Supplementary Notes to "Inequality, Lobbying and Resource Allocation," by Joan
Esteban and Debraj Ray

1. Introduction

In these notes, we supplement the published paper in a number of ways:

[1] We extend the analysis to changes in productivity as well as wealth.

[2] We prove that efficiency increases for large-scale equalizations in the distribution of
wealth.

[3] We provide some sufficient conditions under which an equalization in the wealth
distribution encourages participation.

[4] We discuss the strong single-crossing condition and show that it is automatically
satisfied for the case of a constant elasticity cost function and a Pareto distribution of
wealth, whenever the distribution of wealth is equalized.

2. Productivity Changes and Efficiency

We discuss the assertion in the paper that an across-the-board increase in individual pro-
ductivities will also lead to efficiency gains. This result employs the following strength-
ening of [A.3].

[A.3$'$] For every $\delta > 1$ the ratio
$$\frac{c(w, \delta r)}{c(w, r)}$$
is nondecreasing in w.

[Weaker versions of this assumption will probably suffice but we haven't explored this in
any detail.]

The ratio in [A.3$'$] is the factor increase in the cost by an increase in the bidding expen-
diture by a factor of $\delta > 1$. As wealth increases both costs decrease. Our assumption
simply posits that the cost at the higher level of bidding does not fall faster than the
lower cost. In particular, this assumption excludes the possibility that the costs of ex-
pending δr and r tend to converge to each other as wealth becomes large. Notice that
the cost function that we have proposed as an example in the main text
$$c(w, r) = \hat{c}(r) \left[\frac{1}{w^\theta} + a \right]$$
does satisfy [A.3$'$].

Proposition 1. *Under* [A.1]–[A.3$'$], *a proportional scaling-up of productivities (by $\beta > 1$) cannot reduce allocative efficiency. Indeed, as long as $c(w, \tilde{r}) \neq \beta c(w, r)$ for some w, allocative efficiency must strictly increase.*

To establish the first part of the proposition, we will show that $\tilde{H}$ is (weakly) less risky than H in the sense of second-order stochastic dominance. Observation 2 in the paper then guarantees the result.

To this end, it will suffice to prove that

(1) $\qquad \tilde{H}(z) > H(z)$ for some z implies that $\tilde{H}(z') \geq H(z')$ for all $z' \geq z$,

where

$$\tilde{H}(z) = G(c^{-w}(\tilde{F}^{-1}(1-z), \tilde{r})) = G(c^{-w}(\beta F^{-1}(1-z), \tilde{r})).$$

We first note that $\tilde{r} > r$. With the scaling of productivities, for the given r, there will be strictly more bidders at each wealth level. It follows that the equilibrium bid has to increase up to $\tilde{r}$ where the balance between demand and supply of licenses is restablished.

Secondly, we also note that, since the expected value of the distributions $\tilde{H}$ and H is the same, the two cumulative distributions must intersect at least once.

Let z^* be the smallest z at which the two distributions intersect, so that for w^* we have that

$$\frac{c(w^*, \tilde{r})}{c(w^*, r)} = \beta > 1.$$

We know there is a unique w_1 such that

$$w_1(z) = c^{-w}(F^{-1}(1-z), r)),$$

and a unique w_2 such that

$$w_2(z) = c^{-w}(\tilde{F}^{-1}(1-z), \tilde{r})).$$

Hence (1) is equivalent to $w^* \leq w_1(z) \leq w_2(z)$ for all $z \geq z^*$.

Note that

$$\frac{c(w_2(z'), \tilde{r})}{c(w_1(z'), r)} = \beta = \frac{c(w^*, \tilde{r})}{c(w^*, r)} \text{ for all } z'.$$

Suppose now that contrary to our claim there was $z' > z$ such that $w_1(z') > w_2(z')$. Then

$$\frac{c(w_1(z'), \tilde{r})}{c(w_1(z'), r)} \leq \frac{c(w_2(z'), \tilde{r})}{c(w_1(z'), r)} = \frac{c(w^*, \tilde{r})}{c(w^*, r)} = \beta.$$

But this contradicts our assumption [A.3$'$], so the proof of the first part is complete.

To establish the remainder of the proposition, simply note that if $c(w, \tilde{r}) \neq \beta c(w, r)$ for some w, then $\tilde{H}(z) \neq H(z)$ for *some* z . The strict concavity result established in Observation 2 then assures us that allocative efficiency must *strictly* increase. ∎

3. Large-Scale Reductions in Inequality

In the main text we argue that the effects of wealth redistribution on efficiency are rather complex. However, if the extent of the redistribution is sufficiently large, then efficiency will go up. In the main text we state this result as Proposition 3. We present here the proof of this proposition.

Proof of Proposition 3. Let $\bar{w}$ denote the common mean wealth of G and G^m, $\bar{\lambda}$ the productivity level for which $1 - F(\bar{\lambda}) = \alpha$, and let $\bar{r}$ be such that $c(\bar{w}, \bar{r}) = \bar{\lambda}$. Denote by r^m the equilibrium value of r for each G^m.

We first claim that $r^m \to \bar{r}$ as $m \to \infty$. To establish this, let $\tilde{r}$ be any limit point of r. Observe that $\tilde{r}$ must be finite because the sequence r^m must be bounded.[1] Now notice that for all m,

$$
\begin{aligned}
\alpha \;=\;& \int_0^\infty [1 - F(c(w, r^m))] \, dG^m(w) \\
=\;& \int_0^\infty [1 - F(c(w, \tilde{r}))] \, dG^m(w) + \int_0^\infty [F(c(w, \tilde{r})) - F(c(w, r^m))] \, dG^m(w) \\
=\;& \int_0^\infty [1 - F(c(w, \tilde{r}))] \, dG^m(w) + \int_{W_1}^{W_2} [F(c(w, \tilde{r})) - F(c(w, r^m))] \, dG^m(w) \\
& + \int_0^{W_1} [F(c(w, \tilde{r})) - F(c(w, r^m))] \, dG^m(w) + \int_{W_2}^\infty [F(c(w, \tilde{r})) - F(c(w, r^m))] \, dG^m(w)
\end{aligned}
$$

where W_1 and W_2 are any wealth levels such that $W_1 < \bar{w} < W_2$. The third and last terms on the RHS of the above equation must converge to zero as $m \to \infty$, because $G^m(W_1)$ and $1 - G^m(W_2)$ both converge to 0 (and the integrands in those terms are uniformly bounded). To study the second term on the RHS, observe that $F \circ c$ is a continuous function, so it is *uniformly* continuous at $\tilde{r}$ over all $w \in [W_1, W_2]$. Consequently, the second term must also go to 0 as $m \to \infty$. Finally, by weak convergence,

$$
\int_0^\infty [1 - F(c(w, r^m))] \, dG^m(w) = 1 - F(c(\bar{w}, \tilde{r})).
$$

It follows that $\tilde{r} = \bar{r}$, and the claim is established.

---

[1] It is easy to check that if r^m is unbounded, the equilibrium condition (2) must fail for some m, given the assumptions on G^m.

To complete the proof, let total output produced under G^m be denoted by Y^m. Then

$$
\begin{aligned}
Y^m &= \int_0^\infty \left[\int_{c(w,r^m)}^\infty \lambda dF(\lambda) \right] dG^m(w) \\
&= \int_0^\infty \left[\int_{c(w,\bar{r})}^\infty \lambda dF(\lambda) \right] dG^m(w) + \int_0^\infty \left[\int_{c(w,r^m)}^{c(w,\bar{r})} \lambda dF(\lambda) \right] dG^m(w) \\
&= \int_0^\infty \left[\int_{c(w,\bar{r})}^\infty \lambda dF(\lambda) \right] dG^m(w) + \int_{W_1}^{W_2} \left[\int_{c(w,r^m)}^{c(w,\bar{r})} \lambda dF(\lambda) \right] dG^m(w) \\
&+ \int_0^{W_1} \left[\int_{c(w,r^m)}^{c(w,\bar{r})} \lambda dF(\lambda) \right] dG^m(w) + \int_{W_2}^\infty \left[\int_{c(w,r^m)}^{c(w,\bar{r})} \lambda dF(\lambda) \right] dG^m(w).
\end{aligned}
$$

Once again, the third and last terms go to zero as $m \to \infty$, while uniform continuity can be applied just as before to show that the second term also goes to zero. Finally, by the property of weak convergence,

$$
\int_0^\infty \left[\int_{c(w,\bar{r})}^\infty \lambda dF(\lambda) \right] dG^m(w) \to \int_{c(\bar{w},\bar{r})}^\infty \lambda dF(\lambda)
$$

which simply means that the equilibria under the sequence G^m asymptotically display full allocative efficiency. This implies the proposition. ∎

4. Participation-Encouraging Wealth Redistributions

Proposition 4 in the main paper asserts that a progressive redistribution of wealths will increase efficiency provided that it induces a higher participation of bidders at the old bid r.

We shall now examine the conditions under which a progressive redistribution of wealth does encourage participation (evaluated at the initial equilibrium r).

The equilibrium announcement r solves the equation

$$
(2) \qquad\qquad 1 - \alpha = \int_0^\infty F(c(w,r)) dG(w).
$$

The RHS is strictly increasing in r. Hence, any new distribution of wealth $\tilde{G}$ such that

$$
1 - \alpha = \int_0^\infty F(c(w,r)) dG(w) \geq \int_0^\infty F(c(w,r)) d\tilde{G}(w)
$$

will have an equilibrium $\tilde{r} \geq r$ and hence will *encourage participation*.

By Jensen's inequality, if $\tilde{G}$ has the same expected value as G and Lorenz-dominates G, then $\tilde{r} \geq r$ provided that $F(c(w,r))$ is convex in w.

Upon differentiation of this latter function, we find that

$$\frac{\partial F}{\partial w} = f(c(w,r))c_w(w,r)$$

so that after some manipulation,

$$\frac{\partial^2 F}{\partial w^2} = \frac{f(c(w,r))c_w(w,r)^2}{c(w,r)} \left[\frac{f'(c(w,r))c(w,r)}{f(c(w,r))} + \frac{c(w,r)c_{ww}(w,r)}{c_w(w,r)^2} \right].$$

Since $c_{ww}(w,r) \geq 0$, $f'(\lambda) \geq 0$ is sufficient for the convexity of F with respect to w. The convexity of F can be more problematic at the upper tail of the distribution as f' turns negative. Indeed, if the density falls too sharply, the first term in the braces (negative) might dominate the second term. However, this need not be the case for distributions such as the Pareto — characterized by thick upper tails — or for those which asymptotically approach the Pareto distribution.

This property of the behavior of the upper tails of distributions can be analyzed by means of the "income-share elasticity" of a distribution, $\pi(x)$, as introduced in Esteban (1986). Formally,

$$\pi(x) \equiv 1 + \frac{f'(x)x}{f(x)}.$$

The income-share elasticity is falling with x for all distributions. For many distributions this fall is too fast and $\pi \to -\infty$ as $x \to \infty$. In contrast, the distinctive feature of the Pareto distribution is that $\pi(x) = -\alpha$. This property makes the Pareto distribution particularly suitable to describe many interesting distributions in Economics characterized by tails that are fatter than the ones predicted by the Normal, LogNormal or Exponential distributions. The family of distributions assymptotically "behaving like" a Pareto distribution was first analyzed by Lévy (1927) and later by Mandelbrot (1960). They proposed the "Weak Pareto Law": the ratio of the cumulative distribution function to a Pareto distribution tends to unity as the variable tends to infinity. Esteban (1986) proposed instead the weaker "Weak Weak Pareto Law" (WWPL): the income-share elasticity of a distribution decreases and tends to $-\alpha$ as the variable tends to infinity. Notice that the income-share elasticity of the distributions satisfying the WWPL is always larger than $-\alpha$. Therefore, for distributions satisfying the WWPL[2]

$$\frac{c(w,r)c_{ww}(w,r)}{c_w(w,r)^2} \geq 1 + \alpha$$

is sufficient for the convexity of F.

To examine this further, consider another strengthening of Assumption [A.3]:

---

[2]The WWPL is satisfied by the Pareto distributions of the second and third kind, among many others. By way of illustration, the three-parameter family defined by

$$\pi(x) = -\alpha + \beta x^{-\epsilon}, \text{with } \alpha, \beta, \epsilon > 0$$

satisfies the WWPL and generates the well-known Generalized Gamma distribution, see Esteban (1986).

Assumption [A.3"] The wealth elasticity of the cost function $\epsilon(w, r)$ is non-increasing in w.

We know that

$$\epsilon(w, r) \equiv -\frac{w c_w(w, r)}{c(w, r)}.$$

Differentiating we obtain

$$\begin{aligned}
\frac{\partial \epsilon(w, r)}{\partial w} &= -\frac{c(w, r) c_w(w, r) + w c(w, r) c_{ww}(w, r) - w c_w(w, r)^2}{c(w, r)^2} = \\
&= -\frac{w c_w(w, r)^2}{c(w, r)^2} \left[\frac{c(w, r) c_{ww}(w, r)}{c_w(w, r)^2} - 1 - \frac{1}{\epsilon(w, r)} \right].
\end{aligned}$$

Hence, [A.3"] implies that

$$\frac{c(w, r) c_{ww}(w, r)}{c_w(w, r)^2} \geq \frac{1}{\epsilon(w, r)} + 1.$$

We can conclude that, under [A.3"],

$$\frac{1}{\epsilon(w, r)} \geq \alpha,$$

is a sufficient condition for the convexity of F and hence for a progressive redistribution to be participation encouraging.

5. The Strong Single Crossing Condition

Proposition 5 establishes that a redistribution of wealth satisfying the *Strong Single Crossing Condition* (SSC) will increase efficiency. We examine now the conditions under which the SSC property is satisfied.

We start we recalling the SSC condition [A.4].

[A.4] For every strictly positive (w, r) and $(\tilde{w}, \tilde{r})$ such that $c(w, r) = c(\tilde{w}, \tilde{r})$ and $G(w) = \tilde{G}(w)$,

(3) $$\tilde{G}'(\tilde{w})|c_w(w, r)| > G'(w)|c_w(\tilde{w}, \tilde{r})|.$$

How strong is [A.4]? Consider a family of cost functions in which wealth has a constant-elasticity impact:

$$c(w, r) = \hat{c}(r)/w^\theta,$$

where $\hat{c}(r)$ is some increasing function and $\theta > 0$.[3] Then it is easy to see that

$$|c_w(w, r)| = \frac{\theta \hat{c}(r)}{w^{\theta+1}} = \frac{\theta c(w, r)}{w}.$$

---

[3]The reader will notice that this family is a subclass of the family introduced in (2) of the paper.

Consequently,
$$\frac{G'(w)}{|c_w(w,r)|} = \frac{G'(w)w}{\theta c(w,r)},$$
so that (3) reduces to

(4) $$\tilde{G}'(\tilde{w})\tilde{w} > G'(w)w$$

for every strictly positive pair $(w, \tilde{w})$ such that $G(w) = \tilde{G}(\tilde{w})$. Notice again that this is a specially strong form of second-order stochastic dominance (take $w = \tilde{w}$ and reinspect (4)). However, for several families of distribution functions, the stricter condition is *automatically* satisfied whenever a reduction in inequality (in the sense of Lorenz dominance) takes place.

One such class is the Pareto distribution of the second kind on w. Let
$$G(w) = 1 - m^\delta(w + m)^{-\delta},$$
where $m > 0$ and $\delta > 1$ (to ensure that a mean is well-defined). It is easy to check that w has atomless support on $[0, \infty)$, and that the mean of G is given by $m/(\delta - 1)$. Routine computation also establishes that as m and δ simultaneously increase (say, to $\tilde{m}$ and $\tilde{\delta}$), holding overall mean constant, the distribution becomes progressively more equal in the sense of Lorenz (or equivalently, second-order dominance). The question is: does (4) automatically hold when this change takes place?

To answer this, observe that the restriction $G(w) = \tilde{G}(\tilde{w})$ simply means that

(5) $$m^\delta(w + m)^{-\delta} = \tilde{m}^{\tilde{\delta}}(\tilde{w} + \tilde{m})^{-\tilde{\delta}},$$

so that
$$\frac{m}{w + m} = \left(\frac{\tilde{m}}{\tilde{w} + \tilde{m}}\right)^k,$$
where $k \equiv \tilde{\delta}/\delta$. Now it is easy to see that $kx < 1 - (1 - x)^k$ whenever $x \in (0, 1)$. Using this, we conclude that

(6) $$\frac{w}{w + m} = 1 - \frac{m}{w + m} = 1 - \left(1 - \frac{\tilde{w}}{\tilde{w} + \tilde{m}}\right)^k < k\frac{\tilde{w}}{\tilde{w} + \tilde{m}}.$$

Now $G'(w)w = \delta m^\delta(w + m)^{-(1+\delta)}w$ and $\tilde{G}'(\tilde{w})\tilde{w} = \tilde{\delta}\tilde{m}^{\tilde{\delta}}(\tilde{w} + \tilde{m})^{-(1+\tilde{\delta})}\tilde{w}$, so that — using (5) — (4) will hold if
$$\tilde{\delta}\frac{\tilde{w}}{\tilde{w} + \tilde{m}} > \delta\frac{w}{w + m},$$
which is precisely guaranteed by (6).

References

Esteban, J. (1986), "Income-Share Elasticity and the Size Distribution of Income", *International Economic Review* **27**, 439–444.

Lévy, P. (1927), "Théorie des erreurs. La loi de Gauss et les lois exceptionelles," *Bulletin de la Société Mathematique de France* **52**, 49-85.

Mandelbrot, B. (1960), "The Pareto-Lévy law and the distribution of income," *International Economic Review* **1**, 79–106.