Online Appendix for
POVERTY AND SELF-CONTROL

B. Douglas Bernheim
Stanford University and NBER

Debraj Ray
New York University

Şevin Yeltekin
Carnegie Mellon University

March 2015

This Appendix provides supplementary material to accompany the main text. Section A discusses the psychological foundations for our approach. Section B provides full arguments for all the results in the main text concerning history-dependent equilibria; essentially, up to and including Proposition 4. Section C proves our assertions for the simplified model of Section 5.4 in the paper, and provides associated computational results. Section D provides detailed arguments for results involving Markov perfect equilibria. Section E describes the algorithm for computing subgame-perfect equilibrium values, and the parameter choices for the examples in the main text. Section F provides computed examples with and without poverty traps. Section G shows that a poverty trap is present even when a MPE is used as punishment. Finally, Section H presents the details of the model with taste shocks and lockbox saving regimes. Referenced equations that appear in this Appendix are labeled as (a.1), (a.2), etc. Other equation references are to equations in the main text.

APPENDIX A. PSYCHOLOGICAL FOUNDATIONS

The defining feature of the self-control mechanisms that we model in this paper is that people intentionally create and execute plans for self-reinforcement, establishing incentives by punishing themselves for deviations from (or rewarding themselves for conformity with) desired behavior.[1] Psychological foundations for this mechanism are found in the literatures on self-regulation and behavior modification. For example, in one early study of self-regulation, Bandura and Kupers (1964) observed, "[b]y contrast [to rats or chimpanzees], people typically make self-reinforcement contingent on their performing certain classes of responses which they have come to value as an index of personal merit. They often set themselves relatively explicit criteria of

[1] Certainly, one can frame any form of self-reinforcement as either a reward or a punishment; it is the *difference* in outcomes that creates incentives. Psychologists do not, however, view this framing as neutral, and there is some evidence that people can achieve self-control more effectively by framing self-administered consequences as rewards rather than as punishments; see, e.g., Mahoney, Moura, and Wade (1973). Such framing effects are beyond the scope of our investigation.

achievement, failure to meet which is considered undeserving of self-reward and may elicit self-denial or even self-punitive responses; on the other hand, they tend to reward themselves generously on those occasions when they attain their self-imposed standards."[2] Rehm (1977) notes that "[s]elf-reinforcement has been a major focus of self-control research and many clinical uses of self-administered reward and punishment programs have been described."[3] Likewise, according to Kazdin (2012), "Self-reinforcement and self-punishment techniques have been incorporated into intervention programs and applied to a wide range of problems..."

As explained in the Introduction, we model the decisions of a time-inconsistent individual by studying a dynamic game played by his successive incarnations. In that setting, the individual engages in self-reinforcement by deploying history-dependent strategies, which specify contingent patterns of behavior that serve as rewards and/or punishments, as in the psychological literature. Subgame perfection takes this a step further: it ensures that self-reinforcement is *credibly* implementable. With this interpretation, the scope for exercising self-control through self-punishment/reward is sharply defined by the set of outcomes that can arise in subgame-perfect Nash equilibria.

Not surprisingly, psychologists do not typically employ the language of game theory or the formal logic of subgame perfection. Yet they have long recognized that credibility problems limit the scope for effective self-reinforcement. Ainslie (1975) succinctly summarizes the problem thus: "Self-reward is an intuitively pleasing strategy until one asks how the self-rewarding behavior is itself controlled..."[4]

The logic of using history-dependent strategies to overcome the credibility problem is a recurring theme in Ainslee's work. In particular, he observes that people often successfully adopt personal rules (e.g., "always go to bed early"), which they enforce by construing local deviations to have global significance (e.g., "if I go to bed late today, then I will go to bed late every night"); see Ainslie (1975, 1991). Viewed through our game-theoretic lens, a personal rule is an equilibrium

---

[2] Similarly, Mischel (1973) observes:"The essence of self-regulatory systems is the subject's adoption of contingency rules that guide his behavior in the absence of, and sometimes in spite of, immediate external situational pressures. Such rules specify the kinds of behavior appropriate (expected) under particular conditions, the performance levels (standards, goals) which the behavior must achieve, and the consequences (positive and negative) of attaining or failing to reach those standards." See also Bandura (1971, 1976).

[3]Similarly, Bandura (1976) notes that "[a]mong the various self-regulator phenomena that have been investigated within [the social learning] framework, self-reinforcement occupies a prominent position. In this process, individuals regulate their behavior by making self-reward conditional upon matching self-prescribed standards of performance... [C]ontrol is vested to a large extent in the hands of individuals themselves: they set their own goals, they monitor and evaluate their own performances, and they serve as their own reinforcing agents."

[4] He goes on to argue: "A subject does not actually recruit additional reward by planning to delay a cigarette until he has finished a difficult task. On the contrary, he sets himself a second task: He must both defer smoking and work on his original task on the basis of the same differential reward that has always confronted him." See also Rachlin (1974) and Kazdin (2012).

path for a dynamic intrapersonal game, and the global consequences that support it are the off-equilibrium paths triggered by deviations.[5]

Our interpretation of Ainslie's writings differs from that of both Benabou and Tirole (2004) and Ali (2011). They interpret an individual's contingent beliefs about his own future actions as evidence-based forecasts rather than deliberately contrived arrangements. Some passages in Ainslie's writings are consistent with this interpretation.[6] Yet in other passages, he emphasizes that people intentionally create this conditionality. Ainslie (1991) is particularly instructive on this point. For example, he writes that "…insofar as [the individual] has become aware of this phenomenon, he will be able to induce it where it has not occurred spontaneously, by arbitrarily defining a category of gratification-delaying behaviors that will thereafter prevail or not as a set." Accordingly, he describes personal rules as the mechanism by which "...the person can *arrange* consistent motivation" (emphasis added) for a "prolonged course of action." Indeed, after describing the choices of a time-inconsistent decision maker as involving an "intertemporal prisoner's dilemma," he characterizes personal rules as "a solution to the bargaining problem" between an individual's "successive motivational states." Moreover, citing Klein and Leffler (1981), he notes that "[t]he same logic is the basis for what is called a 'self-enforcing contract' between individuals." To illustrate the use of a personal rule, he examines a simple numerical model of a hyperbolic discounter who, in each of a succession of periods, decides whether to stay up late or go to bed early. Because his model involves no uncertainty concerning preferences, it entails no inference problem. Yet he informally describes a subgame-perfect equilibrium in which the individual exercises self-discipline (going to bed early every night) by contriving a conditional self-punishment (staying up late for ten consecutive nights), and he describes this solution in game theoretic terms: "Insofar as [the individual] sees his current choice as a precedent and not an isolated incident, he will face the incentives of a repeated prisoner's dilemma."

Portions of Ainslie (1991), including the aforementioned numerical example, appear to invoke Nash (or, more generally, Markov) reversion as a solution to the credibility problem. Yet Ainslie (1975) also describes more complex patterns of self-reinforcement. For instance, he discusses the case of an individual who, in order to keep his shoes shined, adopts a personal rule specifying that he must shine them before breakfast (otherwise he will refrain from shining them in the future). Ainslie posits that, upon oversleeping, the subject might be tempted not only to skip this chore, but also to skip the punishment. A secondary punishment is required for that contingency; Ainslie suggests that the subject might skip breakfast (and thereby conform to the letter of his

[5] Laibson (1997), Bernheim, Ray, and Yeltekin (1999), and Benhabib and Bisin (2001) have previously adopted this interpretation.

[6] For example, Ainslie (1991) writes, "If [an individual] makes an impulsive choice, he will have little reason to believe he will not go on doing so, and if he controls his impulse, he has evidence that he may go on doing that." However, in context, one can also read this passage as a reference to the expectations that prevail in a particular intrapersonal equilibrium.

rule). Thus, according to Ainslie (1975), complex patterns of self-reinforcement simply require "skill at private side bets."

The literature also provides insight into the processes by which people arrive at credible schemes of self-reinforcement. Ainslie (1991) discusses trial and error, but there a broader literature emphasizes social learning. For example, one classic experiment shows that "children's patterns and magnitude of self-reinforcement closely matched those of the model to whom they had been exposed. Adults generally served as more powerful modeling stimuli than peers in transmitting self-reinforcing responses" (Bandura and Kupers (1964); see also Bandura (1971, 1976)).

Finally, the psychological literature offers an interesting alternative perspective on the issue of renegotiation-proofness. As this literature notes, an individual may initially arrive at an equilibrium strategy by modeling others. In that event, any subsequent effort to change that strategy may be viewed by later selves as a deviation. Ainslie (1975) certainly recognizes this point; he notes that a decision to call off a "private side bet" might lead the individual to "perceive the bet as having been lost," and thereby jeopardize "the credibility of any similar private side bets."

Ainslie (1975) nevertheless describes one potentially feasible form of renegotiation in the context of his shoe-shining example: the individual can modify his rule for shining shoes as long as he does not do so "just before he was due to shine them again." The principle appears to be that one is always free to revise a personal rule, but not for the current period; to avoid confounding revisions and deviations, any changes must be arms-length and limited to plans for subsequent behavior. Plainly, a continuation equilibrium that reverts to $H(\cdot)$ after a single period is immune to revisions according to this criterion; indeed, it satisfies a strong form of "renegotiation-proofness" (given that the continuation is unimprovable within the entire equilibrium set). Moreover, Ainslie's reasoning arguably implies that reasonable self-punishments *must* have this structure, else they would be revised.

In this context, it is therefore noteworthy that, with an appropriate interpretation of cases in which the continuation value lies above $H^{-}(A)$ but strictly below $H(A)$, the worst self-punishment equilibria have this property. To see this, we adopt a slightly different interpretation from the one offered in Proposition 3. Under this interpretation, the agent binges for one period only, and provided there is some noise in asset returns or she can arrange small side bets (not necessarily fair to her, so that any risk-neutral second party would accept such a bet), she can return to the highest continuation value function in the very next period.[7]

[7]As an example, the equilibrium strategy might specify that, in addition to consuming slightly more than $A - \frac{Y}{\alpha}$, the individual also makes a small wager with another party, leaving her with continuation assets of either Y or $Y - \varepsilon$, with appropriate probabilities. The wager need not be fair. One can also smooth out expected continuation values by introducing a small amount of noise in the return α. While we do not formally consider a stochastic model, the same arguments in the proof of Proposition 3 go through.

APPENDIX B. PROOFS OF RESULTS CONCERNING HISTORY-DEPENDENT EQUILIBRIA

LEMMA **1.** *Let V be an equilibrium value at A, with associated asset choice x. Then*

$$\begin{aligned} V &\geq \left[u\left(A-\frac{B}{\alpha}\right)+\delta L(B)\right]+\frac{1-\beta}{\alpha\beta}u'\left(A-\frac{B}{\alpha}\right)(x-B) \\ &\geq \left[u\left(A-\frac{B}{\alpha}\right)+\frac{\delta}{1-\delta}u\left(\frac{\alpha-1}{\alpha}B\right)\right]+\frac{1-\beta}{\alpha\beta}u'\left(A-\frac{B}{\alpha}\right)(x-B). \end{aligned}$$

Proof. The equilibrium payoff associated with V is $(1-\beta)u\left(A-\frac{x}{\alpha}\right)+\beta V$, so

$$(1-\beta)u\left(A-\frac{x}{\alpha}\right)+\beta V \geq u\left(A-\frac{B}{\alpha}\right)+\beta\delta L(B).$$

Given that u is concave, it follows that

$$\begin{aligned} V &\geq \left[u\left(A-\frac{B}{\alpha}\right)+\delta L(B)\right]+\frac{1-\beta}{\beta}\left[u\left(A-\frac{B}{\alpha}\right)-u\left(A-\frac{x}{\alpha}\right)\right] \\ &\geq \left[u\left(A-\frac{B}{\alpha}\right)+\delta L(B)\right]+\frac{1-\beta}{\alpha\beta}u'\left(A-\frac{B}{\alpha}\right)(x-B). \end{aligned} \tag{a.1}$$

By (5) and $A_t \geq B$ at any date t, we have $u(c_t) \geq u(\upsilon B)$ for any c_t at date t, so that $L(A) \geq (1-\delta)^{-1}u(\upsilon B) > -\infty$. Now, by applying (a.1) to $A = B$ and $V = L(B)$, or (if needed) a sequence of equilibrium values in $\mathcal{V}(B)$ that converge down to $L(B)$,

$$L(B) \geq u\left(B-\frac{B}{\alpha}\right)+\delta L(B). \tag{a.2}$$

Combining (a.1) and (a.2), the proof is complete. ∎

Proof of Observation 1. This is an immediate consequence of Lemma 1. ∎

Proof of Proposition 1. Claim: if $\mathcal{W}$ is nonempty, has closed graph, and satisfies (8), then it generates $\mathcal{W}'$ with the same properties (plus convex-valuedness). We first prove that $\mathcal{W}'$ is nonempty-valued. Consider the function $H_{\mathcal{W}}$ on $[B,\infty)$ defined by $H_{\mathcal{W}}(A) \equiv \max \mathcal{W}(A)$ for all $A \geq B$. It is easy to see that $H_{\mathcal{W}}$ is usc. It follows that $\max_{x\in[0,\alpha(1-\upsilon)A]} u\left(A-x/\alpha\right)+\beta\delta H_{\mathcal{W}}(x)$ is well-defined and admits a (possibly non-unique) solution for every $A \geq B$. Let $x(A)$ denote some solution at A, and define

$$w \equiv u\left(A-\frac{x(A)}{\alpha}\right)+\delta H_{\mathcal{W}}(x(A)).$$

Clearly, w is supported at A by $\mathcal{W}$. (9) is satisfied: pick $x = x(A)$ and $V = H_{\mathcal{W}}(x(A))$. And (10) is satisfied: for each alternative x', take V' to be any element of $\mathcal{W}(x')$.

Claim: $\mathcal{W}'$ has closed graph. Take any sequence $\{A_n, w_n\}$ such that (i) w_n is supported at A_n by $\mathcal{W}$ for all n, and (ii) $(A_n, w_n) \to (A, w)$ (finite) as $n \to \infty$; then w is supported at A by $\mathcal{W}$. To see this, note that for each n, there is x_n feasible for A_n and value $V_n \in \mathcal{W}(x_n)$ such that (9) and (10) are satisfied. Obviously $\{x_n, V_n\}$ is a bounded sequence; pick any limit point (x, V). Then x is certainly a feasible asset choice at A, and $V \in \mathcal{W}(x)$ (because $\mathcal{W}$ has closed graph by assumption). Using the continuation (x, V) at A, it is immediate that (9) is satisfied for w. To prove (10), let x' be any feasible asset choice at A. Then there is $\{x'_n\}$, with x'_n feasible for A_n for all n, such that $x'_n \to x'$. Because w_n is supported at A_n by $\mathcal{W}$, and (x_n, V_n) satisfies (10), there is $V'_n \in \mathcal{W}(x'_n)$ such that

$$u\left(A_n - \frac{x_n}{\alpha}\right) + \beta\delta V_n \geq u\left(A_n - \frac{x'_n}{\alpha}\right) + \beta\delta V'_n \tag{a.3}$$

for every n. Let V' be any limit point of $\{V'_n\}$. Then, because $\mathcal{W}$ has closed graph, $V' \in \mathcal{W}(x')$. Choose an appropriate subsequence of n such that $\{x'_n, V'_n\}$ converges to (x', V'). Passing to the limit in (a.3), we must conclude that (10) holds for (A, w) at x'.

These arguments prove the claim that the limit value w is supported at A by $\mathcal{W}$. With the claim in hand, by taking suitable convex combinations it is easy to prove that the correspondence $\mathcal{W}'$ generated by $\mathcal{W}$ has closed graph. It is trivially convex-valued.

Now, consider the sequence $\{\mathcal{V}_k\}$. Because $\mathcal{V}_0$ is nonempty-valued with closed graph, and satisfies (8), the same is true of the $\mathcal{V}_k$'s. Moreover, for each $t \geq 0$ and all $A \geq B$,

$$\mathcal{V}_k(A) \supseteq \mathcal{V}_{k+1}(A).$$

Take infinite intersections of these nested compact sets (at each A) to argue that

$$\mathcal{V}^*(A) \equiv \bigcap_{t=0}^{\infty} \mathcal{V}_k(A)$$

is nonempty for every A. Furthermore, because $\mathcal{V}_k(A)$ is convex for all $k \geq 0$, so is $\mathcal{V}^*(A)$. Moreover, $\mathcal{V}^*$ has compact graph on any compact interval $[B, D]$,[8] and therefore it has closed graph everywhere. We will show that $\mathcal{V}^*$ generates itself. To this end, we first show for each A, every w supported at A by $\mathcal{V}^*$ lies in $\mathcal{V}^*(A)$. Pick such a value w at A. Then there is a feasible continuation asset choice x at A and $V \in \mathcal{V}^*(x)$ such that (9) holds, and for every feasible choice x' at A, there is $V' \in \mathcal{V}^*(x')$ such that (10) holds. But these continuations are available in $\mathcal{V}_k$ for every k, which means that w is supported at A by every $\mathcal{V}_k$. It follows that $w \in \mathcal{V}_{k+1}(A)$ for every k, so that $w \in \mathcal{V}^*(A)$.

[8] On any compact interval, the (restricted) graphs of the $\mathcal{V}_k$'s are compact and their infinite intersection is the graph of $\mathcal{V}^*$ on the same interval, which must then be compact.

We complete the argument by showing that for every A, $\max \mathcal{V}^*(A)$ and $\min \mathcal{V}^*(A)$ are supportable at A by $\mathcal{V}^*$.[9] The same argument works in either case, so we show this for $\max \mathcal{V}^*(A)$. Because $\mathcal{V}^*(A) = \bigcap_{t=0}^{\infty} \mathcal{V}_k(A)$, the sequence of values $w_k \equiv \max \mathcal{V}_k(A)$ converges to $H(A)$. Moreover, w_k cannot be a proper convex combination of other values in $\mathcal{V}_k(A)$, so w_k is supportable at A by $\mathcal{V}_k$, for every k. That is, for each k, there is x_k feasible for A and value $V_k \in \mathcal{V}_k(x_k)$ such that (9) and (10) are satisfied for w_k. It is easy to see that $\{x_k, V_k\}$ is a bounded sequence. Pick any limit point (x, V) of $\{x_k, V_k\}$. Then x is a feasible choice at A, and $V \in \mathcal{V}^*(x)$.[10] Using the continuation (x, V) at A, then, (9) is satisfied for $w = \max \mathcal{V}^*(A)$ (under $\mathcal{V}^*$).

Now, let x' be any feasible asset choice at A. Because w_k is supported at A by $\mathcal{V}_k$, and (x_k, V_k) has been chosen such that (10) is satisfied, there exists $V'_k \in \mathcal{V}_k(x')$ such that

$$u\left(A - \frac{x_k}{\alpha}\right) + \beta\delta V_k \geq u\left(A - \frac{x'}{\alpha}\right) + \beta\delta V'_k \tag{a.4}$$

for every k. Let V' be any limit point of $\{V'_k\}$. Then, by the argument already used (see footnote 10), $V' \in \mathcal{V}^*(x')$. Choose an appropriate subsequence of n such that $\{x'_n, V'_n\}$ converges to (x', V'). Passing to the limit in (a.4), we see that (10) holds for (A, w) at x'.

This shows that $\mathcal{V}^*$ generates $\mathcal{V}^*$. It is immediate that $\mathcal{V}^*$ contains every correspondence that generates itself,[11] so it is the same as our equilibrium correspondence $\mathcal{V}$. ■

Given Proposition 1, let $H(A)$ and $L(A)$ be the maximum and minimum values of the equilibrium value correspondence $\mathcal{V}$. Because the graph of $\mathcal{V}$ is closed, H is usc and L is lsc. In what follows we take care to account for possible discontinuities in L, which are unfortunately endemic. Let x be a feasible choice of continuation asset at A. Consider all limits of sequences of the form $\{L(x^n)\}$, where $x^n \in [B, \alpha(1-\upsilon)A]$ for all n and $x^n \to x$. Each limit is an equilibrium value at x, because $\mathcal{V}$ has closed graph. Moreover, the collection of all such limits at x (given A) is compact, so a *largest* value $M(x, A)$ exists. That defines the function $M(x, A)$ for $A \geq B$ and $x \in [B, \alpha(1-\upsilon)A]$. An individual can guarantee herself a continuation value that is arbitrarily close to $M(x, A)$, starting from A (by making an asset choice arbitrarily close to x).

LEMMA **2.** *For given A, $M(x, A)$ is usc in x, and for given x, it is nondecreasing in A, and independent of A as long as $x < \alpha(1-\upsilon)A$.*

---

[9] Because $\mathcal{V}^*(A)$ is convex, it equals $[\min \mathcal{V}^*(A), \max \mathcal{V}^*(A)]$. We've shown that all w supportable at A by $\mathcal{V}^*$ must indeed lie in $\mathcal{V}^*(A)$. So, provided we can show that $\max \mathcal{V}^*(A)$ and $\min \mathcal{V}^*(A)$ are supportable at A by $\mathcal{V}^*$, it must follow that $\mathcal{V}^*(A)$ is the convex hull of *all* values supported at A by $\mathcal{V}^*$.

[10] To see why, pick any n in the sequence. Then for $k \geq n$, $V_k \in \mathcal{V}_k(x_k) \subseteq \mathcal{V}_n(x_k)$, so that $V \in \mathcal{V}_n(x)$ by the closed-graph property of $\mathcal{V}_n$. It follows that $V \in \mathcal{V}_n(x)$ for *every* n, so that $V \in \mathcal{V}^*(x)$ as asserted.

[11] Let $\mathcal{V}'$ be any self-generating correspondence. Then if $\mathcal{V}' \subseteq \mathcal{V}_k$, we have $\mathcal{V}' \subseteq \mathcal{V}_{k+1}$. But $\mathcal{V}' \subseteq \mathcal{V}_0$, so it follows that $\mathcal{V}' \subseteq \mathcal{V}_k$ for every k, which implies $\mathcal{V}' \subseteq \mathcal{V}^*$.

Proof. Pick x^n feasible for A such that $x^n \to x \in [B, \alpha(1-\upsilon)A]$ and a corresponding sequence $M^n = M(x^n, A)$. Suppose without loss of generality that $M^n \to M$. For each n, there is $y^n \in [B, \alpha(1-\upsilon)A]$ such that $|y^n - x^n| < 1/n$, and $|L(y^n) - M^n| < 1/n$. It is then easy to see that $y^n \to x$ and $L(y^n) \to M$. So M is a limit value at x, which implies $M(x, A) \geq M$. Therefore $M(x, A)$ is usc in x. To prove that $M(x, A)$ is nondecreasing in A, observe that every sequence of the form $\{L(x^n)\}$, where $x^n \in [B, \alpha(1-\upsilon)A]$, is fully available at $A' > A$, whenever it is available at A. It is also obvious that for any x, exactly the same limit values of $\{L(x^n)\}$ are available when $x < \alpha(1-\upsilon)A$, so that $M(x, A)$ is then unchanging in A whenever the strict inequality holds. ■

Lemma 2 implies that the following "best deviation payoff" at A is well-defined:

(a.5) $$D(A) = \max_x u\left(A - \frac{x}{\alpha}\right) + \beta\delta M(x, A),$$

where it is understood that $x \in [B, \alpha(1-\upsilon)A]$. Lemma 2 also implies that $D(A)$ is an increasing function. Note that D does not necessarily use worst punishments everywhere, but nonetheless a deviant can get payoff arbitrarily close to $D(A)$. That implies:

LEMMA **3.** *The pair* (x, V) *is an equilibrium continuation at* A *if and only if* $x \in [B, \alpha(1-\upsilon)A]$*,* $V \in \mathcal{V}(x)$ *and*

(a.6) $$u\left(A - \frac{x}{\alpha}\right) + \beta\delta V \geq D(A).$$

Proof. Sufficiency: if (x, V) is not an equilibrium continuation, then there exists $y \neq x$ such that $u(A - x/\alpha) + \beta\delta V < u(A - y/\alpha) + \beta\delta L(y)$. But $L(y) \leq M(y, A)$, so $u(A - x/\alpha) + \beta\delta V < u(A - y/\alpha) + \beta\delta M(y, A) \leq D(A)$.

Necessity: if (x, V) is an equilibrium continuation at A, then $x \in [B, \alpha(1-\upsilon)A]$ and $V \in \mathcal{V}(x)$. Moreover, for every feasible y, and sequence of feasible $\{y^n\}$ with $y^n \to y$,

$$u\left(A - \frac{x}{\alpha}\right) + \beta\delta V \geq u\left(A - \frac{y^n}{\alpha}\right) + \beta\delta L(y^n),$$

where the inequality holds trivially for $y^n = x$ (because $V \geq L(x)$) and by incentive compatibility for $y^n \neq x$. Passing to the limit in that inequality, we must conclude that

$$u\left(A - \frac{x}{\alpha}\right) + \beta\delta V \geq u\left(A - \frac{y}{\alpha}\right) + \beta\delta M(y, A).$$

Maximizing the right hand side of this inequality over y, we obtain the desired result. ■

LEMMA **4.** *If* d *solves* (a.5)*, then* $\{d, M(d, A)\}$ *is an equilibrium continuation at* A*.*

Proof. Because $\mathcal{V}$ has closed graph, $M(d, A) \in \mathcal{V}(d)$. Now apply Lemma 3. ■

LEMMA **5.** *$L(A)$ is increasing on $[B,\infty)$.*

Proof. Let $A'' > A' \geq B$. Consider the equilibrium that generates value $L(A'')$ starting from A'', with associated continuation $\{A_1'', V''\}$. By Lemma 3,

$$u\left(A'' - \frac{A_1''}{\alpha}\right) + \beta\delta V'' \geq u\left(A'' - \frac{x}{\alpha}\right) + \beta\delta M(x, A'') \tag{a.7}$$

for $x \in [B, \alpha(1-\upsilon)A'']$. It follows that $V'' > M(x, A'')$ for all $x < A_1''$, which implies

$$L(A'') = u\left(A'' - \frac{A_1''}{\alpha}\right) + \delta V'' > u\left(A'' - \frac{x}{\alpha}\right) + \delta M(x, A'') \tag{a.8}$$

for all $x < A_1''$. Now construct an equilibrium from A': the choice A_1'' (if feasible) is followed by V'', while each $x \in [B, \alpha(1-\upsilon)A']$ is followed by $M(x, A')$.[12] Note that

$$\begin{aligned} u\left(A' - \frac{A_1''}{\alpha}\right) + \beta\delta V'' &> u\left(A' - \frac{x}{\alpha}\right) + \beta\delta M(x, A'') \\ &\geq u\left(A' - \frac{x}{\alpha}\right) + \beta\delta M(x, A'), \end{aligned} \tag{a.9}$$

for $x \in (A_1'', \alpha(1-\upsilon)A']$ (assuming this set is non-empty), where the first inequality uses the strict concavity of u, $A' < A''$ and (a.7), and the second uses Lemma 2.

To complete the description of equilibrium, we must choose a particular continuation at A': pick continuation $\{y, V\}$ to maximize payoff over the specified continuations above. Given (a.9), that is tantamount to choosing from the greatest of the payoffs

$$u\left(A' - \frac{x}{\alpha}\right) + \beta\delta M(x, A')$$

for $x \in [B, \min\{\alpha(1-\upsilon)A', A_1''\}]$, and the payoff at $x = A_1''$ (if feasible), which is

$$u\left(A' - \frac{A_1''}{\alpha}\right) + \beta\delta V'',$$

and a solution is well-defined, because M is usc in x, and the replacement of $M(A_1'', A)$ by V'' at A_1'' (if feasible for A') only increases payoff. The chosen continuation $\{y, V\}$ must be an equilibrium, and by (a.9), $y \leq A_1''$. If $y < A_1''$, then by (a.8) and Lemma 2,

$$L(A'') > u\left(A'' - \frac{y}{\alpha}\right) + \delta M(y, A'') > u\left(A' - \frac{y}{\alpha}\right) + \delta M(y, A') \geq L(A'),$$

and if $y = A_1''$, then again

$$L(A'') = u\left(A'' - \frac{A_1''}{\alpha}\right) + \delta V'' > u\left(A' - \frac{y}{\alpha}\right) + \delta V'' \geq L(A').$$

So in both cases, $L(A'') > L(A')$, as desired. ∎

---

[12] Recall that $M(x, A')$ is indeed an equilibrium value at x because $\mathcal{V}$ has closed graph.

Lemma 5 makes it easy to visualize $M(x, A)$. With L increasing, let $L^+(A)$ denote the right hand limit of L at A; i.e., the common limit of all sequences $\{L(A^n)\}$ as $A^n \downarrow A$, with $A^n > A$ for all n. Clearly, L^+ is an increasing, right-continuous function.

LEMMA **6.** *For any A and $x \in [B, \alpha(1-\upsilon)A)$, $M(x, A)$ equals $L^+(x)$. At $x = \alpha(1-\upsilon)A$, it equals $L(x)$.*

Proof. Obvious, given Lemma 5 and the definitions of L and M. ■

LEMMA **7.** (a) *Let $d(A)$ solve* (a.5). *If $A_1 < A_2$, then $d(A_1) \leq d(A_2)$. Moreover, a largest solution $d^*(A)$ is well-defined for each A, and it is nondecreasing in A.*

(b) *$d^*(A)$ is right-continuous at any A such that $\lim_n d^*(A^n) < \alpha(1-\upsilon)A$ for $A^n \downarrow A$.*

Proof. Let $x_i \equiv d(A_i)$ for $i = 1, 2$. Suppose, on the contrary, that $x_1 > x_2$. Notice that x_1 is feasible at A_2 (because $A_1 < A_2$ and x_1 is feasible at A_1), and that x_2 is feasible at A_1 (because $x_2 < x_1$). Therefore

$$u\left(A_i - \frac{x_i}{\alpha}\right) + \beta\delta M(x_i, A_i) \geq u\left(A_i - \frac{x_j}{\alpha}\right) + \beta\delta M(x_j, A_i)$$

for $i = 1, 2$ and $j \neq i$. Combining these two inequalities, and using Lemma 2 to conclude that $M(x_1, A_2) \geq M(x_1, A_1)$, while $M(x_2, A_2) = M(x_2, A_1)$,[13]

$$\left[u\left(A_2 - \frac{x_2}{\alpha}\right) - u\left(A_2 - \frac{x_1}{\alpha}\right)\right] \geq \left[u\left(A_1 - \frac{x_2}{\alpha}\right) - u\left(A_1 - \frac{x_1}{\alpha}\right)\right].$$

But the above inequality contradicts the strict concavity of u. So $x_1 \leq x_2$, as desired.

Next we show that a largest maximizer $d^*(A)$ exists at each A. Let d^n each solve (a.5) at A, and say that $d^n \to d$. Because $M(x, A)$ is usc in x (Lemma 2),

$$\lim_{n\to\infty} u\left(A - \frac{d^n}{\alpha}\right) + \beta\delta M(d^n, A) \leq u\left(A - \frac{d}{\alpha}\right) + \beta\delta M(d, A),$$

but the left-hand side of this inequality *is* the maximized value of (a.5) for every n, so the right-hand side must have the same value, which shows that d also solves (a.5). That proves the existence of a largest maximizer $d^*(A)$ at every A, and the arguments so far show that $d^*(A)$ is nondecreasing, so the proof of part (a) is complete.

For part (b), fix A and let $d \equiv \lim_n d^*(A^n) < \alpha(1-\upsilon)A$ for $A^n \downarrow A$ (noting that $\{d^*(A^n)\}$ is monotone). Clearly, d is feasible at A. To prove the right continuity of d^* at A, we show that d maximizes (a.5) at A. Suppose not. Let d' maximize (a.5) at A; then

$$u\left(A - \frac{d'}{\alpha}\right) + \beta\delta M(d', A) > u\left(A - \frac{d}{\alpha}\right) + \beta\delta M(d, A). \tag{a.10}$$

[13] Note that $x_2 < x_1 \leq \alpha(1-\upsilon)A_i$ for $i = 1, 2$. By Lemma 2, $M(x_2, A_2) = M(x_2, A_1)$.

Notice that $d' \leq d$ (by part (a), already proved), so $d' < \alpha(1-\upsilon)A \leq \alpha(1-\upsilon)A^n$ for all n. So by Lemma 2, $M(x, A)$ is independent of A at (d', A), and an analogous assertion is true of A^n. Therefore, not only is d' feasible for all A^n, we also have

$$\text{(a.11)} \qquad \lim_n u\left(A^n - \frac{d'}{\alpha}\right) + \beta\delta M(d', A^n) = u\left(A - \frac{d'}{\alpha}\right) + \beta\delta M(d', A).$$

Define $d^n \equiv d^*(A^n)$, and note that for n large, $d^n < \alpha(1-\upsilon)A \leq \alpha(1-\upsilon)A^n$. Using the independence of M in A^n and recalling that $M(x, A)$ is usc in x (Lemma 2),

$$\text{(a.12)} \qquad \lim_n u\left(A^n - \frac{d^n}{\alpha}\right) + \beta\delta M(d^n, A^n) \leq u\left(A - \frac{d}{\alpha}\right) + \beta\delta M(d, A).$$

Combining (a.10)–(a.12), we must conclude that for n large,

$$u\left(A^n - \frac{d'}{\alpha}\right) + \beta\delta M(d', A^n) > u\left(A^n - \frac{d^n}{\alpha}\right) + \beta\delta M(d^n, A^n),$$

which contradicts the fact that d^n maximizes (a.5) for all n. ■

Define the *maintenance value* of an asset level A by $V^s(A) \equiv \frac{1}{1-\delta} u\left(\frac{\alpha-1}{\alpha}A\right)$, and the *maintenance payoff* by $P^s(A) \equiv \left[1 + \frac{\beta\delta}{1-\delta}\right] u\left(\frac{\alpha-1}{\alpha}A\right)$. Say that an asset level S is *sustainable* if there is a stationary equilibrium path from S, or equivalently (by Lemma 3) if $P^s(S) \geq D(S)$.

LEMMA **8** (Observation 2 in main text)**.** *Let $S > B$ be a sustainable asset level, and $\mu \equiv S/B$. Then, if $\{A_t^*\}$ is an equilibrium path from A_0:*

(a) *$\{\mu A_t^*\}$ is an equilibrium path from μA_0.*

(b) *For all t with $\mu A_t^* > S$ and for every $A < S$,*

$$u\left(\mu A_t^* - \frac{\mu A_{t+1}^*}{\alpha}\right) + \beta \sum_{s=t+1}^{\infty} \delta^{s-t} u\left(\mu A_s^* - \frac{\mu A_{s+1}^*}{\alpha}\right) > u\left(\mu A_t^* - \frac{A}{\alpha}\right) + \beta\delta M(A, A_t^*).$$

Proof. Part (a). Let policy ϕ sustain $\{A_t^*\}$ from A_0. Define a new policy ψ:

(i) For any $h_t = (A_0 \ldots A_t)$ with $A_s \geq S$ for $s = 0, \ldots, t$, let $\psi(h_t) = \mu\phi\left(\frac{h_t}{\mu}\right)$.

(ii) For h_t with $A_k < S$ for some smallest $k \leq t$, define $h'_{t-k} = (A_k \ldots A_t)$. Let $\psi(h_t) = \phi_\ell(h'_{t-k})$, where ϕ_ℓ is the equilibrium policy with value $L(A_k)$ at A_k.

For any history h_t with $A_s \geq S$ for $s = 1, \ldots, t$, the asset sequence generated through subsequent application of ψ is the same as the sequence generated through repeated application of ϕ from $\frac{h_t}{\mu}$, but scaled up by the factor μ. It follows that

$$\text{(a.13)} \qquad P_\psi(h_t) = \mu^{1-\sigma} P_\phi\left(\frac{h_t}{\mu}\right) \text{ and } V_\psi(h_t) = \mu^{1-\sigma} V_\phi\left(\frac{h_t}{\mu}\right).$$

We now show that ψ is an equilibrium. First, consider any h_t such that $A_k < S$ at some first $k \leq t$. Then as of period k the policy function ψ shifts to the equilibrium with value $L(A_k)$. So $\psi(h_t)$ is optimal given the continuation policy function.

Next consider any h_t such that $A_s \geq S$ for all $s \leq t$. Consider, first, any deviation to $A \geq S$. Note that h_t/μ is a feasible history under the equilibrium ϕ, while the deviation to $(A/\mu) \geq (S/\mu) = B$ is also feasible. It follows that

$$P_\phi\left(\frac{h_t}{\mu}\right) \geq u\left(\frac{A_t}{\mu} - \frac{A}{\mu\alpha}\right) + \beta\delta V_\phi\left(\frac{h_t.A}{\mu}\right).$$

Multiplying through by $\mu^{1-\sigma}$ and using (a.13), we see that

$$P_\psi(h_t) \geq u\left(A_t - \frac{A}{\alpha}\right) + \beta\delta V_\psi(h_t.A), \tag{a.14}$$

which shows that no deviation to $A \geq S$ can be profitable.

Now consider a deviation to $A < S$. Because S is sustainable,

$$P^s(S) \geq D(S) \geq u\left(S - \frac{A}{\alpha}\right) + \beta\delta M(A, S) \tag{a.15}$$

by Lemma 3. At the same time, (a.14) applied to $A = S$ implies

$$P_\psi(h_t) \geq u\left(A_t - \frac{S}{\alpha}\right) + \beta\delta V_\psi(h_t.S). \tag{a.16}$$

Using (a.13) along with $L(B) \geq V^s(B)$ (see Observation 1), (a.16) becomes

$$\begin{aligned}
P_\psi(h_t) &\geq u\left(A_t - \frac{S}{\alpha}\right) + \beta\delta\mu^{1-\sigma} V_\phi\left(\frac{h_t}{\mu}.B\right) \\
&\geq u\left(A_t - \frac{S}{\alpha}\right) + \beta\delta\mu^{1-\sigma} L(B) \\
&\geq u\left(A_t - \frac{S}{\alpha}\right) + \beta\delta\mu^{1-\sigma} V^s(B) \\
&= u\left(A_t - \frac{S}{\alpha}\right) + \beta\delta V^s(S) \\
&= \left[u\left(A_t - \frac{S}{\alpha}\right) - u\left(S\left(1 - \frac{1}{\alpha}\right)\right)\right] + P^s(S). \qquad \text{(a.17)}
\end{aligned}$$

Combining (a.15) and (a.17),

$$\begin{aligned}
P_\psi(h_t) &\geq \left[u\left(A_t - \frac{S}{\alpha}\right) - u\left(S\left(1 - \frac{1}{\alpha}\right)\right)\right] + u\left(S - \frac{A}{\alpha}\right) + \beta\delta M(A, S) \\
&= \left[u\left(A_t - \frac{S}{\alpha}\right) - u\left(S - \frac{S}{\alpha}\right)\right] - \left[u\left(A_t - \frac{A}{\alpha}\right) - u\left(S - \frac{A}{\alpha}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
& +u\left(A_t - \frac{A}{\alpha}\right) + \beta\delta M(A,S) \\
\text{(a.18)} \qquad \geq \;& u\left(A_t - \frac{A}{\alpha}\right) + \beta\delta M(A,S)
\end{aligned}
$$

where the second inequality follows from the concavity of u and the fact that $A < S \leq A_t$. But, because $M(A,S) \geq L(A) = V_\psi(h_t.A)$, the right hand side of (a.18) is at least as large as the payoff from the deviation, which is $u\left(A_t - [A/\alpha]\right) + \beta\delta V_\psi(h_t.A)$. It follows that the deviation A is unprofitable, so that ψ is an equilibrium.

Part (b). The second inequality in (a.18) holds strictly when $A_t > S$ and $A < S$, by the strict concavity of u. Apply (a.18) (with strict inequality) at date t, with h_t equal to the history on the equilibrium path and setting $M(A,S) = M(A, A_t^*)$ (Lemma 2). ■

LEMMA **9.** *For any asset level A and any path $\{A_t\}$ with $A_t \leq A$ for all $t \geq 0$,*

$$
\text{(a.19)} \qquad V^s(A) - \sum_{t=0}^{\infty} \delta^t u\left(A_t - \frac{A_{t+1}}{\alpha}\right) \geq u'\left(\frac{\alpha-1}{\alpha}A\right)\left(\delta - \frac{1}{\alpha}\right)(A - A_1) \geq 0.
$$

Proof. Let Δ stand for the left hand side of (a.19); then

$$
\begin{aligned}
\Delta &= \sum_{t=0}^{\infty} \delta^t \left[u\left(\frac{\alpha-1}{\alpha}A\right) - u\left(A_t - \frac{A_{t+1}}{\alpha}\right)\right] \\
&\geq u'\left(\frac{\alpha-1}{\alpha}A\right)\sum_{t=0}^{\infty} \delta^t \left[A - \frac{A}{\alpha} - A_t + \frac{A_{t+1}}{\alpha}\right] \\
&= u'\left(\frac{\alpha-1}{\alpha}A\right)\sum_{t=0}^{\infty} \delta^t \left[(A - A_t) - \frac{A - A_{t+1}}{\alpha}\right] \\
&= u'\left(\frac{\alpha-1}{\alpha}A\right)\left[(A - A_0) + \left(\delta - \frac{1}{\alpha}\right)\sum_{t=0}^{\infty} \delta^t (A - A_{t+1})\right] \\
&\geq u'\left(\frac{\alpha-1}{\alpha}A\right)\left(\delta - \frac{1}{\alpha}\right)(A - A_1) \geq 0,
\end{aligned}
$$

where the first inequality uses the concavity of u and the last uses $\delta\alpha > 1$. ■

Let $X(A)$ be the largest and $Y(A)$ the smallest equilibrium asset choice at A.

LEMMA **10.** *$X(A)$ and $Y(A)$ are well-defined and non-decreasing, and X is usc.*

Proof. By Lemma 3, $X(A)$ (resp. $Y(A)$) is the largest (resp. smallest) value of $A' \in [B, \alpha(1-\upsilon)A]$ satisfying

$$u\left(A - \frac{A'}{\alpha}\right) + \beta\delta H(A') \geq D(A) \tag{a.20}$$

$X(A)$ and $Y(A)$ are well-defined because H is usc.

To show that $X(A)$ is nondecreasing, pick $A_1 < A_2$. (a.20) implies that

$$u\left(A_1 - \frac{X(A_1)}{\alpha}\right) + \beta\delta H(X(A_1)) \geq u\left(A_1 - \frac{y}{\alpha}\right) + \beta\delta L(y)$$

for all $y \in [B, \alpha(1-\upsilon)A]$. It follows from the concavity of u that

$$u\left(A_2 - \frac{X(A_1)}{\alpha}\right) + \beta\delta H(X(A_1)) \geq u\left(A_2 - \frac{y}{\alpha}\right) + \beta\delta L(y) \tag{a.21}$$

for all $y \in [B, X(A_1)]$. If the inequality extends to all $y \in [B, \alpha(1-\upsilon)A]$, the claim would be established. Otherwise there is $x' \in (X(A_1), \alpha(1-\upsilon)A_2]$ such that

$$u\left(A_2 - \frac{X(A_1)}{\alpha}\right) + \beta\delta H(X(A_1)) < u\left(A_2 - \frac{x'}{\alpha}\right) + \beta\delta L(x'). \tag{a.22}$$

Combine (a.21) and (a.22) to see that

$$\begin{aligned} u\left(A_2 - \frac{x'}{\alpha}\right) + \beta\delta L(x') \quad &> \quad u\left(A_2 - \frac{X(A_1)}{\alpha}\right) + \beta\delta H(X(A_1)) \\ &\geq \quad u\left(A_2 - \frac{y}{\alpha}\right) + \beta\delta L(y) \end{aligned} \tag{a.23}$$

for all $y \leq X(A_1)$. We now construct an equilibrium starting from A_2 as follows: any choice $A < X(A_1)$ is followed by the continuation equilibrium generating $L(A)$, and any choice $A \geq X(A_1)$ is followed by the continuation equilibrium generating $H(A)$. Because H is usc, there exists some z^* that maximizes $u\left(A_2 - \frac{z}{\alpha}\right) + \beta\delta H(z)$ on $[X(A_1), \alpha(1-\upsilon)A_2]$; in light of (a.23) and the fact that $u\left(A_2 - \frac{x}{\alpha}\right) + \beta\delta H(x) \geq u\left(A_2 - \frac{x}{\alpha}\right) + \beta\delta L(x)$, all choices $A < X(A_1)$ are strictly inferior to z^*. Thus z^* is an equilibrium choice at A_2, so that $X(A_2) \geq z^* \geq X(A_1)$.

To show that $Y(A)$ is non-decreasing, pick $A_1 < A_2$. If $Y(A_2) \geq \alpha[1-\upsilon]A_1$, we're done, so suppose that $Y(A_2) < \alpha[1-\upsilon]A_1$. Construct an equilibrium from A_1 as follows. For any $A \in [B, Y(A_2)]$, assign the continuation value $H(A)$, and for $A \in (Y(A_2), \alpha[1-\upsilon]A_1]$, assign the continuation value $L(A)$. Finally, for the equilibrium asset choice at A_1, assign A', where A' solves

$$\max_{A \in [B, Y(A_2)]} u\left(A_1 - \frac{A}{\alpha}\right) + \beta\delta H(A)$$

(Because H is usc, a solution exists.) We claim that A' maximizes payoff over all the above specifications, so that $\{A', H(A')\}$ is an equilibrium continuation. It certainly does so over choices in $[B, Y(A_2)]$, by construction. For $A \in (Y(A_2), \alpha[1-v]A_1]$,

$$u\left(A_2 - \frac{Y(A_2)}{\alpha}\right) + \beta\delta H(Y(A_2)) \geq u\left(A_2 - \frac{A}{\alpha}\right) + \beta\delta M(A, A_2),$$

so by the concavity of u and Lemma 2,

$$\begin{aligned} u\left(A_1 - \frac{Y(A_2)}{\alpha}\right) + \beta\delta H(Y(A_2)) &\geq u\left(A_1 - \frac{A}{\alpha}\right) + \beta\delta M(A, A_2) \\ &\geq u\left(A_1 - \frac{A}{\alpha}\right) + \beta\delta M(A, A_1), \end{aligned}$$

which proves the claim. Because $A' \leq Y(A_2)$, it follows that $Y(A_1) \leq Y(A_2)$.

Finally, we show that X is usc. For any $A^* \geq B$, $\lim_{A \uparrow A^*} X(A) \leq X(A^*)$ because $X(A)$ is nondecreasing. Now consider any decreasing sequence $A^k \downarrow A^*$, and let X^* denote the (well-defined) limit of $X(A^k)$. For each k, $u\left(A^k - X(A^k)/\alpha\right) + \beta\delta H(X(A^k)) \geq D(A^k)$. Because H is usc and $D(A)$ is nondecreasing, $u\left(A^* - X^*/\alpha\right) + \beta\delta H(X^*) \geq \lim_{k\to\infty} D(A^k) \geq D(A^*)$. That implies $X(A^*) \geq X^* = \lim_{A \downarrow A^*} X(A)$. (In fact, because $X(A)$ is non-decreasing, $X(A^*) = \lim_{A \downarrow A^*} X(A)$.) ■

LEMMA **11.** *If $X(A) = A$, then A is sustainable.*

Proof. Let $A_1 = A$ along with some value V_1 be an equilibrium continuation at A. Then

$$u\left(\frac{\alpha - 1}{\alpha} A\right) + \beta\delta V_1 \geq D(A)$$

by Lemma 3. By Lemmas 9 and 10, $V_1 \leq (1-\delta)^{-1} u\left(\frac{\alpha-1}{\alpha} A\right)$. Using this in the inequality above, we see that $P^s(A) \geq D(A)$, so that A is sustainable. ■

LEMMA **12.** *In the nonuniform case, $\beta\delta(\alpha - 1)/(1-\delta) < 1$.*

Proof. If $\beta\delta(\alpha-1)/(1-\delta) \geq 1$, then by Proposition 6, part (i), there exists a linear Markov equilibrium policy function $\phi(A) = kA$ with $k \geq 1$, which implies uniformity, a contradiction. ■

LEMMA **13.** *Under nonuniformity, the problem*

$$\max_{x \in [0, \alpha(1-v)A]} \left[u\left(A - \frac{x}{\alpha}\right) + \beta\delta V^s(x)\right].$$

has a unique solution $x(A)$ with $x(A) = \Gamma A$, where $0 < \Gamma < 1$. Moreover, the maximand is strictly decreasing in x for all $x \geq x(A)$.

Proof. The maximand is a continuous, strictly concave function, so it has a unique, continuous solution $x(A)$ for each A. Moreover, by strict concavity, the maximand must strictly decline in

x for all $x \geq x(A)$. Define $\xi = \beta\delta(\alpha-1)/(1-\delta)$. By nonuniformity and Lemma 12, we have $\xi < 1$. Routine computation reveals that $x(A) = \Gamma A$, where

$$\Gamma = \frac{\alpha}{1 + \xi^{-\frac{1}{\sigma}}(\alpha - 1)}$$

which (given $\sigma > 0$ and $\xi < 1$) implies $\Gamma < 1$. ■

LEMMA **14.** *For any $A_0 \geq B$, maximize $\sum_{t=0}^{\infty} \delta^t u\left(A_t - \frac{A_{t+1}}{\alpha}\right)$, subject to $A_{t+1} \in [B, \alpha(1-\upsilon)A_t]$, and $A_{t+1} \leq X(A_t)$ for all $t \geq 0$. Then a solution exists, and any solution path $\{A_t^*\}$ is also an equilibrium path starting from A_0, generating the value $H(A_0)$.*

Proof. u is continuous and $X(A_t)$ is usc (Lemma 10), so a solution $\{A_t^*\}$ exists. Let $\{V_t^*\}$ be the sequence of continuation values associated with $\{A_t^*\}$. Consider an equilibrium path from date t, call it $\{A_\tau\}$, sustaining $X(A_t^*)$ at A_t^* and providing continuation value $H(X(A_t^*))$ thereafter. This path necessarily satisfies $A_{\tau+1} \leq X(A_\tau)$ for all $\tau \geq t$, so the definitions of $\{A_t^*\}$ and $\{V_t^*\}$ imply that

$$u\left(A_t^* - \frac{A_{t+1}^*}{\alpha}\right) + \delta V_{t+1}^* \geq u\left(A_t^* - \frac{X(A_t^*)}{\alpha}\right) + \delta H(X(A_t^*)) \tag{a.24}$$

Also, because $A_{t+1}^* \leq X(A_t^*)$ and $\beta < 1$, we have

$$\left(\frac{1}{\beta} - 1\right) u\left(A_t^* - \frac{A_{t+1}^*}{\alpha}\right) \geq \left(\frac{1}{\beta} - 1\right) u\left(A_t^* - \frac{X(A_t^*)}{\alpha}\right) \tag{a.25}$$

Adding (a.24) to (a.25) and multiplying through by β, we obtain

$$u\left(A_t^* - \frac{A_{t+1}^*}{\alpha}\right) + \beta\delta V_{t+1}^* \geq u\left(A_t^* - \frac{X(A_t^*)}{\alpha}\right) + \beta\delta H(X(A_t^*)) \geq D\left(A_t^*\right), \tag{a.26}$$

where the second inequality follows from the fact that $\{X(A_t^*), H(X(A_t^*)\}$ is supportable at A_t^*. Because (a.26) holds for all $t \geq 0$, $\{A_t^*\}$ is an equilibrium path.

Because it is obvious that any equilibrium path must satisfy the constraints of the maximization problem in the statement of the lemma, it follows that the value of this path must be $H(A_0)$. ■

LEMMA **15.** *Suppose that for some $A^* \geq B$, $X(A) > A$ for all $A \geq A^*$. Then starting from any $A \geq A^*$, there is an equilibrium path with monotonic and unbounded accumulation, so that strong self-control is possible. Moreover, some such equilibrium path maximizes value among all equilibrium paths from A.*

Proof. We first claim that for any $A > A^*$ with $\lim_{A' \uparrow A} X(A') = A$, there is $\epsilon > 0$ with

$$X(A') = A \text{ for } A' \in (A - \epsilon, A). \tag{a.27}$$

Suppose on the contrary that there is $A > A^*$ and $\eta > 0$ such that $A' < X(A') < A$ for all $A' \in (A - \eta, A)$. Because $X(A) > A$, Lemma 14 and $\delta\alpha > 1$ together imply

$$H(A) > V^s(A) + \gamma \tag{a.28}$$

for some $\gamma > 0$.[14] Consider any equilibrium continuation $\{X(A'), V_1\}$ from $A' \in (A - \eta, A)$. Because $A'' < X(A'') < A$ for all A'' in that interval, $A'_t < A$ for the resulting equilibrium path. It follows from Lemma 9 that $V^s(A) > V_1$. Combining this inequality with (a.28) and noting that $X(A') \to A$ as $A' \to A$,

$$u\left(A' - \frac{A}{\alpha}\right) + \beta\delta H(A) > u\left(A' - \frac{X(A')}{\alpha}\right) + \beta\delta V_1 \geq D(A')$$

for all $A' < A$ but close to A. So all such A' possess an equilibrium continuation of $\{A, H(A)\}$, which contradicts $X(A') < A'$, and establishes the claim.

We now complete the proof by claiming that any path $\{A_t\}$ from $A \geq A^*$ which solves the problem of Lemma 14 involves monotonic and unbounded accumulation. Suppose this assertion is false. Then at least one of the following must be true:

(i) there exists some date τ such that $A_\tau \geq A_{\tau+1} \leq A_{\tau+2}$, and/or

(ii) the sequence $\{A_t\}$ converges to some finite limit.

Let $\{c_t\}$ be the consumption sequence generated by $\{A_t\}$. In case (i), $c_\tau \geq c_{\tau+1}$. Recalling that $\delta\alpha > 1$, we therefore have

$$u'(c_\tau) < \delta\alpha u'(c_{\tau+1}). \tag{a.29}$$

Moreover, because $X(A_\tau) > A_\tau$ and $A_\tau \geq A_{\tau+1}$, we have

$$A_{\tau+1} < X(A_\tau). \tag{a.30}$$

In case (ii), there exists T such that, for $\tau > T$, (a.29) again holds because c_τ and $c_{\tau+1}$ are close. As far as (a.30) is concerned, there are two subcases to consider:

(a) There is $\tau > T$ with $A_{\tau+1} \leq A_\tau$. Here, (a.30) holds because $X(A_\tau) > A_\tau \geq A_{\tau+1}$.

(b) For $t > T$, A_t is strictly increasing with limit $\bar{A} < \infty$. If $\lim_{t\to\infty} X(A_t) > \bar{A}$, (a.30) plainly holds for some τ sufficiently large. Otherwise $\lim_{t\to\infty} X(A_t) = \bar{A}$. But in this case, we know from the first claim above that for some τ, $X(A_\tau) = \bar{A} > A_{\tau+1}$, so that (a.30) holds yet again for some τ sufficiently large.

In short, (a.29) and (a.30) always hold (for some τ). Now alter the path $\{A_t\}$ by increasing the period-$(\tau + 1)$ asset level from $A_{\tau+1}$ to $A_{\tau+1} + \eta$, leaving asset levels unchanged for all other

[14] If $\delta\alpha > 1$ and $X(A) > A$, then the problem of Lemma 14 isn't solved by the stationary path from A: a small increase in assets followed by asset maintenance would achieve greater value.

periods. Because $X(A)$ is non-decreasing, $A_{\tau+2} \leq X(A_{\tau+1}+\eta)$, and for small η we have $A_{\tau+1}+\eta < X(A_\tau)$ by (a.30); thus, the new path is feasible and also satisfies the constraints that define the value-maximizing path $\{A_t\}$. Taking the derivative of period-τ value with respect to η,

$$\frac{dV_\tau}{d\eta} = \delta^\tau \left[-u'(c_\tau)\frac{1}{\alpha} + \delta u'(c_{\tau+1})\right] > 0,$$

where the inequality holds as a consequence of (a.29). This contradicts the definition of $\{A_t\}$ as a path that solves the problem in Lemma 14, and so establishes the lemma. ■

Proof of Proposition 2. Part (i) is obvious. "Only if" in part (ii) is also obvious, while "if" follows from Lemma 15. Likewise, the "only if" part of part (iii) is obvious, while the "if" part is a consequence of the fact that X is usc. Part (iv) once again is obvious. ■

We set the stage for Proposition 3 by establishing that H is increasing, so that H^- is well-defined:

LEMMA **16.** *$H(A)$ is increasing on $[B,\infty)$.*

Proof. Recall that $H(A)$ is the value of the maximization problem in Lemma 14, with $A_0 = A$. Because X is nondecreasing, It follows that H is increasing in A.[15] ■

Proof of Proposition 3. Let Y be the smallest equilibrium choice of continuation asset at A, and let V be the lowest value such that (Y,V) is a continuation equilibrium from A. By Lemma 3, we have

$$u\left(A - \frac{Y}{\alpha}\right) + \beta\delta V \geq D(A). \tag{a.31}$$

If (a.31) is slack, it is easy to show that Y must equal B and that V can be set equal to $L(B)$.[16] That generates the lowest possible equilibrium value at A and there is nothing left to prove; see the first inequality in Observation 1.

Otherwise (a.31) is binding for Y. In this case,

$$u\left(A - \frac{Y}{\alpha}\right) + \beta\delta V = D(A) \leq u\left(A - \frac{A'}{\alpha}\right) + \beta\delta V'. \tag{a.32}$$

---

[15] Starting from any higher asset level, it is feasible to choose the continuation asset A_1 (and then continuing with the earlier path $\{A_1, A_2, A_3 \ldots\}$).

[16] If strict inequality holds in (a.31), reduce continuation assets, always using a continuation on the upper envelope of the value correspondence, and sliding down the vertical portion of H at any point of discontinuity. (Public randomization allows us to do this.) Note that payoffs and continuation values change continuously as we do this. Eventually we come to $Y = B$ with continuation value $L(B)$.

for any other equilibrium continuation $\{A', V'\}$ at A. Because $A' \geq Y$ by definition, (a.32) shows that $V' \geq V$. It follows that

$$u\left(A - \frac{Y}{\alpha}\right) + \delta V \leq u\left(A - \frac{A'}{\alpha}\right) + \delta V', \tag{a.33}$$

so that once again, $\{Y, V\}$ implements minimum value at A.

To complete the proof of part (i), suppose that $Y > B$ while at the same time, $V < H^{-}(Y)$. Then it is obviously possible to reduce Y slightly while increasing continuation value at the same time.[17] Moreover, the new continuation has higher payoff, so it must be supportable as an equilibrium. Yet it has a lower continuation asset, which contradicts the definition of Y.

For part (ii), we adopt public randomization. Punish at A using Y, and implement the equilibrium continuation value V, which lies between $H^{-}(Y)$ and $H(Y)$ (by part (i)), by randomizing over two specific continuation values: (a) $H(Y)$, and (b) the value obtained by choosing the lowest equilibrium continuation asset at Y — call it Z — and following up with $H(Z)$.

For this randomization to work, two conditions must be met. First, $(Z, H(Z))$ must be an equilibrium continuation at asset level Y. Second, the value V must lie between the value generated by $(Z, H(Z))$, and $H(Y)$. The first condition is trivially met, because Z is the lowest equilibrium continuation at Y, and can certainly be supported by the highest continuation value $H(Z)$. For the second condition, we will now verify that

$$H^{-}(Y) \geq u\left(Y - \frac{Z}{\alpha}\right) + \delta H(Z). \tag{a.34}$$

LEMMA **17.** $Z < (1 - \upsilon)\alpha Y$.

Proof. Proposition 6 in the main text shows that there is always a Markov equilibrium policy ϕ with $\phi(A) < (1 - \upsilon)\alpha A$ for every $A \geq B$.[18] Because $Z \leq \phi(Y)$, the Lemma follows. ■

LEMMA **18.** *For each $\epsilon > 0$, there is $\gamma > 0$ and $\eta > 0$ such that if $A' \in [Y - \gamma, Y]$ and (A'', V'') is a continuation from A' with $A'' \geq Z - \eta$, then Z is a feasible choice at A', and the condition*

$$u\left(A' - \frac{A''}{\alpha}\right) + \beta\delta V'' \geq u\left(A' - \frac{Z}{\alpha}\right) + \beta\delta H(Z) \tag{a.35}$$

implies

$$u\left(A' - \frac{A''}{\alpha}\right) + \delta V'' \geq u\left(A' - \frac{Z}{\alpha}\right) + \delta H(Z) - \epsilon. \tag{a.36}$$

[17] Because $V < H^{-}(Y)$, there exists $Y' < Y$ and $V' \in \mathcal{V}(Y')$ such that $V' > V$.

[18] We should, of course, point out that Proposition 6 is established using separate arguments that rely on none of the lemmas in this Section.

Proof. Given ϵ and invoking Lemma 17, choose γ and η positive but small enough so that Z is feasible for $Y-\gamma$ (and therefore for all $A' \geq Y-\gamma$), and so that

$$\Delta \equiv u\left(Y-\gamma-\frac{Z-\eta}{\alpha}\right)-u\left(Y-\gamma-\frac{Z}{\alpha}\right) \leq \frac{\beta\epsilon}{1-\beta}. \tag{a.37}$$

If (a.35) holds for some $A' \in [Y-\gamma, Y]$ and continuation (A'', V'') with $A'' \geq Z-\eta$, then

$$\begin{aligned} \beta\delta[H(Z)-V''] &\leq u\left(A'-\frac{A''}{\alpha}\right)-u\left(A'-\frac{Z}{\alpha}\right) \\ &\leq u\left(A'-\frac{Z-\eta}{\alpha}\right)-u\left(A'-\frac{Z}{\alpha}\right) \\ &\leq u\left(Y-\gamma-\frac{Z-\eta}{\alpha}\right)-u\left(Y-\gamma-\frac{Z}{\alpha}\right) \\ &= \Delta, \end{aligned} \tag{a.38}$$

where the third inequality follows from the concavity of u. But (a.35) also implies that

$$u\left(A'-\frac{A''}{\alpha}\right)+\delta V'' \geq u\left(A'-\frac{Z}{\alpha}\right)+\delta H(Z)-\delta(1-\beta)[H(Z)-V'']. \tag{a.39}$$

Combining (a.37), (a.38) and (a.39), we see that

$$\begin{aligned} u\left(A'-\frac{A''}{\alpha}\right)+\delta V'' &\geq u\left(A'-\frac{Z}{\alpha}\right)+\delta H(Z)-\frac{(1-\beta)\Delta}{\beta} \\ &\geq u\left(A'-\frac{Z}{\alpha}\right)+\delta H(Z)-\epsilon, \end{aligned}$$

which establishes (a.36), as required. ∎

With Lemma 18 in hand, we return to the verification of (a.34). Let A^n be any sequence of assets, with $A^n < A^{n+1}$ for every n and with $A^n \to Y$. For every n, choose an equilibrium continuation (Z^n, V^n) to maximize equilibrium payoff from A^n:

$$u\left(A^n-\frac{Z^n}{\alpha}\right)+\beta\delta V^n,$$

where $Z^n \in [B, (1-\upsilon)\alpha A^n]$ and $V^n \in \mathcal{V}(Z^n)$ for every n.

Fix $\epsilon > 0$, and look at all indices n with $A^n \geq Y-\gamma$, where γ is given by Lemma 18. We claim that

$$Z^n \geq Z-\eta \text{ for all but finitely many } n. \tag{a.40}$$

where η is given by Lemma 18. For suppose that (a.40) is false along some infinite subsequence. We know that

$$u\left(A^k-\frac{Z^k}{\alpha}\right)+\beta\delta V^k \geq u\left(A^k-\frac{Z}{\alpha}\right)+\beta\delta H(Z)$$

for k along that subsequence. Letting (Z^*, V^*) denote a limit point of (Z^k, V^k), we have:

$$u\left(Y - \frac{Z^*}{\alpha}\right) + \beta\delta V^* \geq u\left(Y - \frac{Z}{\alpha}\right) + \beta\delta H(Z).$$

It follows that (Z^*, V^*) is an equilibrium continuation at Y. But $Z^* \leq Z - \eta$, which contradicts the fact that Z is the *lowest* equilibrium asset choice at Y. Therefore (a.40) holds. Moreover, for all such n, (a.35) holds for $A' = A^n$, $A'' = Z^n$ and $V'' = V^n$. By Lemma 18, (a.36) must hold as well, so that

$$u\left(A^n - \frac{Z^n}{\alpha}\right) + \delta V^n \geq u\left(A^n - \frac{Z}{\alpha}\right) + \delta H(Z) - \epsilon \tag{a.41}$$

for all n large enough. But H is nondecreasing by Lemma 16, so

$$H^-(Y) \geq H(A^n) \geq u\left(A^n - \frac{Z^n}{\alpha}\right) + \delta V^n. \tag{a.42}$$

for all n. Combining (a.41) and (a.42) and passing to the limit in n, we must conclude that

$$H^-(Y) \geq u\left(Y - \frac{Z}{\alpha}\right) + \delta H(Z) - \epsilon.$$

Because $\epsilon > 0$ is arbitrary, (a.34) is established.

It follows that the public randomization described above works to implement the worst punishment at A. Notice that such an implementation returns the individual to her highest continuation value function after at most two periods. ■

Proof of Proposition 4, part (i). First suppose that there is $\epsilon > 0$ with $X(A) \geq A$ on $[B, B+\epsilon]$. By nonuniformity, $X(A') < A'$ for some A'. X is nondecreasing, so $X(S) = S$ for some $S > B$, with $X(A') < A'$ for some $A' \in (S, S+\epsilon')$, for every $\epsilon' > 0$.[19] By Lemma 11, S is sustainable. Define $\mu \equiv S/B$. By Lemma 8 (a), $\mu X(A'/\mu)$ is an equilibrium choice for every $A' \in [S, S+\mu\epsilon]$. But then $X(A') \geq \mu X(A'/\mu) \geq A'$ for all such A', a contradiction.

It follows immediately that $X(B) = B$, and for all $\epsilon > 0$, there exists $A_\epsilon \in (B, B+\epsilon)$ such that $X(A_\epsilon) < A_\epsilon$. But if the result is false, there is also $A'_\epsilon \in (B, A_\epsilon)$ with $X(A'_\epsilon) \geq A'_\epsilon$. Because $X(A)$ is nondecreasing, these observations imply the existence of $S_\epsilon \in (B, B+\epsilon)$ such that $X(S_\epsilon) = S_\epsilon$. By Lemma 11, S_ϵ is sustainable for all $\epsilon > 0$. But for ϵ sufficiently small,

$$D(S_\epsilon) \geq u\left(S_\epsilon - \frac{B}{\alpha}\right) + \beta\delta L(B) \geq u\left(S_\epsilon - \frac{B}{\alpha}\right) + \beta\delta V^s(B) > P^s(S_\epsilon)$$

where the first inequality follows from the definition of D, the second from Lemma 1, and the third from Lemma 13. This is a contradiction. ■

[19] Take S to be the infimum of all A with $X(A) < A$.

LEMMA **19** (Observation 3 in main text)**.** *Suppose that asset levels S_1 and S_2, with $S_1 < S_2$, are both sustainable, and that $X(A) > A$ for all $A \in (S_1, S_2)$. Then there exists $A^* \geq B$ such that $X(A) > A$ for all $A > A^*$.*

Proof. Let $\mu_i \equiv S_i/B$ for $i = 1, 2$; then $\mu_1 < \mu_2$. We claim that there is $A^* \geq B$ such that for all $A > A^*$, there are $\tilde{A} \in (S_1, S_2)$ and integers $(m, n) \geq 0$ with $A = \mu_1^n \mu_2^m \tilde{A}$.

We first show that there is A^* such that for all $A > A^*$, $A \in (\mu_1^k S_1, \mu_2^k S_2)$ for some k. Because $\mu_1 < \mu_2$, there is an integer ℓ with $\mu_1^{k+2} < \mu_2^{k+1}$ for all $k \geq \ell$. For all such k, $(\mu_1^k S_1, \mu_2^k S_2) = (\mu_1^k S_1, \mu_2^{k+1} B)$ overlaps with $(\mu_1^{k+1} S_1, \mu_2^{k+1} S_2) = (\mu_1^{k+2} B, \mu_2^{k+1} S_2)$. So $\cup_{k=\ell}^{\infty} (\mu_1^k S_1, \mu_2^k S_2) = \big(\mu_1^\ell S_1, \infty\big)$. Take A^* to be any number greater than $\mu_1^\ell S_1$.

Next we show that for each integer $k \geq 1$ and $A \in (\mu_1^k S_1, \mu_2^k S_2)$, there is $\tilde{A} \in (S_1, S_2)$ along with an integer $m \in \{0, \ldots, k\}$ such that $A = \mu_1^m \mu_2^{k-m} \tilde{A}$. Divide the interval $(\mu_1^k S_1, \mu_2^k S_2)$ (which is the same as the interval $(\mu_1^{k+1} B, \mu_2^{k+1} B)$) into a sequence of semi-open sub-intervals (preceded by an open interval) that coincide at their endpoints: $(\mu_1^{k+1} B, \mu_1^k \mu_2 B)$, $[\mu_1^k \mu_2 B, \mu_1^{k-1} \mu_2^2 B)$, $\ldots$, $[\mu_1 \mu_2^k B, \mu_2^{k+1} B)$. A must lie in one of these intervals; call it $[\mu_1^{m+1} \mu_2^{k-m} B, \mu_1^m \mu_2^{k-m+1} B)$, which we can rewrite as $[\mu_1^m \mu_2^{k-m} S_1, \mu_1^m \mu_2^{k-m} S_2)$. (The left edge is open if it is the first interval.) Thus, setting $\tilde{A} = A \mu_1^{-m} \mu_2^{m-k}$, we have $\tilde{A} \in (S_1, S_2)$ and $A = \mu_1^m \mu_2^{k-m} \tilde{A}$, as desired.

To complete the proof, pick any $A > A^*$ along with some $\tilde{A} \in (S_1, S_2)$, integer $k \geq 1$ and $m \in \{0, \ldots, k\}$ for which $A = \mu_1^m \mu_2^{k-m} \tilde{A}$. By repeated application of Lemma 8 (a), we see that $X(A) \geq \mu_1^m \mu_2^{k-m} X(\tilde{A})$; noting that $X(\tilde{A}) > \tilde{A}$, we have $X(A) > A$. ■

Let us refer to the assertion of Proposition 4, part (ii), as the Conclusion. Lemma 19 (together with Lemma 15) implies the Conclusion, provided that the supposition of Lemma 19 is satisfied. Via Lemma 19, several other situations also imply the Conclusion. Define $E(A) \equiv P^s(A) - D(A)$.

LEMMA **20.** *$E(A) > 0$ for some $A > B$ implies the Conclusion.*

Proof. Because u is continuous and D is increasing, there is an interval $[S_1, S_2]$ such that $E(A') > 0$ for all $A' \in [S_1, S_2]$ (e.g., take $S_2 = A$ and S_1 to be an asset level slightly below S_2). Clearly, S_1 and S_2 are both sustainable (indeed, every $A' \in [S_1, S_2]$ is).

For each $A' \in [S_1, S_2)$, define $z(A')$ as the largest value in $[S_1, S_2]$ satisfying

$$u\left(A' - \frac{z(A')}{\alpha}\right) + \beta\delta V^s(z(A')) \geq D(A'). \tag{a.43}$$

Because $E(A') > 0$, we have $z(A') > A'$. Moreover, because $E(z(A')) > 0$, we know that $z(A')$ is sustainable. So (a.43) and Lemma 3 imply the existence of an equilibrium starting from A' in which assets increase to $z(A')$ immediately and then remain at $z(A')$ forever. It follows that $X(A') \geq z(A') > A'$ for all $A' \in (S_1, S_2)$. Therefore the condition of Lemma 19 is satisfied: there are assets S_1 and S_2 with $S_1 < S_2$, both sustainable, with $X(A') > A'$ for all $A' \in (S_1, S_2)$. The Conclusion follows. ■

Say that a sustainable asset S is *isolated* if there is an interval around S with no other sustainable asset in that interval.

LEMMA **21.** *If S is sustainable and not isolated, then the Conclusion is true.*

Proof. Assume that S is sustainable and not isolated. By nonuniformity and Lemma 8, there is $A^* > S$ with $X(A^*) > A^*$. If $X(A') > A'$ for all $A' \geq A^*$, the Conclusion follows (Lemma 15). Otherwise, $X(A') \leq A'$ for some $A' > A^*$. Because X is nondecreasing, there is $S^* > A^*$ such that $X(S^*) = S^*$, and $X(A') > A'$ for all $A' \in [A^*, S^*)$.[20] By Lemma 11, S^* is sustainable.

Because S isn't isolated, for every $\epsilon > 0$ there is sustainable S' with $|S' - S| < \epsilon$. Let $\mu \equiv S/B$ and $\mu' \equiv S'/B$. By Lemma 8 (a), $S_1 \equiv \mu S^*$ and $S_2 \equiv \mu' S^*$ are sustainable. Remember that $X(A') > A'$ for all $A' \in [A^*, S^*)$. Using this information, it is easy to see that if S and S' are close enough, then $X(A) > A$ for all $A \in (S_1, S_2)$,[21] because all such A can then be written in the form $\mu' A'$ for some $A' \in (A^*, S^*)$. But now all the conditions of Lemma 19 are met, so that the Conclusion follows. ■

A special case of a sustainable asset level is what we will refer to as an *upper sustainable asset level* $\hat{S}$, one for which $X(\hat{S}) = \hat{S}$, while $X(A) > A$ over an interval of the form $[\hat{S} - \theta, \hat{S})$ for some $\theta > 0$. (Note that by Lemma 11, $\hat{S}$ is sustainable.)

LEMMA **22.** *Let $\hat{S}$ be upper sustainable. Then there is $\epsilon > 0$, such that for every $A \in [\hat{S}, \hat{S} + \epsilon]$, there is an equilibrium which involves first-period continuation asset $A_1 < \hat{S}$, and has value $V(A) < V^s(\hat{S})$.*

Proof. Using Lemma 13 and the fact that $\hat{S}$ is upper sustainable, there are $\zeta > 0$ and $\epsilon_1 > 0$ such that for every $A \in [\hat{S}, \hat{S} + \epsilon_1]$,

$$u\left(A - \frac{\hat{S} - \zeta}{\alpha}\right) + \beta\delta V^s(\hat{S} - \zeta) \geq u\left(A - \frac{A_1}{\alpha}\right) + \beta\delta V^s(A_1) \tag{a.44}$$

whenever $A_1 \geq \hat{S}$, while at the same time,

$$X(A'') > A'' \text{ for all } A'' \in [\hat{S} - \zeta, \hat{S}). \tag{a.45}$$

---

[20] To see this, pick $S > A^*$ such that $X(S) = S$, and now take the infimum over all such values of S; call it S^*. Clearly, $S^* > A^*$ because $X(A^*) > A^*$ and X is nondecreasing.

[21] We presume that $S < S'$ without loss of generality.

By part (i) of this proposition, there is $\tilde{A} > B$ such that every equilibrium from $A \in [B, \tilde{A})$ monotonically descends to B. By Lemma 8 (a) and the fact that $\hat{S}$ is sustainable, there must be a corresponding equilibrium which monotonically descends from A to $\hat{S}$ for every $A \in [\hat{S}, \hat{\mu}\tilde{A})$, where $\hat{\mu} = \hat{S}/B$. Define $\epsilon_2 \equiv \min\{\epsilon_1, \hat{\mu}\tilde{A} - \hat{S}\}$.

Using the first inequality in (a.19) of Lemma 9,

$$V^s(\hat{S}) \geq \sum_{t=0}^{\infty} \delta^t u\left(A_t - \frac{A_{t+1}}{\alpha}\right) + u'\left(\frac{\alpha - 1}{\alpha}\hat{S}\right)\left(\delta - \frac{1}{\alpha}\right)\zeta$$

for any path $\{A_t\}$ starting from $\hat{S}$ with the property that $A_t \leq \hat{S}$ for all $t \geq 0$, and $A_1 \leq \hat{S} - \zeta$. But then there exists $\epsilon_3 > 0$ such that

$$V^s(\hat{S}) > \sum_{t=0}^{\infty} \delta^t u\left(A_t - \frac{A_{t+1}}{\alpha}\right) \tag{a.46}$$

for any path $\{A_t\}$ with $A_t \leq \hat{S}$ for all $t \geq 1$, $A_1 \leq \hat{S} - \zeta$, and $A_0 \leq \hat{S} + \epsilon_3$. Define $\epsilon \equiv \min\{\epsilon_2, \epsilon_3\}$.

Pick any $A \in [\hat{S}, \hat{S} + \epsilon]$, and consider any "descending equilibrium" as described just after (a.45), with payoff $P(A)$. Suppose that it has continuation (A_1, V_1). By Lemma 9, we know that $V_1 \leq V^s(A_1)$, so

$$u\left(A - \frac{A_1}{\alpha}\right) + \beta\delta V^s(A_1) \geq P(A). \tag{a.47}$$

Combining (a.44) and (a.47), we must conclude that

$$u\left(A - \frac{\hat{S} - \zeta}{\alpha}\right) + \beta\delta V^s(\hat{S} - \zeta) \geq P(A). \tag{a.48}$$

Now observe that (a.45), coupled with Lemma 14, implies that $H(\hat{S} - \zeta) \geq V^s(\hat{S} - \zeta)$. Using this information in (a.48), we must conclude that

$$u\left(A - \frac{\hat{S} - \zeta}{\alpha}\right) + \beta\delta H(\hat{S} - \zeta) \geq P(A). \tag{a.49}$$

So the continuation $\{\hat{S} - \zeta, H(\hat{S} - \zeta)\}$ is an equilibrium from every $A \in [\hat{S}, \hat{S} + \epsilon]$. To complete the proof, note that any path $\{A_t\}$ associated with this equilibrium satisfies $A_t \leq \hat{S}$ for all $t \geq 1$,[22] $A_1 \leq \hat{S} - \zeta$, and $A_0 \leq \hat{S} + \epsilon \leq \hat{S} + \epsilon_3$. Therefore (a.46) applies. ■

Recall the definition of $d^*(A)$ as the largest maximizer of (a.5).

[22] This follows from $X(\hat{S}) = \hat{S}$ and the fact that X is nondecreasing.

LEMMA **23.** *If* $d^*(A) = A$ *and* $d^*(A') \le A'$ *over* $A' \in [A, A+\epsilon]$ *for some* $\epsilon > 0$*, then* A *is sustainable.*[23]

Proof. We first show that

$$L^+(A) \le V^s(A). \tag{a.50}$$

By Lemma 5, L is increasing. So there is a sequence $\{A_n\}$ with $A_n \downarrow A$ and $L(A_n)$ (and $L^+(A_n)$) converging to $L^+(A)$. For each n, consider an equilibrium with the lowest value $V(A_n)$ among those that implement $Y(A_n)$.[24] Then

$$(1-\beta)u\left(A_n - \frac{Y(A_n)}{\alpha}\right) + \beta V(A_n) \ge D(A_n), \tag{a.51}$$

for all n. If strict inequality holds along a subsequence of n, then it's easy to see that $L(A_n) \le V(A_n) = u(A_n - B/\alpha) + \delta L(B)$ along that subsequence.[25] Passing to the limit, $L^+(A) \le u(A - B/\alpha) + \delta L(B) \le V^s(A)$, where the second inequality comes from part (i) of the Proposition, already proved, which yields $L(B) = V^s(B)$, together with Lemma 9. So (a.50) holds in this case. In the other case, we may presume that

$$(1-\beta)u\left(A_n - \frac{Y(A_n)}{\alpha}\right) + \beta V(A_n) = D(A_n) \tag{a.52}$$

for all n. But in turn,

$$D(A_n) = u\left(A_n - \frac{d^*(A_n)}{\alpha}\right) + \beta\delta M(d^*(A_n), A_n). \tag{a.53}$$

Combining (a.52) and (a.53), we see that for every n,

$$(1-\beta)u\left(A_n - \frac{Y(A_n)}{\alpha}\right) + \beta V(A_n) = u\left(A_n - \frac{d^*(A_n)}{\alpha}\right) + \beta\delta M(d^*(A_n), A_n). \tag{a.54}$$

Now we pass to the limit in (a.54). By assumption, $d^*(A^n) \le A^n$ for all n large, so $\lim_n d^*(A^n) < \alpha(1-\upsilon)A$.[26] By Lemma 7, d^* is right continuous at A, and so $d^*(A_n)$ converges to $d^*(A) = A$. By Lemma 6, $M(d^*(A_n), A_n) = L^+(d^*(A_n))$ for all n large enough, which converges to $L^+(d^*(A)) = L^+(A)$. Letting (Y, V) denote any limit point of $\{Y(A_n), V(A_n)\}$, we therefore have

$$(1-\beta)u\left(A - \frac{Y}{\alpha}\right) + \beta V = u\left(\frac{\alpha - 1}{\alpha}A\right) + \beta\delta L^+(A). \tag{a.55}$$

[23] In fact, a stronger property holds: if $d^*(A) \ge A$, then A is sustainable. That result follows directly from the existence of an everywhere-non-accumulating Markov-perfect equilibrium. Because we do not use the stronger property, nor do we focus on Markov equilibrium, we omit the proof.

[24] In line with Proposition 3, this value equals $L(A_n)$, but we do not use this fact anywhere in the proofs.

[25] Follow the same argument as in Footnote 16.

[26] That follows from $\alpha(1-\upsilon) > 1$, given $\alpha\delta > 1$ and $1 - \upsilon > \gamma$, where γ is the Ramsey rate of saving.

It follows that

$$\begin{aligned}\beta(1-\delta)L^{+}(A) &\leq \beta V - \beta\delta L^{+}(A)\\ &= u\left(\frac{\alpha-1}{\alpha}A\right) - (1-\beta)u\left(A-\frac{Y}{\alpha}\right)\\ &\leq u\left(\frac{\alpha-1}{\alpha}A\right) - (1-\beta)u\left(\frac{\alpha-1}{\alpha}A\right) = \beta(1-\delta)V^{s}(A), \quad \text{(a.56)}\end{aligned}$$

where the first inequality uses $V(A_n) \geq L(A_n)$ for all n, so that $V \geq L^{+}(A)$, the equality follows from transposing terms in (a.55), and the second inequality uses $d^{*}(A_n) \geq Y(A_n)$ for all n, and $d^{*}(A_n) \to A$, so that $A \geq Y$. But (a.56) again implies (a.50).

With (a.50) in hand, we must conclude that

$$\begin{aligned}u\left(\frac{\alpha-1}{\alpha}A\right) + \beta\delta V^{s}(A) &\geq u\left(\frac{\alpha-1}{\alpha}A\right) + \beta\delta L^{+}(A)\\ &= u\left(\frac{\alpha-1}{\alpha}A\right) + \beta\delta M(A,A)\\ &= D(A)\end{aligned}$$

(where the last equality follows from $d^{*}(A) = A$), which means that A is sustainable. ■

In the rest of the proof, we make the assumption (by way of ultimate contradiction) *that the Conclusion is false*. Note that because many of the steps to follow are based on this presumption, they cannot all be regarded as relationships that truly hold in the model.

LEMMA **24.** *Suppose that the Conclusion is false. Then*

(a) $d^{*}(\hat{S}) < \hat{S}$ *for any upper sustainable asset level* $\hat{S}$*, and*

(b) $d^{*}(A) \leq A$ *for all* $A \geq B$*, with strict inequality whenever* $X(A) \neq A$*.*

Proof. Part (a). Suppose not; then, since $X(\hat{S}) = \hat{S}$ (by the upper sustainability of $\hat{S}$), it follows from Lemma 4 that $d(\hat{S}) = \hat{S}$. We know that $M(\hat{S},\hat{S}) = L^{+}(\hat{S})$ (see footnote 26 and recall Lemma 6), but by Lemma 22,

$$M(\hat{S},\hat{S}) = L^{+}(\hat{S}) < V^{s}(\hat{S}).$$

Invoking (a.5) along with $d(\hat{S}) = \hat{S}$, we must therefore conclude that

$$D(\hat{S}) = u\left(\frac{\alpha-1}{\alpha}\hat{S}\right) + \beta\delta M(\hat{S},\hat{S}) < u\left(\frac{\alpha-1}{\alpha}\hat{S}\right) + \beta\delta V^{s}(\hat{S}) = P^{s}(\hat{S}),$$

or $E(\hat{S}) = P^{s}(\hat{S}) - D(\hat{S}) > 0$. By Lemma 20, the Conclusion follows, a contradiction.

Part (b). If false, then $d^{*}(A) > A$ for some $A \geq B$, or $d^{*}(A) \geq A$ for some $A \geq B$ with $X(A) \neq A$. By Lemma 4, $X(A) \geq d^{*}(A)$, so in either case $X(A) > A$. Note that there is $A' > A$ such that $X(A') \leq A'$, otherwise Lemma 15 assures us that the Conclusion holds. Define $\hat{S}$ by the infimum value of such A'. Then it is immediate that $\hat{S}$ is upper sustainable, and that $X(A'') > A''$ for all $A'' \in [A, \hat{S})$.

Recall that $d^*(A) \geq A$, that d^* is nondecreasing and that $d(\hat{S}) < \hat{S}$ by the upper sustainability of $\hat{S}$ and part (a) of this lemma. So there is $S \in [A, \hat{S})$ with $d^*(S) = S$ *and* $d^*(S') \leq S'$ for all S' in an interval to the right of S.[27] By Lemma 23, S is sustainable.

Set $S = S_1$ and $\hat{S} = S_2$. Recall that $X(A'') > A''$ for all $A'' \in [A, \hat{S})$, so the inequality holds in particular on (S_1, S_2). Now all the conditions of Lemma 19 are satisfied. Together with Lemma 15, we see that the Conclusion must hold, a contradiction. ■

Part (i) of the proposition, along with some of the foregoing lemmas, generates the following construction, *on the assumption that the Conclusion is false*. $X(A)$ starts out below A near B (there is a poverty trap by part (i)). By nonuniformity, $X(A) > A$ for some A; let A_* be the infimum value. $X(A) > A$ on an interval to the right of A_*; if not, sustainable stocks cannot all be isolated, and the Conclusion would follow from Lemma 21.[28] Moreover, by Lemma 15, if the Conclusion is false, there is $S^* < \infty$, defined as the supremum of all asset levels S greater than A_* such that $X(A) > A$ for all $A \in (A_*, S)$. Note that S^* is upper sustainable. (Also note that $X(A_*) > A_*$, otherwise the Conclusion is implied by setting $S_1 = A_*$ and $S_2 = S^*$, and applying Lemma 19.)

Part (i) of the proposition also tells us that $d^*(B) = B$. Let S_* be the largest asset level in $[B, S^*]$ for which $d^*(S) = S$.

LEMMA **25.** *S_* is well-defined, with $B \leq S_* < S^*$, and $X(S_*) = S_*$.*

Proof. By Lemmas 21 and 24, there is a finite set of points in $[B, S^*]$, all strictly smaller than S^*, for which $d^*(S) = S$. (B is one such point.) So S_* is well-defined and $B \leq S_* < S^*$. That $X(S_*) = S_*$ follows from part (b) of Lemma 24 and $d^*(S_*) = S_*$. ■

Figure A.1 summarizes the construction as well as the properties in Lemma 25. Panel A illustrates a case in which $S_* > B$, and Panel B, a case in which $S_* = B$. (Note: it is possible that $X(A) = A$ to the right of S_* and before S^*, though by Lemma 21, this can only happen at isolated points if the Conclusion is false.)

---

[27] To make this entirely clear, let $S \equiv \sup\{S' \in [A, \hat{S}) | d^*(S') > S'\}$. Because d^* is nondecreasing, $d^*(S) \geq S$. Moreover, $d^*(S) > S$ violates the definition of S (again, because d^* is nondecreasing).

[28] By definition of A_*, there is $\{A'_n\}$ converging down to A_* with $X(A'_n) > A'_n$. If the assertion in the text is false, there is $\{A''_n\}$ also converging down to A_* along which $X(A''_n) \leq A''_n$. But then, using the fact that X is nondecreasing, there must be a third sequence along which equality holds, which proves that non-isolated sustainable assets must exist.

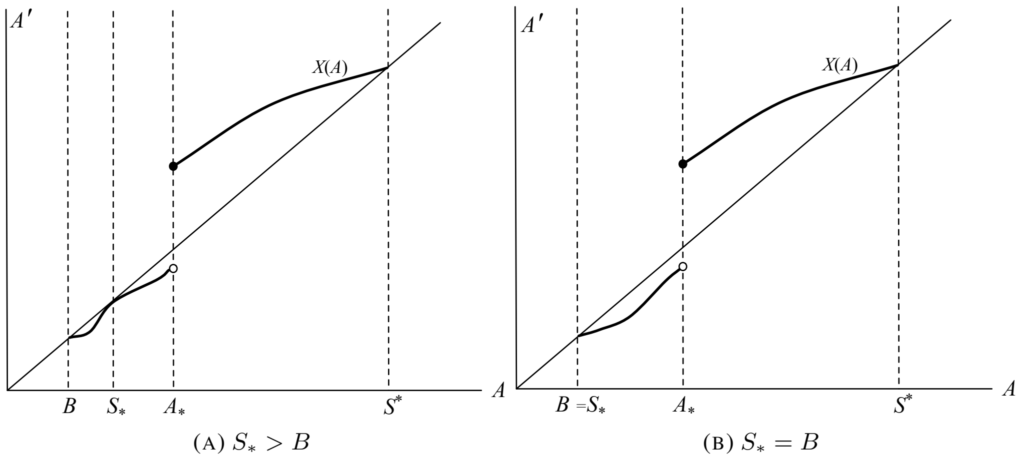

FIGURE A.1. THE TWO SUSTAINABLE ASSETS S_* AND S^*.

Define $Y^+(A)$ as the limit of $Y(A_n)$ as A_n converges down to A. Given Lemma 10, $Y^+(A)$ is well-defined and $Y^+(A) \geq Y(A)$.

LEMMA **26.** *If the Conclusion is false, $Y^+(S_*) \geq S_*$.*

Proof. If $S_* = B$ the result is trivially true, so assume that $S_* > B$. Suppose, on the contrary, that $Y^+(S_*) < S_*$. We first establish a stronger version of (a.50); namely, that

$$L^+(S_*) < V^s(S_*). \tag{a.57}$$

By part (b) of Lemma 24, $d^*(A) \leq A$ in a neighborhood to the right of S_* (indeed, strict inequality holds). With this in mind, carry out exactly the same argument as in the proof of Lemma 23, starting right after (a.50) and leading to (a.56), with S_* in place of A. We need two modifications to ensure that strict inequality in (a.50) holds. First, in case strict inequality holds in (a.51) along a subsequence, then $Y(A_n) = B$ and continuation values equal $L(B)$ along that subsequence, just as in the proof of Lemma 23, with the additional observation that (a.50) must indeed hold strictly, giving us (a.57). Otherwise, equality holds in (a.51), and (a.56) follows as before, with the additional implication that the second inequality in (a.56) — again, with S_* in place of A — must hold strictly, because $S_* > Y^+(S_*) \geq Y(S_*)$. We must therefore conclude that (a.57) holds, and therefore that

$$\begin{aligned} u\left(\frac{\alpha-1}{\alpha}S_*\right) + \beta\delta V^s(S_*) &> u\left(\frac{\alpha-1}{\alpha}S_*\right) + \beta\delta L^+(S_*) \\ &= D(S_*), \end{aligned}$$

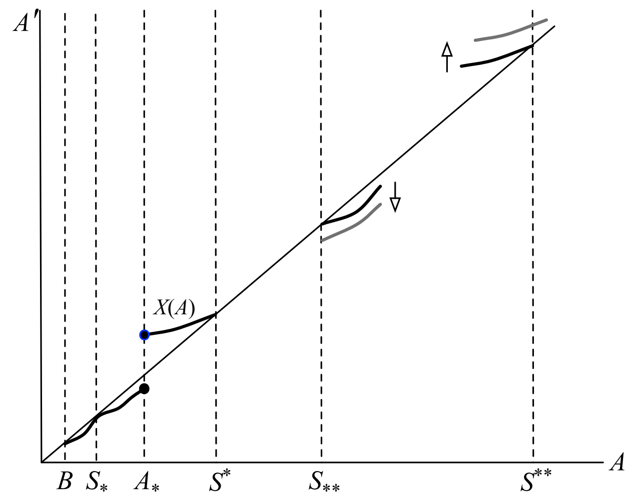

FIGURE A.2. OUTLINE OF THE PROOF STARTING FROM LEMMA 27.

where the equality follows from $d^*(S_*) = S_* < \alpha(1-\upsilon)S_*$, so that $L^+(S_*) = M(S_*, S_*)$ by Lemma 6. In other words, we have $E(S_*) > 0$. But then Lemma 20 assures us that the Conclusion must follow, which is a contradiction. ■

Let $\mu \equiv S^*/B$, and $\rho \equiv S_*/B$; then $\mu > \rho \geq 1$. Let $S_{**} \equiv \mu S_*$, and $S^{**} \equiv \mu S^*$. Note that $S_{**} = \mu S_* = \rho S^*$, so S_{**} is also a scaling of S^* by the factor ρ. (By Lemmas 11 and 25, S_* is sustainable, so Lemma 8 applies with both the scalings μ and ρ.) Here is an outline of the remainder of the proof. Refer to Figure A.2. By Lemma 8 (a), equilibria at assets to the right of S_* and to the left of S^* can be "scaled up" to assets beyond S_{**}, using the factor μ. Asset choices for such equilibria are partly indicated by the upper line to the right of S_{**} and the lower line to the left of S^{**}. But S_{**} is also a scaling of S^* (using ρ), so other equilibrium scalings are possible. In particular, Lemmas 8 and 22 tell us that equilibria with even lower values (and lower continuation assets) are achievable just above S_{**}; see the lower segment to the right of S_{**}. These values serve as punishments for deviations from even higher assets, and so support, in turn, larger asset choices near S^{**} relative to the earlier set of scaled equilibria; see the upper line around S^{**}. That creates a zone beyond S^{**} in which $X(A) > A$. If $X(A) > A$ for *all* $A > S^{**}$, Lemma 15 applies and the proof is complete. Otherwise, there is a first asset level beyond S^{**} at which $X(A) = A$ yet again. Now Lemma 20 applies, and contradicts the starting point of this entire construction: that the Conclusion is false.

Recall the definition of $L^+(x)$, and Lemma 6, which states that $M(x, A) = L^+(x)$ when $x < \alpha(1-\upsilon)A$. This property will play a more active role now.

LEMMA **27.** *Suppose that the Conclusion is false.* (a) *For all $x \geq B$,*

$$L(\mu x) \leq \mu^{1-\sigma} L(x). \tag{a.58}$$

and in particular,

$$M(\mu x, \mu A) \leq \mu^{1-\sigma} M(x, A), \tag{a.59}$$

for all $A \geq B$ and $x \in [B, \alpha(1-\upsilon)A]$. (b) *For every $A > S_*$ with $Y(\mu A) < S_{**}$ and for all $A' \in [S_*, A)$,*

$$L^+(\mu A') < \mu^{1-\sigma} L^+(A'). \tag{a.60}$$

Proof. It is easy to see that Lemma 8 (a) implies (a.58). (a.59) follows for $x \in [B, \alpha(1-\upsilon)A)$ by taking right-hand limits of L, and for $x = \alpha(1-\upsilon)A$ by applying (a.58) directly. To prove part (b), pick $A > S_*$ with $Y(\mu A) < S_{**}$. Let $\tilde{A} \in (S_*, A]$. Because $Y^+(S_*) \geq S_*$ (by Lemma 26), any equilibrium from $\tilde{A}$ that implements $L(\tilde{A})$ has continuation $\{\tilde{A}_1, \tilde{V}_1\}$ with $\tilde{A}_1 \geq S_*$ (by Lemma 10). By Lemma 8 (a), $\{\mu\tilde{A}_1, \mu^{1-\sigma}\tilde{V}_1\}$ is an equilibrium continuation at $\tilde{A}'' \equiv \mu\tilde{A} > S_{**}$. So

$$u\left(\tilde{A}'' - \frac{\mu\tilde{A}_1}{\alpha}\right) + \beta\delta\mu^{1-\sigma}\tilde{V}_1 \geq D(\tilde{A}''), \tag{a.61}$$

and

$$\mu\tilde{A}_1 \geq \mu S_* = S_{**}. \tag{a.62}$$

Consider an equilibrium with the lowest continuation value — call this $\underline{V}$ — among those that implement $Y(\tilde{A}'')$ from $\tilde{A}''$. Then

$$u\left(\tilde{A}'' - \frac{Y(\tilde{A}'')}{\alpha}\right) + \beta\delta\underline{V} \geq D(\tilde{A}''). \tag{a.63}$$

If (a.63) does not bind, then we know that $Y(\tilde{A}'') = B$ and $\underline{V} = L(B)$ (see footnote 16). Recalling that $\tilde{A}'' = \mu\tilde{A}$, we must therefore have

$$\begin{aligned} L(\mu\tilde{A}) &\leq u\left(\mu\tilde{A} - \frac{B}{\alpha}\right) + \delta L(B) \\ &\leq u\left(\mu\tilde{A} - \frac{\mu\tilde{A}_1}{\alpha}\right) + \delta\mu^{1-\sigma}\tilde{V}_1 - \frac{1-\beta}{\alpha\beta} u'\left(\mu\tilde{A} - \frac{B}{\alpha}\right)(\mu\tilde{A}_1 - B) \end{aligned}$$

$$\leq \quad u\left(\mu\tilde{A} - \frac{\mu\tilde{A}_1}{\alpha}\right) + \delta\mu^{1-\sigma}\tilde{V}_1 - \frac{1-\beta}{\alpha\beta}u'\left(\mu A - \frac{B}{\alpha}\right)(S_{**} - B)$$

$$= \quad \mu^{1-\sigma}L(\tilde{A}) - \frac{1-\beta}{\alpha\beta}u'\left(\mu A - \frac{B}{\alpha}\right)(S_{**} - B), \tag{a.64}$$

where the first inequality uses the definition of L, the second inequality uses Lemma 1, and the third inequality invokes (a.62) and $\tilde{A} \leq A$. On the other hand, if (a.63) does bind, then using (a.61) and noting that $\tilde{A}'' = \mu\tilde{A}$,

$$u\left(\mu\tilde{A} - \frac{\mu\tilde{A}_1}{\alpha}\right) + \beta\delta\mu^{1-\sigma}\tilde{V}_1 \geq u\left(\mu\tilde{A} - \frac{Y(\mu\tilde{A})}{\alpha}\right) + \beta\delta\underline{V}. \tag{a.65}$$

Let $\zeta \equiv S_{**} - Y(\mu A)$. Because Y is nondecreasing, we have $Y(\mu\tilde{A}) \leq S_{**} - \zeta \leq \mu\tilde{A}_1 - \zeta$. Using this information in (a.65) and observing that $\mu\tilde{A} \leq \mu A$, we must conclude that there exists $\eta_1 > 0$ with $\mu^{1-\sigma}\tilde{V}_1 \geq \underline{V} + \eta_1$, where η_1 might depend on A but can be chosen independently of $\tilde{A}$. Therefore, using (a.65) again, there is $\eta_2 > 0$ such that

$$u\left(\mu\tilde{A} - \frac{\mu\tilde{A}_1}{\alpha}\right) + \delta\mu^{1-\sigma}\tilde{V}_1 \geq u\left(\mu\tilde{A} - \frac{Y(\mu\tilde{A})}{\alpha}\right) + \delta\underline{V} + \eta_2,$$

or equivalently, $\mu^{1-\sigma}L(\tilde{A}) \geq L(\mu\tilde{A}) + \eta_2$. Combining this inequality with (a.64) and defining $\eta \equiv \min\{\eta_2, [(1-\beta)/\alpha\beta]u'(\mu A - B/\alpha)(S_{**} - B)\}$, we have

$$\mu^{1-\sigma}L(\tilde{A}) \geq L(\mu\tilde{A}) + \eta \tag{a.66}$$

for all $\tilde{A} \in (S_*, A]$. Taking right-hand limits as $\tilde{A} \downarrow A' \in [S_*, A)$ in (a.66) then implies that $L^+(\mu A') < \mu^{1-\sigma}L^+(A')$ for all $A' \in [S_*, A)$. ■

LEMMA **28.** *Suppose that the Conclusion is false, and that for some $A \geq B$,*

$$L^+(d^*(\mu A)) < \mu^{1-\sigma}L^+(d^*(\mu A)/\mu). \tag{a.67}$$

Then

$$D(\mu A) < \mu^{1-\sigma}D(A). \tag{a.68}$$

Proof. By Lemma 24, $d^*(A') \leq A'$ for all $A' \geq B$, so by Lemma 6, $M(A', A') = L^+(A')$. Using this observation along with (a.67), we see that

$$\begin{aligned} D(\mu A) &= u\left(\mu A - \frac{d^*(\mu A)}{\alpha}\right) + \beta\delta M(d^*(\mu A), \mu A) \\ &= \mu^{1-\sigma}u\left(A - \frac{d^*(\mu A)}{\mu\alpha}\right) + \beta\delta M(d^*(\mu A), \mu A) \end{aligned}$$

$$
\begin{aligned}
&= \mu^{1-\sigma} u\left(A - \frac{d^*(\mu A)}{\mu\alpha}\right) + \beta\delta L^+\left(d^*(\mu A)\right) \\
&< \mu^{1-\sigma}\left[u\left(A - \frac{d^*(\mu A)}{\mu\alpha}\right) + \beta\delta L^+\left(\frac{d^*(\mu A)}{\mu}\right)\right] \\
&\leq \mu^{1-\sigma}\left[u\left(A - \frac{d^*(A)}{\alpha}\right) + \beta\delta L^+\left(d^*(A)\right)\right] \\
&= \mu^{1-\sigma}\left[u\left(A - \frac{d^*(A)}{\alpha}\right) + \beta\delta M\left(d^*(A), A\right)\right] = \mu^{1-\sigma} D(A),
\end{aligned}
$$

where the second equality uses the constant-elasticity form of u, the strict inequality invokes (a.67), and the weak inequality follows from the definition of $d^*(A)$. ■

LEMMA **29.** *If the Conclusion is false,* $L^+(\mu A) < \mu^{1-\sigma} L^+(A)$ *for all* $A \in [S_*, S^*]$.

Proof. Because S^* is upper sustainable, Lemma 22 applies, so there is $\epsilon' > 0$ such that for every $A' \in (S^*, S^* + \epsilon']$, $Y(A') < S^*$. Because $S_{**} = \rho S^*$, Lemma 8 (a) implies that $Y(\rho A') < S_{**}$ for all such A'. In turn, this implies that for every $A'' \in (S_*, S_* + \epsilon]$, where $\epsilon \equiv \rho\epsilon'/\mu$, we have $Y(\mu A'') < S_{**}$. By part (b) of Lemma 27, $L^+(\mu A) < \mu^{1-\sigma} L^+(A)$ for all $A \in [S_*, S_* + \epsilon)$.

Suppose, by way of contradiction, that $L^+(\mu A) = \mu^{1-\sigma} L^+(A)$ for some $A \in [S_*, S^*]$. Let A^* be the infimum over such A. Then $A^* \geq S_* + \epsilon$ (by the conclusion of the last paragraph), and by the right-continuity of L^+,

$$L^+(\mu A^*) = \mu^{1-\sigma} L^+(A^*). \tag{a.69}$$

Define $A' \equiv \mu A^*$. There are now two cases to consider. First, if $d^*(A')/\mu > d^*(A^*)$,

$$
\begin{aligned}
D(\mu A^*) = D(A') &= u\left(A' - \frac{d^*(A')}{\alpha}\right) + \beta\delta M(d^*(A'), A') \\
&= \mu^{1-\sigma} u\left(A^* - \frac{d^*(A')}{\mu\alpha}\right) + \beta\delta M(d^*(A'), A') \\
&\leq \mu^{1-\sigma}\left[u\left(A^* - \frac{d^*(A')}{\mu\alpha}\right) + \beta\delta M\left(\frac{d^*(A')}{\mu}, \frac{A'}{\mu}\right)\right] \\
&< \mu^{1-\sigma} D(A^*), \qquad \text{(a.70)}
\end{aligned}
$$

where the weak inequality invokes (a.59), and the strict inequality the fact that $d^*(A^*)$ is the *largest* maximizer of $u\left(A^* - x/\alpha\right) + \beta\delta M\left(x, A^*\right)$, while $d^*(A')/\mu > d^*(A^*)$.

In the second case, $d^*(A')/\mu \leq d^*(A^*)$. Notice that (a.60) fails at $A = A^*$, so using part (b) of Lemma 27, $Y(\mu A) \geq S_{**}$ for all $A > A^*$. At the same time, $d^*(\mu A) \geq Y(\mu A)$ for all A (by Lemma 4). Combining these two observations, $d^*(\mu A) \geq S_{**}$ for all $A > A^*$.

By part (b) of Lemma 24, $d^*(\mu A) \leq \mu A$ for all A, so $\lim_{A \downarrow A^*} d^*(\mu A) \leq \mu A^* < \alpha(1-\upsilon)\mu A^*$. So Lemma 7 (b) applies, and d^* is right continuous at μA^*. Passing to the limit in the last inequality of the previous paragraph as $A \downarrow A^*$, it follows that $S_{**} \leq d^*(\mu A^*) = d^*(A')$, or $S_* \leq d^*(A')/\mu$. So in this second case,

$$S_* \leq d^*(A')/\mu \leq d^*(A^*) < A^*, \tag{a.71}$$

the last inequality following part (b) of Lemma 24, along with the fact that $A^* > S_*$, the latter being the largest value of $A \in [B, S^*]$ with $d^*(A) = A$.

In particular, (a.71) along with the definition of A^* allows us to verify condition (a.67) of Lemma 28 with A set equal to A^*. It follows that (a.68) holds at A^*. Recalling (a.70), we see then that in both cases

$$D(\mu A^*) < \mu^{1-\sigma} D(A^*). \tag{a.72}$$

Let $\{A_1, V_1\}$ be the equilibrium continuation that implements $L(A^*)$. By Lemma 8 (a), $\{\mu A_1, \mu^{1-\sigma} V_1\}$ is an equilibrium at μA^*, it has value equal to $\mu^{1-\sigma} L(A^*)$, and moreover, by the incentive constraint for $\{A_1, V_1\}$ coupled with (a.72),

$$u\left(\mu A^* - \frac{\mu A_1}{\alpha}\right) + \beta\delta\mu^{1-\sigma} V_1 \geq \mu^{1-\sigma} D(A^*) > D(\mu A^*).$$

This strict inequality, along with the fact that $\mu A_1 > B$, proves that one can lower equilibrium value at μA beyond the value created by scaling $\{A_1, V_1\}$, which shows that

$$L(\mu A^*) < \mu^{1-\sigma} L(A^*).$$

This contradicts the definition of A^*, and so completes the proof. ■

Proof of Proposition 4, part (ii). Assume the Conclusion is false. We claim that

$$E(S^{**}) = P^s(S^{**}) - D(S^{**}) > 0. \tag{a.73}$$

There are three possibilities to consider. First, $d^*(S^{**})/\mu \geq S_*$. We verify condition (a.67) of Lemma 28 with S^* in place of A. To do so, note that $d^*(S^{**})/\mu = d^*(\mu S^*)/\mu \geq S_*$, and also that $d^*(\mu S^*)/\mu \leq S^*$ by part (b) of Lemma 24. So we may apply Lemma 29 to $A = d^*(\mu S^*)/\mu$, and conclude that (a.68) is true for $A = S^*$. It follows that

$$D(S^{**}) < \mu^{1-\sigma} D(S^*). \tag{a.74}$$

Because $P^s(S^{**}) = \mu^{1-\sigma} P^s(S^*)$ and $P^s(S^*) \geq D(S^*)$, (a.74) immediately implies (a.73).

The second possibility is that $d^*(S^{**})/\mu < B$, so that $d^*(S^{**}) < \mu B = S^*$. Now apply part (b) of Lemma 8 by setting the path $\{\mu A_t^*\}$ in that lemma to the constant path with asset level $S^{**} = \mu S^*$ at every date.[29] It follows right away that $P^s(S^{**}) > D(S^{**})$, which establishes (a.73).

[29] This is our only use of part (b) of Lemma 8.

So the only remaining possibility is that

$$S_* > d^*(S^{**})/\mu \geq B. \tag{a.75}$$

Let d be a generic continuation asset choice that solves (a.5) at S^*. By Lemma 7 and the fact that $d^*(S_*) = S_*$, it must be the case that $d \geq S_*$. Because S^* is upper sustainable and so sustainable, and $d \geq S_* > d^*(S^{**})/\mu \geq B$, we see that if we define $A_1 \equiv d^*(S^{**})/\mu$, then

$$P^s(S^*) \geq D(S^*) > u\left(S^* - \frac{A_1}{\alpha}\right) + \beta\delta M(A_1, S^*). \tag{a.76}$$

Keeping in mind that $S^{**} = \mu S^*$ and $d^*(S^{**}) = \mu A_1$, we must conclude that

$$\begin{aligned} P^s(S^{**}) = \mu^{1-\sigma} P^s(S^*) &> \mu^{1-\sigma}\left[u\left(S^* - \frac{A_1}{\alpha}\right) + \beta\delta M(A_1, S^*)\right] \\ &= u\left(S^{**} - \frac{d^*(S^{**})}{\alpha}\right) + \beta\delta\mu^{1-\sigma} M(A_1, S^*) \\ &\geq u\left(S^{**} - \frac{d^*(S^{**})}{\alpha}\right) + \beta\delta M(d^*(S^{**}), S^{**}) \\ &= D(S^{**}), \end{aligned}$$

where the first inequality uses (a.76) and the second inequality uses (a.59). That gives us (a.73) again.

By Lemma 20, this immediately precipitates a contradiction, because (a.73) implies that the Conclusion follows, while we have been working with the presumption that the Conclusion is false. ■

APPENDIX C. THE SIMPLIFIED EXAMPLE

Proof of Proposition 5. Here we verify several technical claims which, when paired with the arguments in the main text, constitute a complete proof. Throughout, we remain on the grid $(A^0, A^1, A^2, \ldots)$, where $A^0 \equiv B$, and $A^{k+1} = \lambda A^k$ for all $k \geq 0$.

LEMMA **30.** *There exists $\bar{\lambda}_1 > 1$ such that, for all $\lambda \in (1, \bar{\lambda}_1)$, the unique value-maximizing asset trajectory for the simplified model is $(A^k, A^{k+1}, \ldots)$, and the unique value-minimizing asset trajectory is $(A^k, A^{k-1}, \ldots, A^1, A^0, A^0, \ldots)$.*

Proof. Consider first the alternative problem $\max \sum_{t=0}^{\infty} \delta^t u(c_t)$ for any given A_0, subject to the constraints $c_t = A_t - (A_{t+1}/\alpha) \geq 0$, $A_t \leq A_0\lambda^t$, and $A_t \geq \max\left\{A_0\lambda^{-t}, A^0\right\}$, but do not restrict assets to lie on the grid. We will show that, for $\lambda \in \left(1, (\alpha\delta)^{1/\sigma}\right)$, the unique solution is $A_t = A_0\lambda^t$ for all $t \geq 0$, with the associated consumption path $c_t = \left(1 - \frac{\lambda}{\alpha}\right) A_t$. Consider any other asset path $A_0', A_1', \ldots$ and the associated consumption path $c_0', c_1', \ldots$; we will show that it does not maximize value. (Standard results assure us that a solution exists, so the desired conclusion then follows.) Clearly, there must be a first period, s, in which $c_s' \neq \left(1 - \frac{\lambda}{\alpha}\right) A_0 \lambda^s$, and plainly we have $c_s' > \left(1 - \frac{\lambda}{\alpha}\right) A_0 \lambda^s$ (else $A_{s+1}' > A_0 \lambda^{s+1}$, in violation of the constraint). Let r be the first period after s in which $c_r' < \left(1 - \frac{\lambda}{\alpha}\right) A_0 \lambda^r$. (From the intertemporal budget constraint, we know that such a period exists.) Plainly, $A_t' < A_0 \lambda^t$ for $t = s+1, \ldots, r$. Consider an alternative asset path $A_0'', A_1'', \ldots$ such that $A_t'' = A_t'$ for $t \leq s$ and $t > r$, and $A_t'' = A_t' + \alpha^{t-s} x$ for $t = s+1, \ldots, r$. For the associated consumption path $c_0'', c_1'', \ldots$, we have $c_s'' = c_s' - x$, $c_r'' = c_r' + \alpha^{r-s} x$, and $c_t'' = c_t'$ otherwise. Note that this path is feasible for small x. Let $V(x)$ be the associated value. Then, provided $\lambda < (\delta\alpha)^{\frac{1}{\sigma}}$,

$$
\begin{aligned}
V'(0) &= \delta^r \alpha^{r-s} (c_r')^{-\sigma} - \delta^s (c_s')^{-\sigma} \\
&> \delta^r \alpha^{r-s} \left(\left(1 - \frac{\lambda}{\alpha}\right) A_0 \lambda^r \right)^{-\sigma} - \delta^s \left(\left(1 - \frac{\lambda}{\alpha}\right) A_0 \lambda^s \right)^{-\sigma} \\
&= \delta^s \left(\left(1 - \frac{\lambda}{\alpha}\right) A_0 \lambda^s \right)^{-\sigma} \left[\left(\frac{\delta\alpha}{\lambda^\sigma} \right)^{r-s} - 1 \right] \\
&\geq \delta^s \left(\left(1 - \frac{\lambda}{\alpha}\right) A_0 \lambda^s \right)^{-\sigma} \left[\left(\frac{\delta\alpha}{(\delta\alpha)^{\frac{1}{\sigma}\sigma}} \right)^{r-s} - 1 \right] \\
&= 0.
\end{aligned}
$$

Because $V'(0) > 0$, we know $A_0', A_1', \ldots$ is not value maximizing, which is what we set out to show.

For the simplified model, the set of feasible paths is a subset of the corresponding set for the alternative problem, and it always contains the trajectory with $A_t = A_0 \lambda^t$ for all t. Consequently, that trajectory also maximizes value in the simplified model, which is the first half of the lemma.

To prove the second half of the lemma, we will consider the alternative problem $\min \sum_{t=0}^{\infty} \delta^t u(c_t)$ with an initial asset level of $A_0 = A^k$ for some k, subject to the same constraints. We will show that there exists $\bar{\lambda}_1 \in \left(1, (\alpha\delta)^{1/\sigma}\right)$ such that, for all $\lambda \in \left(1, \bar{\lambda}_1\right)$ value is uniquely minimized with the asset trajectory $A_t = \max\left\{A^0, \frac{A^k}{\lambda^t}\right\}$, along with the consumption trajectory $c_t = \left(1 - \frac{1}{\lambda\alpha}\right) A_t$ for $t < k$ and $c_t = \left(1 - \frac{1}{\alpha}\right) A^0$ for $t \geq k$. Consider any other asset path $A_0', A_1', \ldots$ and the associated consumption path $c_0', c_1', \ldots$; we will show that it does not minimize value. (Standard results assure us that a solution exists, so the desired conclusion then follows.) Clearly, there must be a first period, s, in which $c_s' \neq c_s$, and plainly we have $c_s' < c_s$ (else $A_{s+1}' < \max\left\{A^0, \frac{A^k}{\lambda^{s+1}}\right\}$, in violation of the constraint). Notice that

$$
c_s' \geq \left(1 - \frac{\lambda}{\alpha}\right) A_s
$$

and

$$c'_{s+1} \le \left(1 - \frac{1}{\lambda\alpha}\right)\lambda A_s$$

so that

$$\frac{c'_{s+1}}{c'_s} \le \frac{\lambda - \frac{1}{\alpha}}{1 - \frac{\lambda}{\alpha}}.$$

Define

$$\bar{\lambda}_1 \equiv (\alpha\delta)^{1/\sigma}\left(\frac{\alpha + (\alpha\delta)^{-1/\sigma}}{\alpha + (\alpha\delta)^{1/\sigma}}\right).$$

Clearly, $\bar{\lambda}_1 < (\alpha\delta)^{1/\sigma}$. Using the last inequality, it is easy to check that $\lambda \in \left(0, \bar{\lambda}_1\right)$ implies $\frac{c'_{s+1}}{c'_s} < (\alpha\delta)^{1/\sigma}$.

Consider an alternative asset path $A''_0, A''_1, ...$ such that $A''_t = A'_t$ for $t \neq s+1$, and $A''_{s+1} = A'_t - \alpha x$. For the associated consumption path $c''_0, c''_1, ...$, we have $c''_s = c'_s + x$, $c''_{s+1} = c'_{s+1} - \alpha x$, and $c''_t = c'_t$ otherwise. Note that this path is feasible for small x. Let $V(x)$ be the associated value. Then

$$\begin{aligned} V'(0) &= \delta^s (c'_s)^{-\sigma} - \delta^{s+1}\alpha (c'_{s+1})^{-\sigma} \\ &< \delta^s \left(c'_s\right)^{-\sigma} - \delta^{s+1}\alpha\left((\alpha\delta)^{1/\sigma} c'_s\right)^{-\sigma} \\ &= 0. \end{aligned}$$

Because $V'(0) < 0$, we know $A'_0, A'_1, ...$ is not value minimizing, which is what we set out to show.

For the simplified model, the set of feasible paths is a subset of the corresponding set for the alternative problem, and it always contains the asset trajectory with $A_t = \max\left\{A^0, \frac{A^k}{\lambda^t}\right\}$ for all t. Consequently, that trajectory also minimizes value in the simplified model, which is the second half of the lemma. ■

LEMMA **31.** *There exists $\beta_D > 0$ such that a Markov-perfect equilibrium with decumulation exists iff $\beta \le \beta_D$.*

Proof. In the equilibrium, the individual chooses decumulation regardless of the asset level or history by which the asset level was reached. Let $A_0 = A^k$. Then the value of the decumulation path is

$$D^k \equiv \sum_{t=0}^{k-1} \delta^t \frac{\left[\left(1 - \frac{1}{\lambda\alpha}\right)\left(\frac{1}{\lambda}\right)^t A^k\right]^{1-\sigma}}{1-\sigma} + \frac{\delta^k}{1-\delta}\frac{\left[\left(1 - \frac{1}{\alpha}\right)\left(\frac{1}{\lambda}\right)^k A^k\right]^{1-\sigma}}{1-\sigma}$$

$$= \frac{\left(A^k\right)^{1-\sigma}}{1-\sigma}\left[\left(1-\frac{1}{\lambda\alpha}\right)^{1-\sigma}\left(\frac{1-\left(\frac{\delta}{\lambda^{1-\sigma}}\right)^k}{1-\frac{\delta}{\lambda^{1-\sigma}}}\right)+\frac{\delta^k}{1-\delta}\left[\left(1-\frac{1}{\alpha}\right)\left(\frac{1}{\lambda}\right)^k\right]^{1-\sigma}\right].$$

We will rewrite this as

$$D^k = \frac{\left(A^k\right)^{1-\sigma}}{1-\sigma}\left[\left(1-\frac{1}{\lambda\alpha}\right)^{1-\sigma}\left(\frac{1}{1-\frac{\delta}{\lambda^{1-\sigma}}}\right)+\Gamma(k)\right], \tag{a.77}$$

where

$$\Gamma(k) = \left(\frac{\delta}{\lambda^{1-\sigma}}\right)^k\left[\frac{(1-\frac{1}{\alpha})^{1-\sigma}}{1-\delta}-\frac{\left(\lambda-\frac{1}{\alpha}\right)^{1-\sigma}}{\lambda^{1-\sigma}-\delta}\right]. \tag{a.78}$$

It will be important to know how $\Gamma(k)$ varies with k. Because we have assumed $\frac{\delta}{\lambda^{1-\sigma}} < 1$, the absolute value of $\Gamma(k)$ declines with k. Whether it increases or decreases depends on the sign of

$$\frac{(1-\frac{1}{\alpha})^{1-\sigma}}{1-\delta}-\frac{\left(\lambda-\frac{1}{\alpha}\right)^{1-\sigma}}{\lambda^{1-\sigma}-\delta} \equiv \Phi(\lambda).$$

Notice that $\Phi(1)=0$. Let's calculate $\Phi'(\lambda)$ for $\lambda \geq 1$:

$$\begin{aligned}
\Phi'(\lambda) &= -\left[\frac{(1-\sigma)\left(\lambda-\frac{1}{\alpha}\right)^{-\sigma}(\lambda^{1-\sigma}-\delta)-(1-\sigma)\,\lambda^{-\sigma}\left(\lambda-\frac{1}{\alpha}\right)^{1-\sigma}}{\left(\lambda^{1-\sigma}-\delta\right)^2}\right] \\
&= -(1-\sigma)\frac{\left(\lambda-\frac{1}{\alpha}\right)^{-\sigma}}{\left(\lambda^{1-\sigma}-\delta\right)^2}\left(\left(\lambda^{1-\sigma}-\delta\right)-\lambda^{-\sigma}\left(\lambda-\frac{1}{\alpha}\right)\right) \\
&= (1-\sigma)\frac{\left(\lambda-\frac{1}{\alpha}\right)^{-\sigma}}{\left(\lambda^{1-\sigma}-\delta\right)^2}\left(\delta-\lambda^{-\sigma}\frac{1}{\alpha}\right).
\end{aligned}$$

Clearly, $\lambda-\frac{1}{\alpha} > 0$, and we have assumed $\frac{\delta}{\lambda^{1-\sigma}} < 1$, so $\lambda^{1-\sigma}-\delta > 0$. Furthermore, we have assumed $\lambda < (\delta\alpha)^{1/\sigma}$ and $\lambda > 1$, so $\delta > \lambda^{-\sigma}\frac{1}{\alpha}$. Thus, the derivative has the same sign as $1-\sigma$. Accordingly, for $\lambda > 1$, $\Phi(\lambda) > 0$ if $\sigma < 1$, which means $\Gamma(k)$ is positive and shrinks with k, and $\Phi(\lambda) < 0$ if $\sigma > 1$, which means $\Gamma(k)$ is negative and increases with k.

Now let's determine when decumulation is a Markov-perfect equilibrium, beginning with states $k \geq 1$. Assuming decumulation will always occur in the future, the payoff from decumulation is:

$$D^k_\beta = \frac{\left(A^k\left(1-\frac{1}{\lambda\alpha}\right)\right)^{1-\sigma}}{1-\sigma}+\beta\delta D^{k-1}$$

while the payoff from accumulation (which is followed by decumulation given how future selves play) is:

$$E^k_\beta = \frac{\left(A^k\left(1-\frac{\lambda}{\alpha}\right)\right)^{1-\sigma}}{1-\sigma}+\beta\delta D^{k+1}.$$

Deviating to accumulation is not profitable iff $D^k_\beta \geq E^k_\beta$, or

$$\beta\delta\left(D^{k-1} - D^{k+1}\right) \geq \frac{\left(A^k\right)^{1-\sigma}}{1-\sigma}\left[\left(1-\frac{\lambda}{\alpha}\right)^{1-\sigma} - \left(1-\frac{1}{\lambda\alpha}\right)^{1-\sigma}\right].$$

The right-hand side is negative. Clearly, $D^{k-1} < D^{k+1}$, so the term multiplying β on the left-hand side is also negative. Accordingly, we can rewrite the condition as $\beta \leq \beta_{Dk}$, where

$$\beta_{Dk} = \frac{1}{1-\sigma}\left[\left(1-\frac{1}{\lambda\alpha}\right)^{1-\sigma} - \left(1-\frac{\lambda}{\alpha}\right)^{1-\sigma}\right]\left[\frac{\delta\left(D^{k+1} - D^{k-1}\right)}{\left(A^k\right)^{1-\sigma}}\right]^{-1}. \tag{a.79}$$

Notice that the right-hand side is now a strictly positive number.

Next we observe that

$$D^{k+1} = \frac{\left(\lambda A^k\left(1-\frac{1}{\lambda\alpha}\right)\right)^{1-\sigma}}{1-\sigma} + \delta\frac{\left(A^k\left(1-\frac{1}{\lambda\alpha}\right)\right)^{1-\sigma}}{1-\sigma} + \delta^2 D^{k-1},$$

so

$$\begin{aligned}\frac{D^{k+1} - D^{k-1}}{\left(A^k\right)^{1-\sigma}} &= \frac{\lambda^{1-\sigma}\left(1-\frac{1}{\lambda\alpha}\right)^{1-\sigma}}{1-\sigma} + \delta\frac{\left(1-\frac{1}{\lambda\alpha}\right)^{1-\sigma}}{1-\sigma} \\ &\quad -(1-\delta^2)\frac{\lambda^{\sigma-1}}{1-\sigma}\left[\left(1-\frac{1}{\lambda\alpha}\right)^{1-\sigma}\left(\frac{1}{1-\delta\lambda^{\sigma-1}}\right) + \Gamma(k-1)\right].\end{aligned}$$

From the analysis above, we know that $\frac{1}{1-\sigma}\Gamma(k-1)$ declines with k. Therefore, β_{Dk}, which depends on k only through this term, also declines with k. Because we need the inequality to hold for all k, it must hold in the limit as $k \to \infty$. Using the fact that $\lim_{k\to\infty}\Gamma(k-1) = 0$, we obtain

$$\lim_{k\to\infty}\beta_{Dk} \equiv \beta_{D\infty} = \frac{1}{\delta}\left[\left(1-\frac{1}{\lambda\alpha}\right)^{1-\sigma} - \left(1-\frac{\lambda}{\alpha}\right)^{1-\sigma}\right]\left[\left(1-\frac{1}{\lambda\alpha}\right)^{1-\sigma}\left(\frac{\lambda^{1-\sigma} - \lambda^{\sigma-1}}{1-\delta\lambda^{\sigma-1}}\right)\right]^{-1},$$

which is strictly positive.

Now we turn to the case of $k = 0$. Deviating to accumulation only changes consumption in periods 0 and 1, and is not profitable as long as

$$\beta \leq \beta_{D0} = \frac{1}{\delta}\left[\left(1-\frac{1}{\alpha}\right)^{1-\sigma} - \left(1-\frac{\lambda}{\alpha}\right)^{1-\sigma}\right]\left[\lambda^{1-\sigma}\left(1-\frac{1}{\lambda\alpha}\right)^{1-\sigma} - \left(1-\frac{1}{\alpha}\right)^{1-\sigma}\right]^{-1},$$

which is also strictly positive. From the preceding analysis, the necessary and sufficient condition for a Markov-perfect equilibrium with decumulation is $\beta \leq \beta_D \equiv \min\{\beta_{D0}, \beta_{D\infty}\}$. ∎

LEMMA **32.** *For $k \geq 1$, the inequality $S^k_\beta \geq D^k_\beta$ is equivalent to the condition $\beta \geq \beta_k$, where β_k is strictly positive and decreasing in k.*

Proof. The value from the continual accumulation stream, starting with assets A^k, is

$$S^k = \sum_{t=0}^{\infty} \frac{\delta^t}{1-\sigma}\left[\left(1-\frac{\lambda}{\alpha}\right)\lambda^t A^k\right]^{1-\sigma} = \frac{\left(A^k\right)^{1-\sigma}}{1-\sigma}\frac{\left(1-\frac{\lambda}{\alpha}\right)^{1-\sigma}}{1-\delta\lambda^{1-\sigma}}. \tag{a.80}$$

The payoff from the same stream is

$$S_\beta^k = \frac{1}{1-\sigma}\left[\left(1-\frac{\lambda}{\alpha}\right)A^k\right]^{1-\sigma} + \beta\delta S^{k+1}.$$

Similarly, we can write

$$D_\beta^k = \frac{1}{1-\sigma}\left[\left(1-\frac{1}{\lambda\alpha}\right)A^k\right]^{1-\sigma} + \beta\delta D^{k-1}.$$

Thus we can rewrite the condition $S_\beta^k \geq D_\beta^k$ as

$$\beta\delta\left(S^{k+1}-D^{k-1}\right) \geq \frac{\left(A^k\right)^{1-\sigma}}{1-\sigma}\left[\left(1-\frac{1}{\lambda\alpha}\right)^{1-\sigma} - \left(1-\frac{\lambda}{\alpha}\right)^{1-\sigma}\right].$$

The right-hand side is positive. Clearly, $D^{k-1} < D^{k+1}$, and Lemma 30 tells us that $S^{k+1} > D^{k+1}$. Therefore, the term multiplying β on the left-hand side is also positive. Accordingly, we can rewrite the condition as $\beta \geq \beta_k$, where

$$\beta_k \equiv \frac{1}{1-\sigma}\left[\left(1-\frac{1}{\lambda\alpha}\right)^{1-\sigma} - \left(1-\frac{\lambda}{\alpha}\right)^{1-\sigma}\right]\left[\frac{\delta\left(S^{k+1}-D^{k-1}\right)}{\left(A^k\right)^{1-\sigma}}\right]^{-1} > 0. \tag{a.81}$$

Next we demonstrate that β_k is decreasing in k. Notice that

$$\frac{S^{k+1}}{\left(A^k\right)^{1-\sigma}} = \frac{\lambda^{1-\sigma}}{1-\sigma}\left(\frac{\left(1-\frac{\lambda}{\alpha}\right)^{1-\sigma}}{1-\delta\lambda^{1-\sigma}}\right),$$

which is independent of k. Moreover,

$$\frac{D^{k-1}}{\left(A^k\right)^{1-\sigma}} = \frac{\lambda^{\sigma-1}}{1-\sigma}\left[\left(1-\frac{1}{\lambda\alpha}\right)^{1-\sigma}\left(\frac{1}{1-\delta\lambda^{\sigma-1}}\right) + \Gamma(k-1)\right].$$

In the proof of Lemma 31, we showed that $\frac{\Gamma(k-1)}{1-\sigma}$ decreases with k. It follows that β_k also declines with k, as claimed. ■

LEMMA **33.** *There exists $\bar{\lambda}_2 > 1$ such that for every $\lambda \in (1, \bar{\lambda}_2)$, $\beta_D > \beta_\infty$.*

Proof. The proof proceeds in two steps. The first is to show that there exists $\bar{\lambda}' > 1$ such that, for every $\lambda \in (1, \bar{\lambda}')$, $\beta_{D0} > \beta_{D\infty}$, so that $\beta_D = \beta_{D\infty}$.[30] In the formula for $\beta_{D\infty}$, the numerator

[30] In all of the numerical cases we have considered, $\bar{\lambda}$ is no less than the Ramsey growth rate.

and denominator converge to 0 as $\lambda \to 1$. Applying L'Hospital's rule, we obtain:

$$\begin{aligned}\lim_{\lambda\to 1}\beta_{D\infty} &= \frac{\frac{2}{\alpha}(1-\sigma)\left(1-\frac{1}{\alpha}\right)^{-\sigma}}{2(1-\sigma)\left(1-\frac{1}{\alpha}\right)^{1-\sigma}\left(\frac{\delta}{1-\delta}\right)} \\ &= \frac{1-\delta}{\alpha\delta-\delta}.\end{aligned}$$

In the formula for β_{D0}, the numerator and denominator also converge to 0 as $\lambda \to 1$, so again we apply L'Hospital's rule:

$$\begin{aligned}\lim_{\lambda\to 1}\beta_{D0} &= \frac{\frac{1}{\alpha}(1-\sigma)\left(1-\frac{1}{\alpha}\right)^{-\sigma}}{\delta\left[(1-\sigma)\left(1-\frac{1}{\alpha}\right)^{1-\sigma}+\frac{1}{\alpha}(1-\sigma)\left(1-\frac{1}{\alpha}\right)^{-\sigma}\right]} \\ &= \frac{1}{\delta\alpha}.\end{aligned}$$

Accordingly,

$$\begin{aligned}\lim_{\lambda\to 1}\left(\beta_{D0}-\beta_{D\infty}\right) &= \frac{1}{\delta\alpha}-\frac{1-\delta}{\alpha\delta-\delta} \\ &= \frac{1}{\delta\alpha}\left[1-\frac{\alpha\delta-\delta(\alpha\delta)}{\alpha\delta-\delta}\right] \\ &> 0,\end{aligned}$$

where the final line uses our assumption that $\alpha\delta > 1$, which plainly implies $\frac{\alpha\delta-\delta(\alpha\delta)}{\alpha\delta-\delta} < 1$.

The second step is to show that there exists $\bar{\lambda}'' > 1$ such that, for every $\lambda \in (1, \bar{\lambda}'')$, $\beta_{D\infty} > \beta_\infty$. From a comparison of (a.79) and (a.81), we know that $\beta_{Dk} > \beta_k$ for all $k > 1$, because $S^{k+1} > D^{k+1}$ (Lemma 30). To show that the difference is preserved in the limit, so that $\beta_{D\infty} > \beta_\infty$, we must demonstrate that $\lim_{k\to\infty} S^{k+1}/\left(A^k\right)^{1-\sigma} > \lim_{k\to\infty} D^{k+1}/\left(A^k\right)^{1-\sigma}$. From (a.77), (a.78), and (a.80), we have

$$\lim_{k\to\infty}\frac{S^{k+1}}{\left(A^k\right)^{1-\sigma}} = \frac{\lambda^{1-\sigma}}{1-\sigma}\left(\frac{\left(1-\frac{\lambda}{\alpha}\right)^{1-\sigma}}{1-\delta\lambda^{1-\sigma}}\right)$$

and

$$\lim_{k\to\infty}\frac{D^{k+1}}{\left(A^k\right)^{1-\sigma}} = \frac{\lambda^{1-\sigma}}{1-\sigma}\left(\frac{\left(1-\frac{1}{\lambda\alpha}\right)^{1-\sigma}}{1-\frac{\delta}{\lambda^{1-\sigma}}}\right).$$

Define

$$Q(x) = \frac{\lambda^{1-\sigma}}{1-\sigma}\left(\frac{\left(1-\frac{x}{\alpha}\right)^{1-\sigma}}{1-\delta x^{1-\sigma}}\right).$$

Taking the derivative and simplifying, we obtain:

$$Q'(x) = \lambda^{1-\sigma} \frac{(1-\frac{x}{\alpha})^{-\sigma}}{(1-\delta x^{1-\sigma})^2} \left[\delta x^{-\sigma} - \frac{1}{\alpha} \right].$$

For $x < (\delta\alpha)^{\frac{1}{\sigma}}$, this expression is strictly positive. Thus, setting $\bar{\lambda}'' = (\delta\alpha)^{\frac{1}{\sigma}}$, we have

$$\lim_{k\to\infty} \frac{S^{k+1}}{(A^k)^{1-\sigma}} = Q(\lambda) > Q\left(\frac{1}{\lambda}\right) = \lim_{k\to\infty} \frac{D^{k+1}}{(A^k)^{1-\sigma}}.$$

The desired conclusion follows for $\bar{\lambda}_2 = \min\{\bar{\lambda}', \bar{\lambda}''\}$. ■

LEMMA **34.** *There exists $\bar{\lambda}_3 > 1$ such that if $\lambda \in (1, \bar{\lambda}_3)$ and $\beta < \beta_1$, there is no equilibrium of the simplified model in which the consumer accumulates wealth from A^0.*

Remark: *It follows from the previous claims that, when $\beta < \beta_k$, there is no equilibrium of the simplified model in which the consumer accumulates wealth from $A^1, ..., A^k$. Lemma 32 does not, however, cover the case of $k = 0$, and consequently we must deal with it separately.*

Proof. With $\beta < \beta_1$, the consumer necessarily decumulates from A^1. We claim that, for $\lambda \in (1, \bar{\lambda}_1)$, $\mathbf{A}_a = (A^0, A^1, A^0, A^1, ...)$ (i.e., where alternation between A^0 and A^1 continues forever) is the unique value maximizing asset trajectory starting from A^0. Consider any asset trajectory $A_0', A_1', A_2', ...$ other than $\mathbf{A}_a$. If that trajectory were value-maximizing, we would have a contradiction: contrary to Lemma 30, $\mathbf{A}_c = (A^0, A^0, A^0, ...)$ (i.e., where assets are constant at A^0 forever) would also be value-maximizing. To see why, observe that there must be some s such that $A_s' = A_{s+1}' = A^0$. Define the sequence $\mathbf{A}_1$ such that $A_{1t} = A_{t+s}'$. Plainly, $\mathbf{A}_1$ is also a value-maximizing trajectory (otherwise, we could substitute a continuation path with greater continuation value starting from period s of the original trajectory, thereby increasing its value). If this new trajectory is $\mathbf{A}_c$, then we are done, so assume it is not. Then there is a first period $r_1 > 0$ in which $A_{1r_1} = A^0$ and $A_{1,r_1+1} = A^1$. Construct a new asset trajectory $\mathbf{A}_2$ such that $A_{2t} = A_{1t}$ for $t \leq r_1$, and $A_{2t} = A_{1,t-r_1}$ for $t > r_1$. Plainly, $\mathbf{A}_2$ must also be a value-maximizing trajectory. Iterating this step, we generate a series of value-maximizing trajectories, where the asset level for the k-th trajectory is A^0 through period r_k, and where $\lim_{k\to\infty} r_k = \infty$. Because the transversality condition holds, these trajectories must generate the same value as $\mathbf{A}_c$. Therefore, $\mathbf{A}_c$ must also be a value-maximizing trajectory, as claimed.

In light of what we have just shown, accumulation is sustainable from A^0 if and only if the payoff associated with $\mathbf{A}_a$ (which provides the highest-value continuation path for a candidate equilibrium with initial accumulation) is no less than the payoff associated with $\mathbf{A}_c$ (which serves as the worst possible punishment path in light of Lemma 32).

The payoff associated with trajectory $\mathbf{A}_a$ is

$$P_a = \left(A^0\right)^{1-\sigma}\left[u\left(1-\frac{\lambda}{\alpha}\right)+\left(\frac{\beta\delta}{1-\delta}\right)\left(\frac{u\left(\lambda-\frac{1}{\alpha}\right)+\delta u\left(1-\frac{\lambda}{\alpha}\right)}{1+\delta}\right)\right].$$

The payoff associated with trajectory $\mathbf{A}_c$ is

$$P_c = \left(A^0\right)^{1-\sigma}\left[u\left(1-\frac{1}{\alpha}\right)+\frac{\beta\delta}{1-\delta}u\left(1-\frac{1}{\alpha}\right)\right].$$

Accumulation is sustainable from A^0 iff $P_a \geq P_c$, or equivalently (after some manipulation):

$$\beta \geq \beta_0' = \left(\frac{1-\delta}{\delta}\right)\frac{u\left(1-\frac{1}{\alpha}\right)-u\left(1-\frac{\lambda}{\alpha}\right)}{\left(\frac{u\left(\lambda-\frac{1}{\alpha}\right)+\delta u\left(1-\frac{\lambda}{\alpha}\right)}{1+\delta}-u\left(1-\frac{1}{\alpha}\right)\right)}.$$

As λ approaches unity, both the numerator and the denominator converge to zero. Applying L'Hospital's rule, we obtain:

$$\begin{aligned}\lim_{\lambda\to1}\beta_0' &= \left(\frac{1-\delta}{\delta}\right)\frac{\frac{1}{\alpha}u'\left(1\right)}{\frac{u'(1)-\frac{\delta}{\alpha}u'(1)}{1+\delta}}\\ &= \left(\frac{1-\delta}{\delta}\right)\frac{1+\delta}{\alpha-\delta}.\end{aligned}$$

To establish the claim, we must show that $\lim_{\lambda\to1}\beta_0' \geq \lim_{\lambda\to1}\beta_1$. The formula for the latter can be written as follows:

$$\beta_1 = \left(\frac{1-\delta}{\delta}\right)\frac{u\left(1-\frac{1}{\alpha\lambda}\right)-u\left(1-\frac{\lambda}{\alpha}\right)}{\frac{1-\delta}{1-\delta\lambda^{1-\sigma}}u\left(\lambda-\frac{\lambda^2}{\alpha}\right)-u\left(\frac{1}{\lambda}-\frac{1}{\alpha\lambda}\right)}.$$

Once again, as λ approaches unity, both the numerator and the denominator converge to zero. Applying L'Hospital's rule and simplifying, we obtain

$$\begin{aligned}\lim_{\lambda\to1}\beta_1 &= \left(\frac{1-\delta}{\delta}\right)\frac{\frac{2}{\alpha}u'\left(1-\frac{1}{\alpha}\right)}{\left(1-\frac{2}{\alpha}\right)u'\left(1-\frac{1}{\alpha}\right)+\left(1-\frac{1}{\alpha}\right)u'\left(1-\frac{1}{\alpha}\right)+\delta\left(1-\sigma\right)\frac{1}{1-\delta}u\left(1-\frac{1}{\alpha}\right)}\\ &= \left(\frac{1-\delta}{\delta}\right)\frac{2(1-\delta)}{2\alpha+2\delta-\alpha\delta-3}.\end{aligned}$$

We note in passing that $2\alpha+2\delta-\alpha\delta-3 = 2(\alpha-1)(1-\delta)+(\alpha\delta-1) > 0$.

Now observe that

$$\begin{aligned}\lim_{\lambda\to1}\beta_0' - \lim_{\lambda\to1}\beta_1 &= \frac{1+\delta}{\alpha-\delta}-\frac{2(1-\delta)}{2\alpha+2\delta-\alpha\delta-3}\\ &= \frac{(3-\delta)(\alpha\delta-1)}{(\alpha-\delta)\left(2\alpha+2\delta-\alpha\delta-3\right)} \;>\; 0.\end{aligned}$$

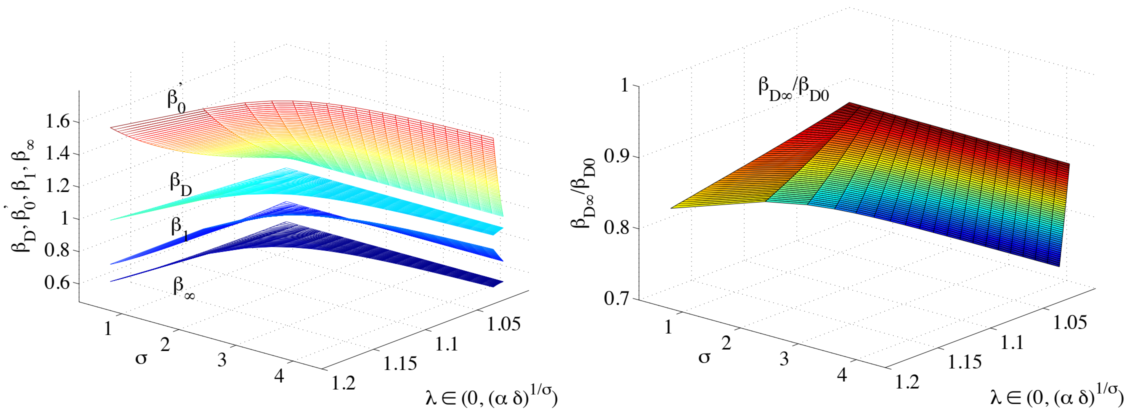

FIGURE A.3. EXAMPLE: β VALUES AND RATIOS, $\alpha = 1.3$, $\delta = 0.8$.

Consequently, there is some $\bar{\lambda}_3 \in (1, \bar{\lambda}_1)$ for which the lemma holds. ■

Remark: For the purpose of the proposition, $\bar{\lambda} = \min\left\{\bar{\lambda}_2, \bar{\lambda}_3\right\}$.

Numerical Examples. Although the claims in Lemma 30-34 are proved for λ close to unity, a wide range of numerical simulations support the claims for all $\lambda \in (0, (\alpha\delta)^{1/\sigma})$. Figure A.3 shows the values of $\beta_0', \beta_D, \beta_1$ and β_∞ for 20 different σ ranging between 0.25 and 4, and 50 different λ values with $\lambda \in (0, (\alpha\delta)^{1/\sigma})$.[31] The left panel of Figure A.3 confirms that for the full range of $\lambda \in (0, (\alpha\delta)^{1/\sigma})$, $\beta_1 > \beta_\infty$, $\beta_0' > \beta_1$ (Lemma 34). The right panel shows that $\beta_{D0} > \beta_{D\infty}$, and the two panels combined together confirm that $\beta_D = \beta_{D\infty} > \beta_\infty$ (Lemma 33).

APPENDIX D. MARKOV EQUILIBRIA

Proof of Proposition 6, part (i). We will show that there exists a linear Markov equilibrium policy function $\phi(A) = kA$ with $k \geq 1$. To this end, assume that all "future selves" employ the policy function $\phi(A) = kA$ with $k \in [1, \alpha]$ for all $A \geq B$. That yields the value function

$$V(A) = \frac{Q}{1-\sigma} A^{1-\sigma},$$

where

$$Q \equiv \frac{(\alpha - k)^{1-\sigma}}{\alpha^{1-\sigma}\left(1 - \delta k^{1-\sigma}\right)}. \tag{a.82}$$

[31] In these examples, δ and α are fixed at 0.8 and 1.3 respectively, as in the other numerical examples in this appendix and the main text. Changes in α and δ do not alter the qualitative nature of these pictures.

The individual's current problem is therefore to solve

$$\max_{x\in[B,\alpha(1-\upsilon)A]} \frac{1}{1-\sigma}\left[\left(A-\frac{x}{\alpha}\right)^{1-\sigma}+\beta\delta Q x^{1-\sigma}\right].$$

The corresponding necessary and sufficient first-order condition is

$$\frac{1}{\alpha}\left(A-\frac{x}{\alpha}\right)^{-\sigma}=\beta\delta Q x^{-\sigma}.$$

After some manipulation, we obtain

$$\frac{A}{x}=\frac{1}{\alpha}+\left(\frac{1}{\alpha\beta\delta Q}\right)^{1/\sigma}\equiv\frac{1}{k^*} \tag{a.83}$$

Note that $x=k^*A$. Accordingly, the policy function is an equilibrium if $k^*=k$. Substituting (a.82) into (a.83) and rearranging yields

$$k^{\sigma}=\alpha\beta\delta+(1-\beta)\,\delta k. \tag{a.84}$$

Define $\Lambda(k)\equiv k^{\sigma}$ and $\Phi(k)=\alpha\beta\delta+(1-\beta)\,\delta k$. Notice that $\Lambda(1)\leq\Phi(1)$ (given that $\beta\delta(\alpha-1)/(1-\delta)\geq 1$), and $\Lambda(\alpha)>\Phi(\alpha)$ (given the transversality condition $\delta\alpha^{1-\sigma}<1$). By continuity, it follows that there exists a solution on the interval $[1,\alpha)$, which establishes the proposition. ■

Proof of Proposition 6, part (ii). The proof proceeds in two steps.

Step 1. An individual best-responds to a stationary Markov policy, subject to an additional (artificial) constraint $A'\leq A$, where A' is the continuation asset choice. For this modified game, there exists a non-decreasing usc Markov policy function.

To establish this step, we construct a sequence of policy functions $\{\phi^n\}$ and value functions $\{V^n\}$ as follows:

$\phi^0(A)=B$ for all $A\geq B$ and $V^0(A)=u\left(A-\frac{B}{\alpha}\right)+\frac{\beta\delta}{1-\delta}u\left(B\left(1-\frac{1}{\alpha}\right)\right)$, and for $n>0$,

$$\phi^n(A)=\max\Phi^n(A) \tag{a.85}$$

where

$$\Phi^n(A)=\arg\max_{A'\in[B,A]}\left[u\left(A-\frac{A'}{\alpha}\right)+\beta\delta V^{n-1}(A')\right],$$

and

$$V^n(A)=u\left(A-\frac{\phi^n(A)}{\alpha}\right)+\delta V^{n-1}\left(\phi^n(A)\right). \tag{a.86}$$

We claim that, for all n, ϕ^n and V^n are well-defined and usc, and ϕ^n is non-decreasing. Clearly, ϕ^0 and V^0 have these properties. Now suppose ϕ^{n-1} and V^{n-1} have these properties; we will show that ϕ^n and V^n also have them. Because u is continuous and V^{n-1} usc, we know that

$\Phi^n(A)$ is a nonempty-valued, compact-valued correspondence.[32] Therefore $\phi^n(A) \equiv \max \Phi^n(A)$ is well-defined. The fact that ϕ^n is non-decreasing follows from a standard single-crossing argument that relies on the strict concavity of u. It remains to prove that ϕ^n and V^n are usc.

To this end, consider a sequence $A^k \to A$ with $\phi^k \equiv \phi^n(A^k)$ converging to some asset level ϕ. We claim that $\phi^n(A) \geq \phi$. If not, then in particular, ϕ is not optimal at A, and there exists another continuation asset $x < \phi$ with

$$u\left(A - \frac{x}{\alpha}\right) + \beta\delta V^{n-1}(x) > u\left(A - \frac{\phi}{\alpha}\right) + \beta\delta V^{n-1}(\phi).$$

At the same time, because V^{n-1} is usc,

$$u\left(A - \frac{\phi}{\alpha}\right) + \beta\delta V^{n-1}(\phi) \geq \limsup_k \left[u\left(A^k - \frac{\phi^k}{\alpha}\right) + \beta\delta V^{n-1}(\phi^k)\right],$$

but these two inequalities allow us to conclude that for k large enough, $x < A^k$ and

$$u\left(A^k - \frac{x}{\alpha}\right) + \beta\delta V^{n-1}(x) > u\left(A^k - \frac{\phi^k}{\alpha}\right) + \beta\delta V^{n-1}(\phi^k),$$

which contradicts the optimality of ϕ^k at A^k. So we have established that $\phi^n(A)$ is usc, and in particular, that $\phi^n(A) \geq \phi$ and

$$u\left(A - \frac{\phi^n(A)}{\alpha}\right) + \beta\delta V^{n-1}\left(\phi^n(A)\right) \geq u\left(A - \frac{\phi}{\alpha}\right) + \beta\delta V^{n-1}\left(\phi\right). \tag{a.87}$$

Because $\phi^n(A) \geq \phi$, (a.87) implies that $V^{n-1}\left(\phi^n(A)\right) \geq V^{n-1}\left(\phi\right)$, so that adding $(1-\beta)\delta V^{n-1}\left(\phi^n(A)\right)$ and $(1-\beta)\delta V^{n-1}\left(\phi\right)$ respectively to the left- and right-hand sides of (a.87), we have

$$u\left(A - \frac{\phi^n(A)}{\alpha}\right) + \delta V^{n-1}\left(\phi^n(A)\right) \geq u\left(A - \frac{\phi}{\alpha}\right) + \delta V^{n-1}\left(\phi\right). \tag{a.88}$$

Using (a.88) and the fact that V^{n-1} is usc, we must conclude that

$$\begin{aligned} V^n(A) &= u\left(A - \frac{\phi^n(A)}{\alpha}\right) + \delta V^{n-1}\left(\phi^n(A)\right) \\ &\geq u\left(A - \frac{\phi}{\alpha}\right) + \delta V^{n-1}\left(\phi\right) \\ &\geq \lim_{k\to\infty}\left[u\left(A^k - \frac{\phi^n\left(A^k\right)}{\alpha}\right) + \delta V^{n-1}\left(\phi^n(A^k)\right)\right] \\ &= \lim V^n(A^k), \end{aligned}$$

as desired. This completes our inductive claim.

---

[32] It suffices to prove that $\Phi^n(A)$ is closed-valued, which is a standard exercise given that V^{n-1} is usc.

Next, for each $n > 1$, define

$$\theta^n \equiv \sup\left\{A \geq B \mid \phi^n(A') = \phi^{n-1}(A') \text{ for all } A' < A\right\}.$$

Because $\beta < \beta^*$, we have $\frac{\beta\delta}{1-\delta}(\alpha - 1) < 1$, so that $\theta^1 > B$ and $V^1(A) = V^0(A)$ for $A \in \left[B, \theta^1\right)$. Now we argue that if $\theta^n > B$ and $V^n(A) = V^{n-1}(A)$ for $A \in [B, \theta^n)$, then $\theta^{n+1} \geq \theta^n > B$ and $V^{n+1}(A) = V^n(A)$ for $A \in \left[B, \theta^{n+1}\right)$. Recall that

$$\Phi^{n+1}(A) = \arg \max_{A' \in [B,A]} \left[u\left(A - \frac{A'}{\alpha}\right) + \beta\delta V^n(A')\right].$$

But for all $A \in [B, \theta^n)$ and $A' \in [B, A]$, we have

$$u\left(A - \frac{A'}{\alpha}\right) + \beta\delta V^n(A') = u\left(A - \frac{A'}{\alpha}\right) + \beta\delta V^{n-1}(A'),$$

which implies $\Phi^{n+1}(A) = \Phi^n(A)$ for such A, and hence $\phi^{n+1}(A) = \phi^n(A)$. Thus, $\theta^{n+1} \geq \theta^n > B$. Moreover, for $A \in \left[B, \theta^{n+1}\right)$, we have

$$\begin{aligned} V^{n+1}(A) &= u\left(A - \frac{\phi^{n+1}(A)}{\alpha}\right) + \beta\delta V^{n-1}\left(\phi^{n+1}(A)\right) \\ &= u\left(A - \frac{\phi^n(A)}{\alpha}\right) + \beta\delta V^{n-1}\left(\phi^n(A)\right) \\ &= V^n(A), \end{aligned}$$

as desired.

From the preceding argument, it follows that θ^n is a non-decreasing sequence. There are two possibilities: (i) θ^n increases without bound, and (ii) θ^n converges to a finite bound, θ^*.

In case (i), we take $\phi^R(A) = \lim_{n\to\infty} \phi^n(A)$ for all A. For any finite interval $[B, \theta]$, there exists n' such that $\theta^n > \theta$ for $n \geq n'$, which implies $\phi^R(A) = \phi^n(A)$ for such n and all $A \in [B, \theta]$. It follows that ϕ^R is well-defined. Defining

$$V^R(A) = \sum_{k=0}^{\infty} \delta^k u\left(\left(\phi^R\right)^k(A) - \frac{\left(\phi^R\right)^{k+1}(A)}{\alpha}\right),$$

we plainly have $V^R(A) = V^{n'}(A)$ for all $A \in [B, \theta]$. We know that $\phi^{n+1}(A)$ solves

$$\max_{A' \in [B,A]} \left[u\left(A - \frac{A'}{\alpha}\right) + \beta\delta V^n(A')\right];$$

in light of the fact that, for $n \geq n'$ and $A \in [B, \theta]$, we have $\phi^{n+1}(A) = \phi^R(A)$ and $V^{n+1}(A) = V^n(A) = V^R(A)$, it follows that $\phi^R(A)$ solves

$$\max_{A' \in [B,A]} \left[u\left(A - \frac{A'}{\alpha}\right) + \beta\delta V^R(A')\right]$$

for $A \in [B, \theta]$. Because the preceding statement holds for all θ, ϕ^R is an equilibrium policy function.

Now consider case (ii). Define $\mu = \frac{\theta^*}{B}$. Consider a sequence of intervals $I^k = [\mu^k B, \mu^{k+1} B)$ for $k = 0, 1, ...$ For any $A \in I^k$, we take $\phi^R(A) = \mu^k \lim_{n\to\infty} \phi^n \left(\frac{A}{\mu^k}\right)$. Using precisely the same argument as in case (i), it follows that, for $A \in I^0$, $\phi^R(A)$ is a best choice at A when future decisions are governed by ϕ^R. In the next paragraph, we show that $\phi^R(\theta^*) = \mu \lim_{n\to\infty} \phi^n \left(\frac{\theta^*}{\mu}\right) = \theta^*$ is likewise a best choice at θ^* when future decisions are governed by ϕ^R.

Consider any sequence $A^k \downarrow \theta^*$. We claim there cannot exist k and n such that $\phi^{n+1}(A^k) < \theta^n$. Suppose on the contrary that the preceding inequality holds for some k and n. From the concavity of u, we would then have $\phi^{n+1}(A) < \theta^n$ for all $A \leq A^k$. But that implies $\phi^{n+1}(A) = \phi^n(A)$ for all $A \leq A^k$ (because $V^n(A) = V^{n-1}(A)$ for $A < \theta^n$), so $\theta^{n+1} > \theta^*$, a contradiction, which establishes the claim. It follows that $\left(A^k, \phi^k(A^k)\right)$ converges to (θ^*, θ^*). Because $\frac{\beta\delta}{1-\delta}(\alpha - 1) < 1$ and all asset trajectories induced by ϕ^k are non-increasing, we have

$$\begin{aligned}
\text{(a.89)} \qquad & \lim_{k\to\infty} \sup \left[u\left(A^k - \frac{\phi^k(A^k)}{\mu}\right) + \beta\delta V^{k-1}\left(\phi^k(A^k)\right)\right] \\
\leq \quad & \lim_{k\to\infty} \sup \left[u\left(A^k - \frac{\phi^k(A^k)}{\mu}\right) + \frac{\beta\delta}{1-\delta} u\left(\phi^k(A^k)\left(1 - \frac{1}{\alpha}\right)\right)\right] \\
= \quad & \left(1 + \frac{\beta\delta}{1-\delta}\right) u\left(\theta^*\left(1 - \frac{1}{\alpha}\right)\right)
\end{aligned}$$

Now suppose some $A^+ < \theta^*$ is a strictly better choice than θ^* from θ^* when future decisions are governed by ϕ^R. For some $\Delta > 0$, we therefore have

$$u\left(\theta^* - \frac{A^+}{\alpha}\right) + \beta\delta V^R(A^+) > \left(1 + \frac{\beta\delta}{1-\delta}\right) u\left(\theta^*\left(1 - \frac{1}{\alpha}\right)\right) + \Delta.$$

For k sufficiently large, it must therefore also be the case that

$$u\left(A^k - \frac{A^+}{\alpha}\right) + \beta\delta V^R(A^+) > u\left(A^k - \frac{\theta^*}{\mu}\right) + \frac{\beta\delta}{1-\delta} u\left(\theta^*\left(1 - \frac{1}{\alpha}\right)\right) + \Delta.$$

Taking k large enough so that $\theta^k > A^+$ (which implies $V^R(A^+) = V^k(A^+)$), we then have

$$u\left(A^k - \frac{A^+}{\alpha}\right) + \beta\delta V^{k-1}(A^+) > u\left(A^k - \frac{\theta^*}{\mu}\right) + \frac{\beta\delta}{1-\delta} u\left(\theta^*\left(1 - \frac{1}{\alpha}\right)\right) + \Delta.$$

Taking limits, we have

$$\text{(a.90)} \qquad \lim_{k\to\infty} \left[u\left(A^k - \frac{A^+}{\alpha}\right) + \beta\delta V^{k-1}(A^+)\right] \geq \left(1 + \frac{\beta\delta}{1-\delta}\right) u\left(\theta^*\left(1 - \frac{1}{\alpha}\right)\right) + \Delta$$

Combining (a.89) and (a.90) we see that, for sufficiently large k, A^+ is a better choice that $\phi^k(A^k)$ starting from A^k when future decisions are governed by ϕ^{k-1}, a contradiction. Thus, θ^* is indeed a best choice at θ^* when future decisions are governed by ϕ^R.

We have established that, for $A \in I^0 \cup \{\theta^*\}$, $\phi^R(A)$ is a best choice at A when future decisions are governed by ϕ^R. Now suppose that, for $A \in \left(\cup_{n=0}^k I^n\right) \cup \{\mu^{n+1}B\}$, $\phi^R(A)$ is a best choice at A when future decisions are governed by ϕ^R. We claim that the same statement holds for $A \in I^{k+1} \cup \{\mu^{k+2}B\}$. Take any such A. For a deviation to any $A' \in I^{k+1}$, we have by construction

$$\begin{aligned}
u\left(A - \frac{A'}{\alpha}\right) + \beta\delta V_{\phi^R}(A') &= \left(\mu^{k+1}\right)^{1-\sigma}\left[u\left(\frac{A}{\mu^{k+1}} - \frac{A'}{\mu^{k+1}\alpha}\right) + \beta\delta V_{\phi^R}\left(\frac{A'}{\mu^{k+1}}\right)\right] \\
&\leq \left(\mu^{k+1}\right)^{1-\sigma} P_{\phi^R}\left(\frac{A}{\mu^{k+1}}\right) \\
&= P_{\phi^R}(A),
\end{aligned}$$

which implies that $\phi^R(A)$ is at least as good a choice as A'. Now consider any $A' \in [B, \mu^{k+1}B)$. Recalling that $\phi^R\left(\mu^{k+1}B\right) = \mu^{k+1}B$, we have by hypothesis

$$u\left(\mu^{k+1}B - \frac{A'}{\alpha}\right) + \beta\delta V_{\phi^R}(A') \leq u\left(\mu^{k+1}B - \frac{\mu^{k+1}B}{\alpha}\right) + \beta\delta V_{\phi^R}(\mu^{k+1}B).$$

But then, by the concavity of u (and using $\mu^{k+1}B \in I^{k+1}$), for $A \in I^{k+1} \cup \{\mu^{k+2}B\}$ we have

$$\begin{aligned}
u\left(A - \frac{A'}{\alpha}\right) + \beta\delta V_{\phi^R}(A') &\leq u\left(A - \frac{\mu^{k+1}B}{\alpha}\right) + \beta\delta V_{\phi^R}(\mu^{k+1}B) \\
&\leq u\left(A - \frac{\phi^R(A)}{\alpha}\right) + \beta\delta V_{\phi^R}(\phi^R(A)),
\end{aligned}$$

which again implies that $\phi^R(A)$ is at least as good a choice as A'. Applying induction, we see that ϕ^R is an equilibrium policy function.

Step 2. Now we construct a Markov equilibrium policy function ϕ for the original game. There are two possibilities to consider: (i) $\phi^R(A) < A$ for all $A > B$, and (ii) $\phi^R(A) = A$ for some $A > B$.

For case (i), we simply take $\phi = \phi^R$. For any asset level A, we claim $\phi^R(A)$ solves the maximization problem $\max_{A' \in [B, \alpha(1-\nu)A]} u\left(A - \frac{A'}{\alpha}\right) + \beta\delta V_{\phi^R}(A')$. Because ϕ^R is usc and we are in case (i), there exists $\varepsilon > 0$ such that $\phi^R(A') < A' - \varepsilon$ for all $A' \in [A, \alpha(1-\nu)A]$. Divide $[A, \alpha(1-\nu)A]$ into $N > \frac{1}{\varepsilon}A\left(\alpha(1-\nu) - 1\right)$ consecutive intervals, $I_1, ..., I_N$, of the same length $\ell \equiv \frac{A(\alpha(1-\nu)-1)}{N} < \varepsilon$, with $I_n = (A + (n-1)\,\ell, A + n\ell]$. Also define $I_0 = [B, A]$.

We claim that, for any $A' \in I_n$ with $n > 0$, there exists $A'' \in I_m$ for $m < n$ such that $u\left(A - \frac{A'}{\alpha}\right) + \beta\delta V_{\phi^R}(A') < u\left(A - \frac{A''}{\alpha}\right) + \beta\delta V_{\phi^R}(A'')$. To see why, simply take $A'' = \phi^R(A')$, observe that $u\left(A' - \frac{A'}{\alpha}\right) + \beta\delta V_{\phi^R}(A') \leq u\left(A' - \frac{A''}{\alpha}\right) + \beta\delta V_{\phi^R}(A'')$, and use the concavity of u.

From the claim, it follows that, for any $A' \in (A, \alpha(1-\upsilon)A]$, there exists $A'' \in I_0$ such that $u\left(A - \frac{A'}{\alpha}\right) + \beta\delta V_{\phi^R}(A') < u\left(A - \frac{A''}{\alpha}\right) + \beta\delta V_{\phi^R}(A'')$. Because $\phi^R(A)$ solves the problem $\max_{A' \in [B,A]} u\left(A - \frac{A'}{\alpha}\right) + \beta\delta V_{\phi^R}(A')$ by construction, it therefore also solves the problem $\max_{A' \in [B,\alpha(1-\nu)A]} u\left(A - \frac{A'}{\alpha}\right) + \beta\delta V_{\phi^R}(A')$.

For case (ii), we define $P_M(A) = \left(1 + \frac{\beta\delta}{1-\delta}\right) u\left(A\left(1 - \frac{1}{\alpha}\right)\right)$; this represents the payoff from maintaining asset level A forever. From Lemma 1 it follows that $P_{\phi^R}(A) \geq u\left(A - \frac{B}{\alpha}\right) + \frac{\beta\delta}{1-\delta} u\left(B\left(1 - \frac{1}{\alpha}\right)\right)$. Given that $\frac{\beta\delta}{1-\delta}(\alpha - 1) < 1$, there is some A^0 such that $P_M(A) < P_{\phi^R}(A)$ for $A \in (B, A^0]$. Moreover, for any A with $\phi^R(A) = A$, we obviously have $P_M(A) = P_{\phi^R}(A)$. Let $A^1 = \min\left\{A > B \mid \phi^R(A) = A\right\}$; because ϕ^R is usc, we know A^1 exists, and moreover we plainly have $A^1 > A^0$. Hence we can define $A^* \equiv \sup\left\{A \mid P_M(A') < P_{\phi^R}(A') \text{ for all } A' < A\right\}$; clearly, $A^* \in (A^0, A^1]$.

We now claim that $P_M(A^*) = P_{\phi^R}(A^*)$. This is obvious in the case where $A^* = A^1$. Suppose $A^* < A^1$. Then there exists a sequence $A^k \to A^*$ with $A^k \geq A^*$ such that $P_M(A^k) = P_{\phi^R}(A^k)$, and $\lim_{k\to\infty} \phi^R(A^k) < A^*$ (otherwise, because ϕ^R is usc, we would have $\phi^R(A^*) = A^* < A^1$, a contradiction). Because $\phi^R(A^*)$ is a feasible choice from $A^k \geq A^*$, we have, for all k,

$$u\left(A^k - \frac{\phi^R(A^*)}{\alpha}\right) + \beta\delta V_{\phi^R}(\phi^R(A^*)) \leq P_{\phi^R}(A^k) = P_M(A^k)$$

Noting that u and P_M are both continuous and taking limits, we have $P_{\phi^R}(A^*) \leq P_M(A^*)$. Also, because $\phi^R(A^k)$ is a feasible choice from A^* for sufficiently large k, we have$\neq$

$$\begin{aligned}
P_{\phi^R}(A^*) &\geq u\left(A^* - \frac{\phi^R(A^k)}{\alpha}\right) + \beta\delta V_{\phi^R}(\phi^R(A^k)) \\
&= \left[u\left(A^* - \frac{\phi^R(A^k)}{\alpha}\right) - u\left(A^k - \frac{\phi^R(A^k)}{\alpha}\right)\right] + P_{\phi^R}(A^k) \\
&= \left[u\left(A^* - \frac{\phi^R(A^k)}{\alpha}\right) - u\left(A^k - \frac{\phi^R(A^k)}{\alpha}\right)\right] + P_M(A^k)
\end{aligned}$$

Noting that u and P_M are both continuous and taking limits, we have $P_{\phi^R}(A^*) \geq P_M(A^*)$. Combining the last two arguments, we have $P_{\phi^R}(A^*) = P_M(A^*)$, as claimed.

Define $\mu \equiv \frac{A^*}{B}$, divide $[B, \infty)$ into intervals of the form $I^n = [\mu^n B, \mu^{n+1} B)$ for $n = 0, 1, 2, ...$, and construct the policy function ϕ as follows: for $A \in I^n$, let $\phi(A) = \mu^n \phi^R\left(\frac{A}{\mu^n}\right)$. It is easy to check that, for any $A \in I^n$, the path generated by ϕ starting from A remains entirely within I^n, and indeed is the same as the path generated from ϕ starting from $\frac{A}{\mu^n}$, scaled up by the factor μ^n.

Before establishing that ϕ is an equilibrium, we prove two claims.

Claim 1: For every $A \in I^n$, we have $P_\phi(A) > u\left(A - \frac{\mu^{n+1}B}{\alpha}\right) + \beta\delta V_\phi(\mu^{n+1}B)$. We will show this for $n = 0$ (in which case $\mu^{n+1}B = A^*$); the argument for $n > 0$ is essentially identical (only the scaling changes). We know that

$$\left[u\left(A^* - \frac{\phi^R(A^*)}{\alpha}\right) + \beta\delta V_{\phi^R}(\phi^R(A^*))\right] - \left[u\left(A^* - \frac{A^*}{\alpha}\right) + \beta\delta V_{\phi^R}(A^*)\right] = P_{\phi^R}(A^*) - P_M(A^*) = 0$$

Because u is concave and $\phi^R(A^*) \leq A^*$, we therefore have, for $A \in I^0$,

$$\left[u\left(A - \frac{\phi^R(A^*)}{\alpha}\right) + \beta\delta V_{\phi^R}(\phi^R(A^*))\right] - \left[u\left(A - \frac{A^*}{\alpha}\right) + \beta\delta V_{\phi^R}(A^*)\right] > 0$$

But then

$$\begin{aligned} P_\phi(A) &= P_{\phi^R}(A) \\ &\geq u\left(A - \frac{\phi^R(A^*)}{\alpha}\right) + \beta\delta V_{\phi^R}(\phi^R(A^*)) \\ &> u\left(A - \frac{A^*}{\alpha}\right) + \beta\delta V_{\phi^R}(A^*) \end{aligned}$$

as desired.

Claim 2: For $A \in I^n$, we have $P_\phi(\mu^{n+1}B) > u\left(\mu^{n+1}B - \frac{A}{\alpha}\right) + \beta\delta V_\phi(A)$. For the case of $n = 0$ (for which $\mu B = A^*$), the claim follows because, for $A \in [B, A^*) = I^0$, we have

$$u\left(A^* - \frac{A}{\alpha}\right) + \beta\delta V_\phi(A) = u\left(A^* - \frac{A}{\alpha}\right) + \beta\delta V_{\phi^R}(A) < P_{\phi^R}(A^*),$$

and in addition $P_{\phi^R}(A^*) = P_M(A^*) = P_\phi(A^*)$. The argument for $n > 0$ is essentially identical; only the scaling changes.

Now we show that ϕ is a Markov equilibrium policy function. Consider any $A \geq B$, and suppose A lies in I^n. Observe that, by construction of ϕ and by Claim 1, for all other $A' \in I^n \cup \{\mu^{n+1}B\}$, we have

$$u\left(A - \frac{\phi(A)}{\alpha}\right) + \beta\delta V_\phi(\phi(A)) \geq u\left(A - \frac{A'}{\alpha}\right) + \beta\delta V_\phi(A'). \tag{a.91}$$

Suppose that, starting at A, (a.91) holds for all $A' \in I^m \cup \{\mu^{m+1}B\}$, for some $m \geq n$. We will show it also holds for all $A' \in I^{m+1} \cup \{\mu^{m+2}B\}$. By construction, Claim 1, and the fact that

$\phi\left(\mu^{m+1}B\right) = \mu^{m+1}B$, we know that for all $A' \in I^{m+1} \cup \{\mu^{m+2}B\}$,

$$\begin{aligned} P_\phi(\mu^{m+1}B) &= u\left(\mu^{m+1}B - \frac{\mu^{m+1}B}{\alpha}\right) + \beta\delta V_\phi(\mu^{m+1}B) \\ &\geq u\left(\mu^{m+1}B - \frac{A'}{\alpha}\right) + \beta\delta V_\phi(A'). \end{aligned}$$

Because u is concave and $A < \mu^{m+1}B$, it follows that, for all $A' \in I^{m+1} \cup \{\mu^{m+2}B\}$, we have

$$u\left(A - \frac{\mu^{m+1}B}{\alpha}\right) + \beta\delta V_\phi(\mu^{m+1}B) > u\left(A - \frac{A'}{\alpha}\right) + \beta\delta V_\phi(A').$$

Combining this inequality with the premise that (a.91) holds for all $A' \in I^m \cup \{\mu^{m+1}B\}$ – and specifically for $A' = \mu^{m+1}B$ – implies that (a.91) holds for all $A' \in I^{m+1} \cup \{\mu^{m+2}B\}$ as well. Applying induction on m, we see that, starting at A, there is no $A' > A$ that yields a higher payoff than $\phi(A)$.

Having already shown that, starting at $A \in I^n$, (a.91) holds for all other $A' \in I^n$, we will now show that if it holds for all $A' \in I^m$ with $m \leq n$, then it also holds for all $A' \in I^{m-1}$. By Claim 2 and the fact that $\phi\left(\mu^m B\right) = \mu^m B$, we know that for all $A' \in I^{m-1}$,

$$\begin{aligned} P_\phi(\mu^m B) &= u\left(\mu^m B - \frac{\mu^m B}{\alpha}\right) + \beta\delta V_\phi(\mu^m B) \\ &\geq u\left(\mu^m B - \frac{A'}{\alpha}\right) + \beta\delta V_\phi(A'). \end{aligned}$$

Because u is concave and $A > \mu^m B$, it follows that, for all $A' \in I^{m-1}$, we have

$$u\left(A - \frac{\mu^m B}{\alpha}\right) + \beta\delta V_\phi(\mu^m B) > u\left(A - \frac{A'}{\alpha}\right) + \beta\delta V_\phi(A').$$

Combining this inequality with the premise that (a.91) holds for all $A' \in I^m$ – and specifically for $A' = \mu^m B$ – implies that (a.91) holds for all $A' \in I^{m-1}$ as well. Applying induction on m, we see that, starting at A, there is no $A' < A$ that yields a higher payoff than $\phi(A)$.

It follows that ϕ is in fact a Markov equilibrium policy function. ■

Proof of Proposition 7. By Proposition 6, part (i), when $\beta \geq \beta^*$ there exists a linear Markov equilibrium with non-decumulation at all asset levels (and strict accumulation when $\beta > \beta^*$). Hence, if the proposition is false, it must be that $\beta < \beta^*$, which we will assume throughout the remainder of this proof, as we work towards a contradiction.[33]

Step 1. If ϕ is a Markov equilibrium strategy, then ϕ is nondecreasing.

---

[33] For $\sigma \geq 1$, we have shown that with $\beta \leq \beta^*$, there exists no Markov equilibrium with $\phi(A) > A$ for any asset level A. We conjecture that the same is true for $\sigma \in (0, 1)$, and have numerical support for this conjecture, but have not proven it.

Suppose on the contrary that for some $A_1 > A_2$, we have $\phi(A_1) < \phi(A_2)$. Let V_i be the value of the continuation consumption stream starting from asset level $\phi(A_i)$. Then

$$\begin{aligned}
\beta\delta\left(V_2 - V_1\right) &\geq u\left(A_2 - \frac{\phi(A_1)}{\alpha}\right) - u\left(A_2 - \frac{\phi(A_2)}{\alpha}\right) \\
&> u\left(A_1 - \frac{\phi(A_1)}{\alpha}\right) - u\left(A_1 - \frac{\phi(A_2)}{\alpha}\right) \\
&\geq \beta\delta\left(V_2 - V_1\right),
\end{aligned}$$

where the first inequality follows from the fact that $\phi(A_2)$ is (weakly) chosen over $\phi(A_1)$ at asset level A_2, the second inequality follows from the strict concavity of u combined with $A_1 > A_2$ and $\phi(A_1) < \phi(A_2)$, and the third inequality follows from the fact that $\phi(A_1)$ is weakly chosen over $\phi(A_2)$ at asset level A_1. But this sequence of inequalities plainly involves a contradiction.

In what follows, $\phi(A') > A'$ for some $A' \geq B$, as given in the proposition.

Step 2. Suppose there exists $A'' > A'$ such that $\phi(A'') \leq A''$. Then there exists $A^* \in (A', A'']$ such that $\phi(A^*) = A^*$ and $\phi(A) > A$ for all $A \in (A', A^*)$.

Let $A^* \equiv \inf\{A \in [A', A''] \mid \phi(A) \leq A\}$. Because ϕ is non-decreasing, we have $\phi(A) > A$ for $A \in [A', \phi(A'))$, from which it follows that $A^* > A'$. By construction, $\phi(A) > A$ for all $A \in (A', A^*)$. If $\phi(A^*) > A^*$, we would have $\phi(A) > A$ for some interval above A^*, and if $\phi(A^*) < A^*$, we would have $\phi(A) < A$ for some interval below A^* (in each case because ϕ is non-decreasing); in either case, A^* would not equal $\inf\{A \in [A', A''] \mid \phi(A) \leq A\}$, a contradiction. Therefore, $\phi(A^*) = A^*$.

Step 3. For $\beta < \beta^*$, there exists $\gamma_1 < 1$ such that, for any $\gamma \in (\gamma_1, 1)$, we have

$$\left(1 + \frac{\beta\delta}{1-\delta}\right) u\left(1 - \frac{1}{\alpha}\right) < u\left(1 - \frac{\gamma}{\alpha}\right) + \left(\frac{\beta\delta}{1-\delta}\right) u\left(\gamma\left(1 - \frac{1}{\alpha}\right)\right) \equiv \psi_1(\gamma).$$

Observe that

$$\psi_1'(1) = u'\left(1 - \frac{1}{\alpha}\right)\frac{1}{\alpha}\left[(\alpha - 1)\left(\frac{\beta\delta}{1-\delta}\right) - 1\right] < 0,$$

(given $\beta < \beta^*$), from which the desired conclusion follows.

Step 4. There exists $\gamma_2 > 1$ such that, for any $\gamma \in (1, \gamma_2)$, we have

$$\left(\frac{1}{1-\delta}\right) u\left(1 - \frac{1}{\alpha}\right) < u\left(1 - \frac{\gamma}{\alpha}\right) + \left(\frac{\delta}{1-\delta}\right) u\left(\gamma\left(1 - \frac{1}{\alpha}\right)\right) \equiv \psi_2(\gamma).$$

Observe that

$$\psi_2'(1) = u'\left(1 - \frac{1}{\alpha}\right)\frac{1}{\alpha}\left(\frac{\alpha\delta - 1}{1-\delta}\right) > 0$$

(given $\alpha\delta > 1$), from which the desired conclusion follows.

Step 5. $\phi(A) > A$ for all $A \geq A'$.

Suppose not. Then, by Step 2, there exists $A^* > A'$ such that $\phi(A^*) = A^*$ and $\phi(A) > A$ for all $A \in (A', A^*)$. Starting from A^*, the equilibrium generates a constant asset trajectory.

Now consider $A'' \in (A', A^*)$ with $A'' > \max\{\gamma_1, 1/\gamma_2\} A^*$. Then, starting from A'', ϕ generates a non-decreasing asset trajectory A^* (strictly increasing as long as it remains below A^*), for which the asset growth rate is always less than γ_2.

By step 3,

$$\left(1+\frac{\beta\delta}{1-\delta}\right)u\left(A^*\left(1-\frac{1}{\alpha}\right)\right) < u\left(A^*-\frac{A''}{\alpha}\right)+\left(\frac{\beta\delta}{1-\delta}\right)u\left(A''\left(1-\frac{1}{\alpha}\right)\right) \tag{a.92}$$

By step 4, for $k \geq 0$, we have

$$\left(\frac{1}{1-\delta}\right)u\left(\phi^k(A'')\left(1-\frac{1}{\alpha}\right)\right) \leq u\left(\phi^k(A'')-\frac{\phi^{k+1}(A'')}{\alpha}\right)+\left(\frac{\delta}{1-\delta}\right)u\left(\phi^{k+1}(A'')\left(1-\frac{1}{\alpha}\right)\right)$$

(with strict inequality when $\phi^k(A'') < A^*$, so that $\phi^{k+1}(A'') > \phi^k(A'')$). By recursively substituting this inequality into (a.92), we obtain

$$\left(1+\frac{\beta\delta}{1-\delta}\right)u\left(A^*\left(1-\frac{1}{\alpha}\right)\right) < u\left(A^*-\frac{A''}{\alpha}\right)+\beta\delta\sum_{k=0}^{\infty}\delta^k u\left(\phi^k(A'')-\frac{\phi^{k+1}(A'')}{\alpha}\right),$$

from which it follows that the individual would deviate from $\phi(A^*)$ to A'', contradicting the supposition that ϕ is an Markov equilibrium.

Step 6. For $A \geq B$, let $\widehat{\phi}(A) = \phi(\mu A)/\mu$, where $\mu = A'/B$. Then (a) $\widehat{\phi}(A) > A$ for all $A \geq B$, and (b) $\widehat{\phi}$ is an Markov equilibrium.

Part (a) follows immediately from Step 3: for $A \geq B$, we have $\mu A \geq A'$, so $\widehat{\phi}(A) = \phi(\mu A)/\mu > \mu A/\mu = A$. Part (b) follows from the homotheticity of utility, combined with the fact that, if a continuation asset choice A_1 generates the consumption trajectory $(c_0', c_1', ...)$ starting from A under $\widehat{\phi}$, then the choice μA_1 generates the consumption trajectory $(\mu c_0', \mu c_1', ...)$ starting from μA under ϕ. Thus, if a continuation asset choice A_1 yielded a strictly higher payoff than $\widehat{\phi}(A)$ starting from A under $\widehat{\phi}$, then the choice μA_1 would yield a strictly higher payoff than $\phi(\mu A_1)$ under ϕ, a contradiction. ■

Appendix E. Algorithm

This section describes the iterative computational algorithm for obtaining an approximation to the equilibrium value correspondence $\mathcal{V}(A)$ through the sequence of correspondences $\{\mathcal{V}_k\}$ (See Section 3). Our initial correspondence is

$$\mathcal{V}_0(A) = \left[u\left(A-\frac{B}{\alpha}\right)+\frac{\delta}{1-\delta}u\left(\frac{\alpha-1}{\alpha}B\right), R(A)\right]$$

in light of Observation 1.

The computational algorithm proceeds in four steps.[34] First, we consider a finite grid on the action and utility spaces. Second, given that continuation payoffs are governed by some correspondence $\mathcal{V}_k$, we determine the best-deviation payoffs at each asset level A (assuming the worst feasible punishments in the continuation set, which are well-defined given the discrete grid).

Third, we maximize and minimize value at each A subject to the no-deviation constraint and constraints on continuation utilities (that they be suitably drawn from $\mathcal{V}_k$). For this optimization step, we think of the individual as choosing the continuation level of assets rather than current consumption. This is convenient from a computational perspective.[35]

Finally, we use public randomization to construct $\mathcal{V}_{k+1}$ from the maximum and minimum values in Step 3, and test to see if convergence has occurred. The convergence criterion measures the largest difference (in the L^∞ norm) in utility bounds for each asset level between successive approximations. We end our iterations when this difference is "small," or more precisely, when

$$\max_{A\in\mathcal{A}}\{\max\{\ |L_k(A)-L_{k+1}(A)|, |H_k(A)-H_{k+1}(A)|\ \}\} < \epsilon$$

for some given precision parameter $\epsilon > 0$, where $\mathcal{A}$ is the discretized, finite action set from Step 1.

More formally, for a given set of parametric assumptions, our computational algorithm repeatedly applies the following four steps until convergence is achieved:

Step 1. *Initialization.*

1.1. Let $\mathcal{A}$ be a finite set of assets, chosen suitably fine and with a large upper bound.

1.2. Determine initial utility bounds $[L_0(A), H_0(A)]$ for each $A \in \mathcal{A}$.

Step 2. *Best Deviations.*

2.1. Let $\mathcal{A}(A_j) = \{A_i \in \mathcal{A} | A_j \geq c(A_i, A_j) \geq \nu A_j\}$ where $c(A_i, A_j) = A_j - A_i/\alpha$.

2.2. For each $A_i \in \mathcal{A}(A_j)$ compute

$$\tilde{D}(A_i, A_j) = \ u(c(A_i, A_j)) + \ \beta\delta L_k(A_i).$$

2.3. For each $A_j \in \mathcal{A}$ compute $D(A_j) = \max_{A_i\in\mathcal{A}(A_j)} \tilde{D}(A_i, A_j)$.

Step 3. *Highest and Lowest Values.*

---

[34] This iterative numerical algorithm is a variation of the method of computing equilibria of supergames developed by Judd, Yeltekin and Conklin (2003).

[35] If consumption remains the choice variable, then we would need to discretize the consumption set. Additionally, the technology would have to be modified to ensure that for each current asset level and consumption choice, next period's assets are in the discretized asset set.

3.1. Compute

$$H_{k+1}(A_j) = \max_{A_i \in \mathcal{A}(A_j)} \{u(c(A_i, A_j)) + \delta V_i\}$$

subject to the no-deviation constraint:

$$u(c(A_i, A_j)) + \beta\delta V_i \geq D(A_j), \tag{a.93}$$

and the feasibility condition on continuation value:

$$V_i \in \mathcal{V}_k(A_i). \tag{a.94}$$

3.2. Compute

$$L_{k+1}(A_j) = \min_{A_i \in \mathcal{A}(A_j)} \{u(c(A_i, A_j)) + \delta V_i\}$$

subject to exactly the same constraints (a.93) and (a.94).

Step 4. *Public Randomization and Convergence.*

4.1. Set $V_{k+1}(A) = [L_{k+1}(A), H_{k+1}(A)]$ (public randomization). Stop if convergence is reached; else return to Step 2.

Note that, in the maximization problem of Step 3.1, we must always set $V_i = H_k(A_i)$ as the continuation utility. After all, if any continuation value satisfies the no-deviation constraint (a.93), then so does the highest feasible continuation value, and that raises the overall value of the maximand as well. In contrast, in the minimization problem of Step 3.2, we do not generally use $L_k(A_i)$ as the continuation utility, because the lowest feasible continuation value does not necessarily satisfy the no-deviation condition (a.93).[36]

For the results reported in Figure 1, we set $\sigma = 0.5$, so that

$$u(c) = \frac{1}{2}c^{1/2}.$$

Assets take on 8001 values between $[B, \bar{A}]$. We set $\bar{A} = 200$ and $B = 0.5$.[37] For the exercise depicted in Figure 1, we set the rate of return equal to 30%, the discount factor equal to 0.8, the hyperbolic parameter (β) equal to 0.4. Figure 1 Panel A plots the highest equilibrium asset choice, $X(A)$ and lowest equilibrium asset choice, $Y(A)$. Panel B plots the equilibrium value

[36] However, Proposition 2 in the main text can be adapted to show that a carrot-and-stick structure obtains, so that often the highest continuation value (or some minor variant thereof) is also chosen in this problem.

[37] The analytical results allow for unbounded asset accumulation. An unbounded state space is not feasible computationally, but to ensure that the asset bound does not impact the policy and value functions reported in any significant way, we proceed in the following way. We choose an initial asset bound and note the asset level below this bound where the value and policy functions converge to the Ramsey solution ($\beta = 0$ case). We use the analytical Ramsey solution to approximate the value and policy functions beyond this intermediate asset value. We repeat this for a variety of intermediate asset values and initial asset bounds to check the robustness of the results for asset values below the intermediate asset level.

correspondence. For this particular exercise, a poverty trap exists below an asset level of 3.47. For initial asset levels above 3.47, however, there is indefinite accumulation.

Appendix F. Uniformity and Nonuniformity: Parametric Examples

Figures (A.4), (A.5) and (A.6) display the continuation asset choices associated with the best SPE values. With the exception of β, each policy function has been generated by the same parameterization of the intrapersonal game: $\sigma = 0.5,\ B = 0.5,\ \alpha = 1.3$ and $\delta = 0.8$. The hyperbolic discount factor β takes values from the set $\{0.1, 0.2, 0.3, 0.35, 0.4, 0.45, 0.5, 0.55, 0.6, 0.7, 0.8, 0.9, 1.0\}$. At high β values, indefinite accumulation can be achieved from any initial A. At low β values, accumulation is not possible from any initial A. These are the uniform cases. For intermediate values of β, however, a poverty trap is present, with the threshold level of A dependent on the value of β. The higher the β (in the range of nonuniform cases), the lower the threshold A that allows an individual to escape poverty.[38]

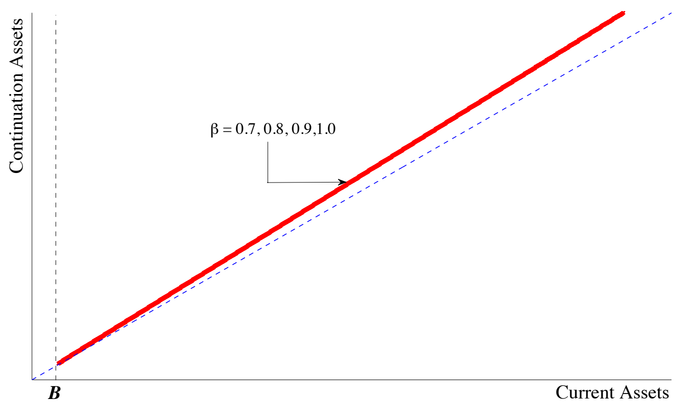

Figure A.4. High β: Indefinite accumulation.

Appendix G. Markov Punishments

Figure A.7 shows the highest equilibrium asset choices possible with reversion to strictly decumulating Markov equilibria (also plotted). It shows that a poverty trap is possible when such Markov equilibria are used as punishments.

[38] Although we only report cases from a specific, $\{\alpha,\ \delta,\ B\}$ combination, as long as $\alpha\delta > 1$ (i.e. the Ramsey problem leads to accumulation), we find regions of uniformity (low and high β) and a region of non uniformity (intermediate values of β) for other parameterizations of the model.

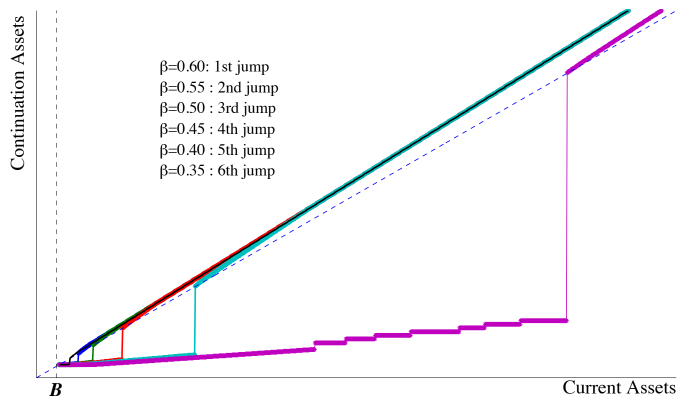

FIGURE A.5. INTERMEDIATE β: POVERTY TRAP

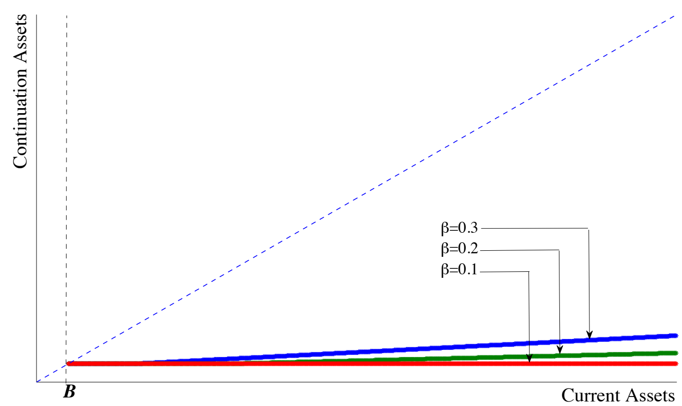

FIGURE A.6. LOWβ: DECUMULATION .

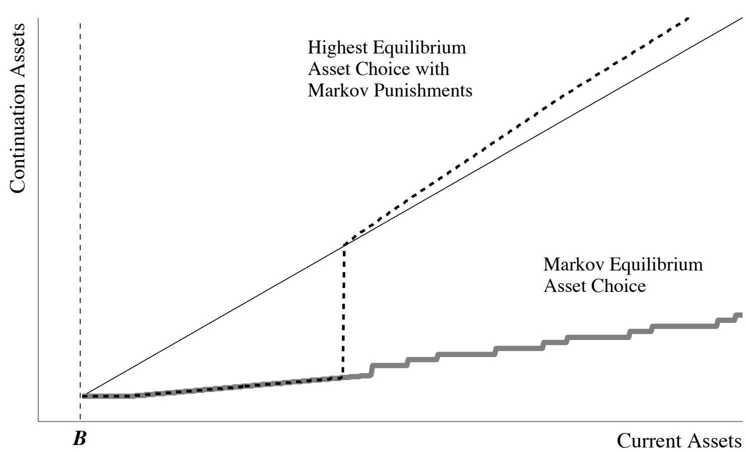

FIGURE A.7. MARKOV PUNISHMENTS.

Appendix H. Policy Regimes

In this section, we describe in more detail the extended model with taste shocks used in Section 7.3, as well as the policy regimes displayed in Figures 6 and 7. These regimes have a *lockbox* feature: assets are kept in an account with a rule specifying when and how much of the funds can be accessed. Each regime considers a different rule.

When $\alpha\delta > 1$, complete reliance on a lockbox always dominates internal rules provided that all consumption expenditures are perfectly foreseen; see discussion in main text. For these examples to have non-trivial solutions, we extend the original model to include an iid taste shock η (with probability distribution $p(\eta)$) that takes values in some finite set N and affects the flow utility in a multiplicative way. In every period, individuals make their saving/consumption decision after the realization of the current taste shock.

We first describe the baseline solution of this model without any lockboxes; it is a straightforward extension of the solution with no taste shocks. Specifically, we can think of an *expected value correspondence* $\mathcal{V}^*(A;B)$ at the start of any date that defines the set of expected equilibrium values, the expectation taken over the taste shock which is about to be realized at that date, for every asset level. (For reasons that will become clear below, we explicitly carry the lower bound B, to be thought of as unchanging for all dates.) Because η is iid, $\mathcal{V}^*$ is the same at all dates. Thinking of these as continuation values from, say, date $t+1$, we can now define $\mathcal{V}^*(A,\eta;B)$ as the set of generated values at date t for any individual with asset level $A \geq B$, who has just experienced the taste shock η. The fixed-point logic of equilibrium generation then tells us that

$$\mathcal{V}^*(A;B) = \sum_{\eta\in N} p(\eta)\mathcal{V}^*(A,\eta;B)$$

for every $A \geq B$, where we define the above convex combination of sets as the collection of all elements that are themselves the same convex combinations of elements drawn from the individual sets.[39]

This value correspondence can be generated by a variation of the same iterated procedure described in Appendix E.

Now we consider regimes with lockboxes and thresholds. All the regimes we consider have the following lockbox properties: interest can always be withdrawn from the lockbox, which pays the same rate $\alpha - 1$ as a conventional savings account. No conventional savings is allowed until

---

[39] Under public randomization, each set is an interval and so all we need to do is convexify the best elements, and likewise the worst elements, and then draw the interval between these two numbers.

a threshold (A^T) is reached.[40] At that point, some or all of the lockbox principal is unlocked and made available. Let $\hat{B}$ denote the amount that still remains locked.

Recall that by convention, A includes non-financial labor income assets and an amount B is *always* "locked up" by the imperfect credit market. Therefore, we must constrain all our regimes by the property that $A^T \geq \hat{B} \geq B$.[41] In particular, we recover the standard problem by setting $A^T = \hat{B} = B$. Note that once past the threshold, the remainder of the problem facing the individual is *exactly* as in the standard case, without a lockbox feature, provided we replace the lower bound on assets by $\hat{B}$. So we can conceive of the overall problem as follows: at any date t, an individual is either "free" or "locked", depending on whether she has *ever* crossed the asset threshold A^T before date t. If she is free, then her (expected) value correspondence from that date onwards is governed by $\mathcal{V}^*(A, \hat{B})$. We can use this fact to anchor the construction of her value correspondence in the locked state. Denote this latter correspondence by $\hat{\mathcal{V}}$. It is to be noted that $\hat{\mathcal{V}}$ depends on the three parameters $(B, A^T, \hat{B})$, but we don't need to carry this dependence explicitly in the notation and so suppress it.

We can now determine best deviation payoffs (for every realization of the taste shock), as well as highest and lowest values, in the locked state. For every η and A in the locked state, consider the problem of finding

$$\hat{D}(A, \eta) \equiv \sup_{A' \in [A, \alpha(1-\upsilon)A]} \eta u\left(A - \frac{A'}{\alpha}\right) + \beta\delta L(A'), \tag{a.95}$$

subject to

$$L(A') = \begin{cases} \inf \mathcal{V}^*(A', \hat{B}) & \text{if } A' \geq A^T \\ \inf \hat{\mathcal{V}}(A') & \text{if } A' < A^T \end{cases} \tag{a.96}$$

Notice how the constraint in (a.95) requires $A' \geq A$: assets cannot be run down in the locked state. The second constraint describes where worst punishments following the deviation come from: if the choice of A' "frees" the individual, then it is drawn from the equilibrium value correspondence $\mathcal{V}^*(A', \hat{B})$ corresponding to the subsequent free state, and if the individual is still locked, it must come from the lowest value in $\hat{\mathcal{V}}(A')$. As a matter of fact, both infima in (a.96) can be shown to be attained, while in the discretized, finite computational problem under consideration, the "sup" in (a.95) can be replaced by "max".

[40] The exercises we conduct are meant to be illustrative, and so we do not allow for contemporaneous savings while the lockbox is "active". These more realistic modifications can be easily studied, at least numerically.

[41] So, really, the *financial* assets in the lockbox are given by $A - B$, and all thresholds and locked amounts must be reinterpreted accordingly.

With $\hat{D}$ in hand, we can turn to the problem of generating values at each A and η in the locked state. It is possible to generate any value V such that

$$V = \eta u \left(A - \frac{A'}{\alpha} \right) + \delta V'$$

for some A' with $A' \geq A$, and V' satisfying

$$V' \in \begin{cases} \mathcal{V}^*(A', \hat{B}) & \text{if } A' \geq A^T \\ \hat{\mathcal{V}}(A') & \text{if } A' < A^T \end{cases}$$

as long as the no-deviation constraint is also met:

$$\eta u \left(A - \frac{A'}{\alpha} \right) \beta\delta V' \geq \hat{D}(A, \eta).$$

Let $\hat{H}(A, \eta)$ and $\hat{L}(A, \eta)$ be the largest and smallest such values,[42] and recalling public randomization, define

$$\hat{\mathcal{V}}(A, \eta) \equiv [\hat{L}(A, \eta), \hat{H}(A, \eta)].$$

These are the "η-specific" value correspondences, and now we impose the fixed point consideration that

$$\hat{\mathcal{V}}(A) = \sum_{\eta \in N} p(\eta) \hat{\mathcal{V}}(A, \eta)$$

for every $A \in [B, A^T]$.

From a computational perspective, we discretize the space of assets and proceed exactly as in Appendix E to calculate $\hat{\mathcal{V}}$. That is, a two-stage procedure is employed, the first to determine the standard value correspondence $\mathcal{V}^*$ (for the lower bounds B and $\hat{B}$), followed by a similar process to obtain $\hat{\mathcal{V}}$. We omit the details here.

The text considers four regimes, (a)-(d), all drawn from the class above. In Regime (a), there is no lockbox. This represents our standard case and corresponds to setting $A^T = B$. In Regime (b), the principal in the locked account is fully accessible after a specified $A^T > B$ is reached; so $\hat{B} = B$.

In Regime (c), the threshold is eliminated. This corresponds to setting A^T equal to infinity in the above problem (the value of $\hat{B}$ is irrelevant). The individual can always withdraw current interest, but can never access the principal.

In Regime (d), contributions to the lock-up account stop once the threshold is reached, a conventional account becomes accessible, but the principal in the lock-up account remains out of reach forever. That is, $A^T = \hat{B} > B$. In this case, a switch to the standard problem occurs once the

[42] Once again, we disregard questions of attaining the maximum and minimum, which are trivial in the current finite context, but which can be affirmatively settled anyway.

threshold is passed, but to a *different* standard problem, one characterized by the lower bound A^T on assets.

For the results displayed in Figures 6 and 7, the taste shock η takes two values, $\{0.8, 1.1\}$, with the associated probabilities $p(\eta = 0.8) = 0.3$ and $p(\eta = 1.1) = 0.7$. All other parameters are the same as in the earlier numerical results: the hyperbolic discount factor (β) is 0.4, the geometric discount factor (δ) is 0.8, the constant elasticity parameter (σ) is 0.5, and B and $\bar{A}$ are set to 0.5 and 200 respectively. The standard problem with no lockbox features a poverty trap at low asset values. For $\eta = 0.8$, there is a poverty trap for $A < 4.42$ and for the high shock $\eta = 1.1$, a poverty trap exists when $A < 5.35$. For the lock-up regimes (b) and (d), A^T is set to 5.5, slightly above the poverty threshold for the high taste shock state.

REFERENCES

Ainslie, G. 1975. "Specious Reward: A Behavioral Theory of Impulsiveness and Impulse Control." *Psychological Bulletin* 82:463–496.

———. 1991. "Derivation of "Rational" Economic Behavior from Hyperbolic Discount Curves." *American Economic Review* 81:134–140.

Ali, S. N. 2011. "Learning Self-Control." *Quarterly Journal of Economics* 126(2):857–893.

Bandura, A. 1971. *Social Learning Theory*. General Learning Press: New York.

———. 1976. "Self-reinforcement: Theoretical and Methodological Considerations." *Behaviorism* 4(2):135–155.

Bandura, A. and C. J. Kupers. 1964. "Transmission of Patterns of Self-Reinforcement Through Modeling." *Journal of Abnormal and Social Psychology* 69 (1):1–9.

Benabou, R. and J. Tirole. 2004. "Willpower and Personal Rules." *Journal of Political Economy* 112:848–886.

Benhabib, J. and A. Bisin. 2001. "Self-Control and Consumption-Saving Decisions: Cognitive Perspectives." Mimeo., Department of Economics, New York University.

Bernheim, B. D., D. Ray, and S. Yeltekin. 1999. "Self-Control, Saving, and the Low Asset Trap." URL `http://www.econ.nyu.edu/user/debraj/Papers/selfcontrol.pdf`.

Kazdin, A. E. 2012. *Behavior Modification in Applied Settings, Seventh Ed.* Waveland Press.

Klein, B. and K. B. Leffler. 1981. "The Role of Market Forces in Assuring Contractual Performance." *Journal of Political Economy* 89 (4):615–641.

Laibson, D. 1997. "Golden Eggs and Hyperbolic Discounting." *Quarterly Journal of Economics* 112:443–477.

Mahoney, M. J., N. G. Moura, and T. C. Wade. 1973. "Relative Efficacy of Self-reward, Self-punishment, and Self-monitoring Techniques for Weight Loss." *Journal of Consulting and Clinical Psychology* 40 (3):404–407.

Mischel, W. 1973. “Toward a Cognitive Social Learning Reconceptualization of Personality.” *Pscyhological Review* 80 (4):252–283.

Rachlin, H. 1974. “Self-Control.” *Behaviorism* 2:94–107.

Rehm, L. P. 1977. “A Self-Control Model of Depression.” *Behavior Therapy* 8:787–804.