm{CB}(V)$. Let $S = \{1, 2\}$. Then

$$V_{x,S}(S) = 1.97$$
$$V_{x,S}(\{1\}) = 0.03$$
$$V_{x,S}(\{2\}) = 1.06.$$

It is clear that the only objection to x_S is from $\{2\}$ and, therefore, $x_S \notin \mathrm{CB}(V_{x,S})$.

REFERENCES

1. R. AUMANN AND M. MASCHLER, The bargaining set for cooperative games, *in* "Advances in Game Theory" (M. Dresher, L. S. Shapley, and A. W. Tucker, Eds.), Annals of Mathematical Studies No. 52, Princeton Univ. Press, Princeton, NJ, 1964.
2. M. DAVIS AND M. MASCHER, The kernel of a cooperative game, *Naval Res. Logist. Quart.* **12** (1965), 223–259.
3. J. GREENBERG, Cores of convex games without sidepayments, mimeo, University of Haifa, 1982.
4. J. GREENBERG, The core and the solution as abstract stable sets, mimeo, University of Haifa, 1987.
5. M. MASCHLER, B. PELEG, AND L. S. SHAPLEY, The kernel and bargaining sets for convex games, *Internat. J. Game Theory* **1** (1972), 73–93.
6. A. MAS-COLELL, An equivalence theorem for a bargaining set, *J. Math. Econ.*, in press.
7. B. PELEG, A proof that the core of an ordinal convex game is a von Neumann–Morgenstern solution, *Math. Soc. Sci.* **11** (1986), 83–87.
8. D. RAY, Credible coalitions and the core, *Internat. J. Game Theory*, in press.
9. W. W. SHARKEY, Convex games without sidepayments, *Internat. J. Game Theory* **11** (1981), 101–106.
10. V. V. VILKOV, Convex games without sidepayments, *Vestnik Leningradskiva Universitata* **7** (1977), 21–24.
11. R. VOHRA, An existence theorem for a bargaining set, mimeo, Brown University, 1989.